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# Estimation of the trend function and auto-covariance for spatial models

*Estimation de la tendance et de l'auto-covariance pour les modèles spatiaux*

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ABSTRACT

We first establish, through a Berry–Esseen-type bound, the asymptotic normality of a local linear estimate of the regression function in a fixed design setting when the errors are stationary isotropic spatial random fields. On the other hand, we investigate the weak convergence of an empirical estimate of the variance of these errors in a general $\alpha$-mixing setting.



RÉSUMÉ

Nous établissons tout d'abord, à travers une borne de type Berry–Esseen, la normalité asymptotique d'un estimateur localement linéaire de la fonction de régression dans le cadre d'un design déterministe, lorsque les erreurs sont des champs aléatoires spatiaux isotropiques stationnaires. Nous établissons ensuite la convergence faible d'un estimateur de la variance de ces erreurs dans un cadre spatial $\alpha$-mélangeant.



## Version française abrégée

Nous considérons le modèle de régression spatiale à design déterministe, défini par :

$$Y_{\mathbf{i}} = r(x_{\mathbf{i}}) + \epsilon_{\mathbf{i}}, \ \ \mathbf{i} \in \mathbb{N}^N, \ x_{\mathbf{i}} \in [0,1]^N.$$

Nous souhaitons estimer $r$ à l'aide des observations $(x_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}}$, où $\mathbf{n} = (n, ..., n)^\tau \in (\mathbb{N}^*)^N$, $N \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $\mathcal{I}_{\mathbf{n}} = \{\mathbf{i} = (i_1, ..., i_N)^\tau \in (\mathbb{N}^*)^N : 1 \leq i_k \leq n, k = 1, ..., N\}$, $x_{\mathbf{i}} = \frac{\mathbf{i}}{n} = (i_1/n, ..., i_N/n)^\tau \in [0,1]^N$, $r$ est à valeurs réelles, défini sur $[0,1]^N$ et appartient à la classe $\sum(\beta, L, M)$, $\beta > \frac{N}{4}$, $L > 0$, $M > 0$, qui est définie dans (2). De plus, $(\epsilon_{\mathbf{i}}, \mathbf{i} \in \mathcal{I}_{\mathbf{n}})$ est un processus aléatoire réel strictement stationnaire de moyenne nulle et de variance inconnue $\sigma^2 > 0$, $\mathrm{Cov}(\epsilon_{\mathbf{i}}, \epsilon_{\mathbf{j}}) = \sigma^2 \exp(-a\|\mathbf{i} - \mathbf{j}\|)$, où $a$ est une constante positive connue. Notre objectif est d'établir, d'une part, la normalité asymptotique d'un estimateur

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localement linéaire de $r$ et, d'autre part, la convergence uniforme en probabilité d'un estimateur empirique de la variance des erreurs $(\epsilon_{\mathbf{i}})$. Pour ce faire, nous proposons les estimateurs de la régression (en un site $\mathbf{s}_0 \in [0,1]^N$) et de la variance suivants :

$$\widehat{r}(\mathbf{s}_0) = (1, \mathbf{0}^\tau)\left(\frac{1}{n^N}\mathcal{X}^\tau W_0 \mathcal{X}\right)^{-1}\left(\frac{1}{n^N}\mathcal{X}^\tau W_0 \mathcal{Y}\right), \ \widehat{\sigma}^2 = \frac{1}{2(n^{N-1}(n-p))}\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}^p}(Y_{\mathbf{i}} - Y_{\mathbf{i}-pe_1})^2$$

où $\mathcal{I}_{\mathbf{n}}^p = \{\mathbf{i} = (i_1, ..., i_N) \in (\mathbb{N}^*)^N : p+1 \le i_1 \le n, \ 1 \le i_k \le n, k = 2, ..., N\}$, $\mathcal{Y} = (Y_{\mathbf{1}}, \cdots, Y_{\mathbf{n}})^\tau_{n^N\times 1}$,

$$\mathcal{X} = \begin{pmatrix} 1 & \left(\frac{\mathbf{1}/n - \mathbf{s}_0}{h}\right)^\tau \\ \vdots & \vdots \\ 1 & \left(\frac{\mathbf{n}/n - \mathbf{s}_0}{h}\right)^\tau \end{pmatrix}_{n^N\times(N+1)}, \ W_0 = \operatorname{diag}\left\{\frac{1}{h^N}K\left(\frac{\mathbf{1}/n - \mathbf{s}_0}{h}\right), \cdots, \frac{1}{h^N}K\left(\frac{\mathbf{n}/n - \mathbf{s}_0}{h}\right)\right\}_{n^N\times n^N},$$

$K$ étant un noyau multivarié et $h$ une fenêtre. Après avoir introduit le contexte dans lequel nous travaillons, nous montrons que

$$\sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\frac{\widehat{r}(\mathbf{s}_0) - \mathbb{E}\left(\hat{r}(\mathbf{s}_0)\right)}{\sqrt{\operatorname{Var}\left(\hat{r}(\mathbf{s}_0)\right)}} \le z\right) - \mathbb{P}\left(D \le z\right)\right| = O\left(n^{-N/2}h^{-2N}(\log n)^{2N}\right), \text{ où } D \sim \mathcal{N}(0,1), \text{ et}$$

$$\forall\tau > 0, \sup_{r\in\sum(\beta,L,M)} \mathbb{P}_r(|\widehat{\sigma}^2 - \sigma^2| > \tau) \longrightarrow 0 \text{ lorsque } n \longrightarrow +\infty.$$

## 1. Introduction

In many fields such as econometrics, epidemiology, oceanography, and geology, the fixed-design regression model is a fundamental tool in data statistical analysis. Some works on estimation and specification tests have allowed one to solve various concrete problems from this model (see, for instance, [10], [8]). They highlight the interest of considering this model for spatial data. In this paper, we investigate the estimation of the trend function and auto-covariance for a fixed-design spatial regression model defined by

$$Y_{\mathbf{i}} = r(x_{\mathbf{i}}) + \epsilon_{\mathbf{i}}, \ \mathbf{i} \in \mathbb{N}^N, \ x_{\mathbf{i}} \in [0,1]^N. \tag{1}$$

We want to estimate $r$ from observations $(x_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}}$, where $\mathbf{n} = (n, ..., n)^\tau \in (\mathbb{N}^*)^N$, $N \in \mathbb{N}^* = \mathbb{N}\setminus\{0\}$, $\mathcal{I}_{\mathbf{n}} = \{\mathbf{i} = (i_1, ..., i_N)^\tau \in (\mathbb{N}^*)^N : 1 \le i_k \le n, k = 1, ..., N\}$, $x_{\mathbf{i}} = \frac{\mathbf{i}}{n} = (i_1/n, ..., i_N/n)^\tau \in [0,1]^N$. We suppose that $r$ is a function from $[0,1]^N$ to $\mathbb{R}$ that belongs to the class $\sum(\beta, L, M)$, $\beta > \frac{N}{4}$, $L > 0$, $M > 0$, defined from the Hölder class $H(\beta, L)$ (resp. the class $G(L)$) of functions $f$ satisfying $|f(x) - f(y)| \le L\,\|x-y\|_\infty^\beta$ (resp. $|f(x) - f(y)| \le L\,\|x-y\|_\infty$) as:

$$\sum(\beta, L, M) = \begin{cases} \{r \in H(\beta, L) : \|r\|_\infty \le M\} \text{ if } \frac{N}{4} < \beta \le 1 \\ \left\{r \in G(L) : \|r\|_\infty \le M, \left\|r'\right\|_\infty \le M\right\} \text{ if } \beta > 1 \end{cases}, \tag{2}$$

where $r'$ denotes the gradient of $r$. Besides, $(\epsilon_{\mathbf{i}}, \mathbf{i} \in \mathbb{N}^N)$ is a strictly stationary real random process with zero mean and unknown variance $\sigma^2 > 0$. The points $x_{\mathbf{i}}$ are deterministic on $[0,1]^N$ and numbered in such a way that $\|x_{\mathbf{i}} - x_{\mathbf{i}-e_1}\|_\infty = \|\mathbf{i}/n - (\mathbf{i} - e_1)/n\|_\infty = \frac{1}{n}$ (see [10] for an example of some fixed spatial design), $\|.\|_\infty$ is the supremum-norm in $\mathbb{Z}^N$, $\mathbf{i} \in \mathcal{I}_{\mathbf{n}} \setminus \{\mathbf{1}\}$ with $\mathbf{1} = (1, 1, ..., 1)^\tau$ and $e_1 = (1, 0, \cdots, 0)^\tau \in \mathbb{N}^N$. Our first aim in this paper is to establish, through a Berry–Eseen-type bound, the asymptotic normality of a local linear estimate of the regression function $r$. Secondly, we investigate the weak convergence of an empirical estimator of the variance of the errors $\epsilon_{\mathbf{i}}$ in a general $\alpha$-mixing setting. A nonparametric local linear estimator of $r$ was developed in [6] and its weak consistency has been studied. Moreover, Wang and Wang [10] proposed a local linear estimate of $r$ in the spatio-temporal context. They established, for their estimator, the weak convergence as well as the asymptotic normality through Bernstein's small-block–large-block technique, but the condition imposed on the bandwidth $h$ (i.e. $n^N h^{N(4N+5)} \to \infty$) is not a minimal condition. In this paper, we impose a weaker condition on $h$ (i.e. $n^N h^{4N} \to \infty$). Besides, the asymptotic normality, through Lindeberg's method, of the random-design regression local linear estimate for $\alpha$-mixing spatial data has been given in [5] and the condition verified by the bandwidth $h$ is $n^N h^{3N} \to \infty$, but errors $\epsilon_{\mathbf{i}}$ are assumed to be independent random fields: that is the main difference with this present work, where errors are spatially correlated. On the other hand, the estimator of the auto-covariance received an interest in the literature: one can refer to [6] for a more detailed discussion. We then propose an empirical estimator of the variance of the errors that extends to the spatial case where the estimator in [8] is given for independent variables, and we study its weak convergence. This estimator may be interesting when we investigate specification tests of the regression function (see for instance [8] for the independent data case): it has the advantage of being directly computed from data compared to the one proposed in [6], which is based on a local linear estimator of $r$. Furthermore, for $N \leq 4$, the convergence rate $O\left(\max\left(\frac{\log n}{n^4}, \frac{\log n}{n^N}\right)\right)$ of our estimator is better than the one of [6], i.e. $O\left((n^N h^N)^{-1}\right)$, with $h^N = n^{-\delta N}$ and $0 < \delta < 1$. This paper is organized as follows. In the next section, we give the regression and covariance estimators; assumptions and results are in the Section 3. Section 4 is reserved to some indications concerning the proofs of the results presented in this note.

## 2. Definition of the regression and covariance estimators

A local linear estimate $\widehat{r}$ ([6]) of the true regression function $r$ of (1) can be obtained in the following way. We approximate $r$ locally by a linear function by using the Taylor expansion in a neighbourhood of $\mathbf{s}_0 \in [0,1]^N$, that is $r(\mathbf{s}) \approx \beta_0 + \beta_1^\tau(\mathbf{s} - \mathbf{s}_0)$, $\quad \mathbf{s} \in [0,1]^N$, where $\beta_0 = r(\mathbf{s}_0)$ and $\beta_1 = r'(\mathbf{s}_0)$, and we estimate $\beta_0$ by the solution of the following least squares minimization problem

$$\min_{\beta_0,\beta_1} \sum_{\mathbf{i}\in\mathcal{I}_\mathbf{n}} \left\{Y_\mathbf{i} - \beta_0 - \beta_1^\tau\left(\frac{\mathbf{i}}{n} - \mathbf{s}_0\right)\right\}^2 \frac{1}{h^N} K\left(\frac{\mathbf{i}/n - \mathbf{s}_0}{h}\right)$$

where $K : \mathbb{R}^N \longrightarrow \mathbb{R}_+$ is a kernel function. Put $\mathcal{Y} = (Y_\mathbf{1}, \cdots, Y_\mathbf{n})^\tau_{n^N\times 1}$,

$$\mathcal{X} = \begin{pmatrix} 1 & \left(\frac{\mathbf{1}/n - \mathbf{s}_0}{h}\right)^\tau \\ \vdots & \vdots \\ 1 & \left(\frac{\mathbf{n}/n - \mathbf{s}_0}{h}\right)^\tau \end{pmatrix}_{n^N\times(N+1)}, \quad W_0 = \operatorname{diag}\left\{\frac{1}{h^N}K\left(\frac{\mathbf{1}/n - \mathbf{s}_0}{h}\right), \cdots, \frac{1}{h^N}K\left(\frac{\mathbf{n}/n - \mathbf{s}_0}{h}\right)\right\}_{n^N\times n^N}$$

and $R = \left(\beta_0, h\beta_1^\tau\right)^\tau_{(N+1)\times 1}$. The estimator $\widehat{R}$ of $R$ is $\widehat{R} = \left(\frac{1}{n^N}\mathcal{X}^\tau W_0 \mathcal{X}\right)^{-1}\left(\frac{1}{n^N}\mathcal{X}^\tau W_0 \mathcal{Y}\right)$. Then, the nonparametric kernel estimator of the regression function is

$$\widehat{r}(\mathbf{s}_0) = (1, \mathbf{0}^\tau)\widehat{R}, \quad \mathbf{s}_0 \in [0,1]^N \text{ where } \mathbf{0}^\tau = (0, \cdots, 0) \in \mathbb{N}^N.$$

In practice, we often consider that $(\epsilon_\mathbf{i})_{\mathbf{i}\in\mathbb{N}^N}$ is a stationary isotropic zero-mean random field such that

$$\operatorname{Var}(\epsilon_\mathbf{i}) = \sigma^2, \operatorname{Cov}(\epsilon_\mathbf{i}, \epsilon_\mathbf{j}) = \gamma_0(\|\mathbf{i} - \mathbf{j}\|) = \sigma^2 \exp(-a\|\mathbf{i} - \mathbf{j}\|) \text{ for } \mathbf{i} \neq \mathbf{j}, \tag{3}$$

where $a$ is a positive constant assumed to be known in our context. Classically, the covariance function is estimated by maximum likelihood or weighted least squares ([3], p. 92). An alternative is to consider the empirical estimator of $\sigma^2$: let us fix an integer $p \geq 1$ (possibly function of $n$) such that $p < n$, then

$$\widehat{\sigma}^2 = \frac{1}{2(n^{N-1}(n-p))} \sum_{\mathbf{i}\in\mathcal{I}^p_\mathbf{n}} (Y_\mathbf{i} - Y_{\mathbf{i}-pe_1})^2,$$

where $\mathcal{I}^p_\mathbf{n} = \{\mathbf{i} = (i_1, ..., i_N) \in (\mathbb{N}^*)^N : p+1 \leq i_1 \leq n,\ 1 \leq i_k \leq n, k = 2, ..., N\}$.

## 3. Assumptions and results

Assume the following.

**Assumption 1.** There exists some positive constant $b$ such that $\sup_{\mathbf{u}\in\mathbb{N}^N} |\epsilon_\mathbf{u}| < b$.

**Assumption 2.** For all $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, $\mathbf{i}_4$, we have:

$$\operatorname{Cov}(\epsilon_{\mathbf{i}_1}\epsilon_{\mathbf{i}_2}, \epsilon_{\mathbf{i}_3}\epsilon_{\mathbf{i}_4}) = \operatorname{Cov}(\epsilon_{\mathbf{i}_1}, \epsilon_{\mathbf{i}_3})\operatorname{Cov}(\epsilon_{\mathbf{i}_2}, \epsilon_{\mathbf{i}_4}) + \operatorname{Cov}(\epsilon_{\mathbf{i}_1}, \epsilon_{\mathbf{i}_4})\operatorname{Cov}(\epsilon_{\mathbf{i}_2}, \epsilon_{\mathbf{i}_3}), \tag{4}$$

$$\mathbb{E}\left[|\epsilon_{\mathbf{i}_1}|\epsilon_{\mathbf{i}_2}\epsilon_{\mathbf{i}_3}\right] \geq 0 \text{ and } \mathbb{E}\left(|\epsilon_{\mathbf{i}_1}|^3\right) = \kappa > 0. \tag{5}$$

Assumption 1 is a classical assumption. It has been used in [4] to establish a large deviation inequality for bounded spatial random variables. Assumption 2 is satisfied by the Gaussian distribution and is made for achieving a Berry–Esseen-type bound. Relation (4) has been assumed in [6] for establishing the consistency of the empirical semivariogram.

For any collection of sites $E \subset \mathbb{N}^N$, denote by $\mathcal{B}(E)$ the Borel $\sigma$-field generated by $\{\epsilon_\mathbf{i}, \mathbf{i} \in E\}$; for each couple $E'$, $E''$, let $dist(E', E'') = \min\{||\mathbf{i}' - \mathbf{i}''|| : \mathbf{i}' \in E', \mathbf{i}'' \in E''\}$ be the distance between $E'$ and $E''$, where $||\mathbf{i}|| := (i_1^2 + ... + i_N^2)^{1/2}$ stands for the

Euclidean norm. Finally, write Card$(E)$ for the cardinality of $E$. The field $(\epsilon_{\mathbf{i}})_{\mathbf{i}\in(\mathbb{N}^*)^N}$ satisfies the following mixing condition. There exists a function $g:\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $g(t)\searrow 0$ as $t\to\infty$, such that for any $E', E''\subset(\mathbb{N}^*)^N$ with finite cardinals

$$\alpha(\mathcal{B}(E'),\mathcal{B}(E''))\le \widehat{f}(\mathrm{Card}(E'),\mathrm{Card}(E''))g(dist(E',E'')), \tag{6}$$

where $\widehat{f}:\mathbb{N}^2\to\mathbb{R}^+$ is a symmetric positive function that is non-decreasing in each variable, and $\alpha(\mathcal{B}(E'),\mathcal{B}(E''))=\sup_{B\in\mathcal{B}(E'),C\in\mathcal{B}(E'')}|\mathbb{P}(B\cap C)-\mathbb{P}(B)\mathbb{P}(C)|$. We assume that $\widehat{f}$ satisfies

$$\widehat{f}(i,j)\le c\min(i,j),\ \ \forall i,j\in\mathbb{N} \tag{7}$$

for some constant $c>0$. If $\widehat{f}\equiv 1$, then the field $(\epsilon_{\mathbf{i}})_{\mathbf{i}\in(\mathbb{N}^*)^N}$ is called strongly mixing. Many stochastic processes, among them various useful time series models satisfy strong mixing properties, which are relatively easy to check. Condition (7) is weaker than the strong mixing condition and have been used for finite-dimensional variables in [1]. We consider that the process satisfies a polynomial mixing condition:

$$g(t)\le Ct^{-\xi},\ \ C>0,\ \ t\in\mathbb{R},\ \text{with}\ \xi\ge\frac{N+1}{1-\gamma}\ge 2(N+1),\ \gamma\in\left]\frac{1}{2},1\right[. \tag{8}$$

Condition (3) can be satisfied by stationary Gaussian strongly mixing random fields. For that, one can construct such Gaussian fields with a sufficiently large polynomial decay (8) of correlation. In what follows, we introduce assumptions on kernel $K$. They have also been used in [6].

**Assumption 3.** $K(\cdot)$ is symmetric, Lipschitz, continuous and bounded. The support of $K(\cdot)$ is $[-1,1]^N$, $\int K(\mathbf{u})\,\mathrm{d}\mathbf{u}=1$, $\int \mathbf{u}K(\mathbf{u})\,\mathrm{d}\mathbf{u}=\mathbf{0}$, $\int \mathbf{u}\mathbf{u}^\tau K(\mathbf{u})\,\mathrm{d}\mathbf{u}=\nu_2(K)I$ with $\nu_2(K)\ne 0$ and $m\le K(\mathbf{u})\le M$, where $m$ and $M$ are strictly positive constants.

**Theorem 1.** *(asymptotic normality of $\hat{r}(\mathbf{s}_0)$)*
*Assume that Assumptions 1–3 and condition (3) hold and h is such that $h\to 0$, $n^N h^{4N}\to\infty$ and $(\log n)^{4N}/(n^N h^{4N})\to 0$ as $n\to\infty$. Then, for n large enough and $\mathbf{s}_0\in[0,1]^N$, we have*

$$\sup_{z\in\mathbb{R}}\left|\mathbb{P}\left(\frac{\hat{r}(\mathbf{s}_0)-\mathbb{E}\left(\hat{r}(\mathbf{s}_0)\right)}{\sqrt{\mathrm{Var}\left(\hat{r}(\mathbf{s}_0)\right)}}\le z\right)-\mathbb{P}\left(D\le z\right)\right|=O\left(n^{-N/2}h^{-2N}(\log n)^{2N}\right)\ \ \textit{where}\ \ D\sim\mathcal{N}(0,1).$$

**Theorem 2.** *(weak convergence of $\widehat{\sigma}^2$)*
*Choose p such that $p=\lfloor(\log n)^{1/(4N)}\rfloor$, where $\lfloor x\rfloor$ stands for the integer part of a real number x. Assume that conditions (6)-(8) hold. In addition, assume that there exists some positive constants a and b such that $\mathbb{E}[\exp(a|\epsilon_{\mathbf{i}}|^b)]<\infty$. Then, for all $\tau>0$, we have*

$$\sup_{r\in\sum(\beta,L,M)}\mathbb{P}_r(|\widehat{\sigma}^2-\sigma^2|>\tau)=O\left(\frac{\log n}{n^{N-1}(n-p)}\right)+O\left(\max\left(\frac{\log n}{n^4},\frac{\log n}{n^N}\right)\right)\to 0\ \ \textit{as}\ \ n\to+\infty.$$

## 4. Brief outline of the proofs

**Proof of Theorem 1.** Putting $A_n:=\frac{1}{n^N}\mathcal{X}^\tau W_0\mathcal{X}$, $K_{\mathbf{i}}:=\frac{1}{h^N}K\left(\frac{\mathbf{i}/n-\mathbf{s}_0}{h}\right)$,

$$k_n(\mathbf{i})=\frac{1}{\det(A_n)}\left[\det\left(\frac{1}{n^N}\sum_{\mathbf{i}\in\mathcal{I}_\mathbf{n}}\left(\frac{\mathbf{i}/n-\mathbf{s}_0}{h}\right)\left(\frac{\mathbf{i}/n-\mathbf{s}_0}{h}\right)^\tau K_{\mathbf{i}}\right)\right]$$
$$-\frac{1}{\det(A_n)}\left(\frac{1}{n^N}\sum_{\mathbf{i}\in\mathcal{I}_\mathbf{n}}\left(\frac{\mathbf{i}/n-\mathbf{s}_0}{h}\right)^\tau K_{\mathbf{i}}\right)\left(\frac{\mathbf{i}/n-\mathbf{s}_0}{h}\right),$$

and $X_{\mathbf{i}}=K\left(\frac{\mathbf{i}/n-\mathbf{s}_0}{h}\right)\epsilon_{\mathbf{i}}k_n(\mathbf{i})$, one has $\mathbb{E}\left(X_{\mathbf{i}}\right)=0$ and $\lim_{n\to\infty}n^Nh^N\mathrm{Var}\left(\widehat{r}(\mathbf{s}_0)\right)<+\infty$. Since, from Assumption 3, we have $\left|\frac{\mathbf{i}/n-\mathbf{s}_0}{h}\right|\le\mathbf{1}$ and $\lim_{n\to\infty}k_n(\mathbf{i})=1$, then for $n$ large enough, we have $k_n(\mathbf{i})>0$ and, since $K\left(\frac{\mathbf{i}/n-\mathbf{s}_0}{h}\right)>0$, then, with Assumption 2, we have $\mathbb{E}\left(|X_{\mathbf{i}}|^3\right)=\kappa K^3\left(\frac{\mathbf{i}/n-\mathbf{s}_0}{h}\right)k_n^3(\mathbf{i})>0$, $\mathbb{E}\left[|X_{\mathbf{i}_1}|X_{\mathbf{i}_2}X_{\mathbf{i}_3}\right]=\Pi_{l=1}^3K\left(\frac{\mathbf{i}_l/n-\mathbf{s}_0}{h}\right)k_n(\mathbf{i}_l)\mathbb{E}\left[|\epsilon_{\mathbf{i}_1}|\epsilon_{\mathbf{i}_2}\epsilon_{\mathbf{i}_3}\right]\ge 0$. Further, since

$$\mathbb{E}\left[\epsilon_{\mathbf{i}_1}\epsilon_{\mathbf{i}_2}\epsilon_{\mathbf{i}_3}\epsilon_{\mathbf{i}_4}\right]=\mathrm{Cov}(\epsilon_{\mathbf{i}_1},\epsilon_{\mathbf{i}_3})\mathrm{Cov}(\epsilon_{\mathbf{i}_2},\epsilon_{\mathbf{i}_4})+\mathrm{Cov}(\epsilon_{\mathbf{i}_1},\epsilon_{\mathbf{i}_4})\mathrm{Cov}(\epsilon_{\mathbf{i}_2},\epsilon_{\mathbf{i}_3})+\mathrm{Cov}(\epsilon_{\mathbf{i}_3},\epsilon_{\mathbf{i}_4})>0,$$

then $\mathbb{E}\left[X_{\mathbf{i}_1}X_{\mathbf{i}_2}X_{\mathbf{i}_3}X_{\mathbf{i}_4}\right]\ge m^4\mathbb{E}\left[\epsilon_{\mathbf{i}_1}\epsilon_{\mathbf{i}_2}\epsilon_{\mathbf{i}_3}\epsilon_{\mathbf{i}_4}\right]\ge 0$. Moreover, we have $\frac{\widehat{r}(\mathbf{s}_0)-\mathbb{E}(\widehat{r}(\mathbf{s}_0))}{\sqrt{\mathrm{Var}(\widehat{r}(\mathbf{s}_0))}}=\sum_{\mathbf{i}\in\mathcal{I}_\mathbf{n}}\frac{X_{\mathbf{i}}}{\sqrt{V_\mathbf{n}}}=S_\mathbf{n}$ and $h\to 0$, such that $n^Nh^{4N}\to\infty$ and $(\log n)^{4N}/(n^Nh^{4N})\to 0$. Applying Theorem 2 in [2], we obtain the result of Theorem 1. □

**Proof of Theorem 2.** Put $\xi_{\mathbf{i}} := A_{\mathbf{i}} + B_{\mathbf{i}} + C_{\mathbf{i}}$, with $A_{\mathbf{i}} := (\epsilon_{\mathbf{i}} - \epsilon_{\mathbf{i}-pe_1})^2 - 2\sigma^2$, $B_{\mathbf{i}} := (r(x_{\mathbf{i}}) - r(x_{\mathbf{i}-pe_1}))^2$, $C_{\mathbf{i}} := 2(\epsilon_{\mathbf{i}} - \epsilon_{\mathbf{i}-pe_l})(r(x_{\mathbf{i}}) - r(x_{\mathbf{i}-pe_1}))$, then

$$\mathbb{E}[(\widehat{\sigma}^2 - \sigma^2)^2] = \frac{1}{4(n^{N-1}(n-p))^2}\left[\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}^p}\mathbb{E}(\xi_{\mathbf{i}}^2) + \sum_{\mathbf{i}_0\neq\mathbf{i}_1}\mathbb{E}(\xi_{\mathbf{i}_0}\xi_{\mathbf{i}_1})\right] := D_n + E_n.$$

By using Lemma 2.1 in [9] together with the order defined in [7], we obtain

$$D_n = O\left(\frac{1}{n^{N-1}(n-p)}\right) \text{ and } E_n = O\left(\frac{\log n}{n^{N-1}(n-p)}\right) + O\left(\max\left(\frac{\log n}{n^4}, \frac{\log n}{n^N}\right)\right).$$

Therefore,

$$\sup_{r\in\sum(\beta,L,M)}\mathbb{P}_r(|\widehat{\sigma}^2 - \sigma^2| > \tau) = O\left(\frac{\log n}{n^{N-1}(n-p)}\right) + O\left(\max\left(\frac{\log n}{n^4}, \frac{\log n}{n^N}\right)\right) \to 0 \text{ as } n \to +\infty. \quad \square$$


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