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Differential K-theory, η-invariant, and localization

K-théorie différentielle, invariant η et localisation

Bo Liu [a], Xiaonan Ma [b]

[a] School of Mathematical Sciences, Shanghai Key Laboratory of PMMP, East China Normal University, Shanghai, 200241, PR China
[b] Université Paris-Diderot (Paris-7), UFR de mathématiques, case 7012, 75205 Paris cedex 13, France

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ABSTRACT

We establish a version of a localization formula for equivariant η-invariants by combining an extension of Goette's result on the comparison of two types of equivariant η-invariants and a localization formula in differential K-theory for S^1-actions. An important step is to construct a pre-λ-ring structure in differential K-theory.

RÉSUMÉ

Nous établissons un résultat de comparaison de deux versions naturelles de l'invariant η équivariant par une formule locale. En combinant ce résultat avec une formule de localisation en K-théorie différentielle, nous obtenons une formule de localisation pour l'invariant η équivariant. Une étape importante est la construction d'une structure de pré-λ-anneau sur la K-théorie différentielle.

0. Introduction

In this note, we give various refinements of the fixed-point formulas in equivariant K-theory of Atiyah–Segal at the level of certain global spectral invariants: the equivariant η-invariants.

More precisely, if Y is a compact Riemannian manifold equipped with the action of a compact Lie group, and if D is a Dirac operator on Y, Atiyah and Segal [4] gave an expression for the equivariant index of D in terms of the K-theory of the fixed-point set.

On the other hand, η-invariants of Dirac operators are global spectral invariants of odd dimensional compact manifolds, which appear in the index theorem of Atiyah–Patodi–Singer (APS) for manifolds with boundary [3]. In the equivariant version [18] of the theorem of APS, the contribution of the boundary is given by the equivariant η-invariant of the boundary.

E-mail addresses: boliumath@outlook.com, bliu@math.ecnu.edu.cn (B. Liu), xiaonan.ma@imj-prg.fr (X. Ma).

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In this note, when the group is just S^1, we establish an analogue of the Atiyah–Segal localization formula for such equivariant η-invariants. More precisely, in Theorem 3.3, we show that as functions on S^1, up to a rational function with integral coefficients, the equivariant η-invariant coincides with the equivariant η-invariant of the fixed-point set.

To prove our result, we proceed in two steps. In a first step, of independent interest, we extend to equivariant η-invariants what was done by Bismut–Goette for equivariant holomorphic torsion [13]. In the same way as fixed-point formulas have two equivalent versions, the Lefschetz fixed-point formulas and Kirillov-like formulas of Berline–Vergne [6], the same is true for equivariant η-invariants. Our first step consists in showing that the difference between the two versions is given by an explicit local formula, involving natural Chern–Simons currents. The techniques used in this first step are inspired by Bismut–Goette [13].

In a second step, by developing methods of differential K-theory, we prove our final formula, by first showing that it holds for any element in the complement of a finite set, modulo the values at this element of rational functions with integral coefficients; and we use the first step to finally obtain our final result over S^1.

Our results on equivariant η-invariants should be compared with the results of Köhler–Roessler [24], [25] for equivariant holomorphic torsion on arithmetic varieties.

Note that the holomorphic analytic torsion (and its families version: the torsion forms of Bismut–Köhler [15]) is the analytic counterpart to the direct image in Arakelov geometry [36], whose foundation was developed by Gillet-Soulé and Bismut in the 1980s. The η-invariant (and its families version: the η-forms of Bismut–Cheeger [11]) is now the analytic counterpart to the direct image in differential K-theory, developed by Hopkins–Singer [23], Simons–Sullivan [35], Bunke–Schick [17], Freed–Lott [19], etc.

In the arithmetic context, Köhler and Roessler's results [24, Theorem 4.4] give a relation of the equivariant holomorphic torsion of a complex manifold to the analytic torsion of the fixed-point set for n-th roots of unity. In [25, Lemma 2.3], they discussed in detail this problem and made a conjecture for complex manifolds [25, Conjecture, p. 82]. Köhler–Roessler [25] did not use the comparison formula of Bismut–Goette [13], but they used instead their arithmetic equivariant Riemann–Roch formula. For more applications of the arithmetic equivariant Riemann–Roch formula, see Maillot-Roessler [32] and later references.

Details will be developed in [28,29].

Notation: For $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, we denote by $\Omega^{\bullet}(X,\mathbb{K})$ the space of smooth $\mathbb{K}$-valued differential forms on a manifold X and its subspaces of even/odd degree forms by $\Omega^{\mathrm{even/odd}}(X,\mathbb{K})$. Let d be the exterior differential, then the image of d is the space of exact forms, $\mathrm{Im}\,\mathrm{d}$.

1. Comparison formula for equivariant η-invariants

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Let Y be an odd-dimensional compact oriented G-manifold. Let g^{TY} be a G-invariant metric on TY. Assume that Y has a G-equivariant spin^c structure [26, Appendix D], with the associated G-equivariant Hermitian line bundle (L,h^L). We denote by $\mathcal{S}(TY,L)$ the corresponding spinor bundle on Y. Let (E,h^E) be a G-equivariant Hermitian vector bundle on Y. Let ∇^{TY} be the Levi-Civita connection on (TY,g^{TY}). Let ∇^L and ∇^E be G-invariant Hermitian connections on (L,h^L) and (E,h^E). Put

$$\underline{TY}=(TY,g^{TY},\nabla^{TY}),\quad \underline{L}=(L,h^L,\nabla^L),\quad \underline{E}=(E,h^E,\nabla^E). \tag{1}$$

We call $\underline{E}$ a G-equivariant geometric triple. Let $\nabla^{\mathcal{S}_Y\otimes E}$ be the connection on $\mathcal{S}(TY,L)\otimes E$ induced by ∇^{TY}, ∇^L and ∇^E.

Let $c(\cdot)$ be the Clifford action of TY on $\mathcal{S}(TY,L)$. The Dirac operator is defined by

$$D^Y\otimes E=\sum_i c(e_i)\nabla^{\mathcal{S}_Y\otimes E}_{e_i}:\mathscr{C}^\infty(Y,\mathcal{S}(TY,L)\otimes E)\to\mathscr{C}^\infty(Y,\mathcal{S}(TY,L)\otimes E). \tag{2}$$

Here $\{e_i\}$ is a locally orthonormal frame of TY. Let $\mathrm{d}v_Y(x)$ be the Riemannian volume form of (Y,g^{TY}). Then $D^Y\otimes E$ is a first-order self-adjoint elliptic operator on Y with respect to the Hermitian product

$$\langle s,s'\rangle=\int_Y\langle s,s'\rangle(x)\,\mathrm{d}v_Y(x),\quad \text{for } s,s'\in\mathscr{C}^\infty(Y,\mathcal{S}(TY,L)\otimes E). \tag{3}$$

Its kernel $\mathrm{Ker}(D^Y\otimes E)$ is a finite-dimensional G-complex vector space. Let $\exp(-u(D^Y\otimes E)^2)$, $u>0$, be the heat semi-group of $(D^Y\otimes E)^2$.

For $g\in G$, the equivariant (reduced) η-invariant associated with $\underline{TY},\underline{L},\underline{E}$ is defined by [18],

$$\bar{\eta}_g(\underline{TY},\underline{L},\underline{E})=\int_0^{+\infty}\mathrm{Tr}\left[g(D^Y\otimes E)\exp(-u(D^Y\otimes E)^2)\right]\frac{\mathrm{d}u}{2\sqrt{\pi u}}+\frac{1}{2}\mathrm{Tr}|_{\mathrm{Ker}(D^Y\otimes E)}[g]\in\mathbb{C}. \tag{4}$$

When $g=e$, the identity element of G, $\bar{\eta}_g(\underline{TY},\underline{L},\underline{E})$ is just the reduced η-invariant $\bar{\eta}(\underline{TY},\underline{L},\underline{E})$. The convergence of the integral at $u=0$ in (4) is nontrivial (see, e.g., [12, Theorem 2.6], [18], [37, Theorem 2.1]).

The G-action on $\mathscr{C}^\infty(Y,E)$ is given by $(g.s)(x)=g(s(g^{-1}x))$ for $g\in G$, $s\in\mathscr{C}^\infty(Y,E)$. For $K\in\mathfrak{g}$, let $K^Y(x)=\frac{\partial}{\partial t}\big|_{t=0}\mathrm{e}^{tK}\cdot x$ be the induced vector field on Y, and $\mathcal{L}_K$ be the corresponding Lie derivative given by $\mathcal{L}_K s=\frac{\partial}{\partial t}\big|_{t=0}\left(\mathrm{e}^{-tK}.s\right)$ for $s\in\mathscr{C}^\infty(Y,E)$. The associated moment maps are defined by [5, Definition 7.5]

$$\begin{aligned} m^E(K)&:=\nabla^E_{K^Y}-\mathcal{L}_K|_E\in\mathscr{C}^\infty(Y,\mathrm{End}(E)),\\ m^{TY}(K)&:=\nabla^{TY}_{K^Y}-\mathcal{L}_K|_{TY}=\nabla^{TY}_{\cdot}K^Y\in\mathscr{C}^\infty(Y,\mathrm{End}(TY)). \end{aligned}\tag{5}$$

For $g\in G$, let Y^g be the fixed-point set of g. Observe that $m^{TY}(K)|_{Y^g}$ preserves the decomposition of real vector bundles on Y^g

$$TY|_{Y^g}=TY^g\oplus\oplus_{0<\theta\leq\pi}N(\theta),\tag{6}$$

where $\mathrm{d}g|_{N(\pi)}=-\mathrm{Id}$ and for each θ, $0<\theta<\pi$, $N(\theta)$ is the underlying real vector bundle of a complex vector bundle over Y^g on which $\mathrm{d}g$ acts by multiplication by $\mathrm{e}^{\mathrm{i}\theta}$. Let $m^{TY^g}(K)$ and $m^{N(\theta)}(K)$ be the restrictions of $m^{TY}(K)|_{Y^g}$ to TY^g and $N(\theta)$. Since ∇^{TY} is G-invariant, it preserves the splitting (6). Let ∇^{TY^g} and $\nabla^{N(\theta)}$ be the corresponding induced connections on TY^g and $N(\theta)$, with curvatures R^{TY^g} and $R^{N(\theta)}$. Let R^E be the curvature of ∇^E. Let $R_K^{TY^g}$, $R_K^{N(\theta)}$ and R_K^E be the equivariant curvatures of TY^g, $N(\theta)$, and E defined by

$$\begin{aligned} R_K^{TY^g}&=R^{TY^g}-2\mathrm{i}\pi m^{TY^g}(K),\quad R_K^{N(\theta)}=R^{N(\theta)}-2\mathrm{i}\pi m^{N(\theta)}(K),\\ R_K^E&=R^E-2\mathrm{i}\pi m^E(K). \end{aligned}\tag{7}$$

Let R_K^L be the corresponding equivariant curvature on L as in (7). We assume that g acts on $L|_{Y^g}$ by multiplication by $\mathrm{e}^{\mathrm{i}\theta_1}$, $0\leq\theta_1<2\pi$.

For $g\in G$, let $Z(g)\subset G$ be the centralizer of g, and let $\mathfrak{z}(g)$ be its Lie algebra. For $K\in\mathfrak{z}(g)$, $|K|$ small enough, set

$$\begin{aligned} \widehat{\mathrm{A}}_{g,K}(TY,\nabla^{TY}):&=\det{}^{1/2}\left(\frac{\frac{\mathrm{i}}{4\pi}R_K^{TY^g}}{\sinh\left(\frac{\mathrm{i}}{4\pi}R_K^{TY^g}\right)}\right)\\ &\quad\times\prod_{0<\theta\leq\pi}\left(\mathrm{i}^{\frac12\dim N(\theta)}\det{}^{1/2}\left(1-g\exp\left(\frac{\mathrm{i}}{2\pi}R_K^{N(\theta)}\right)\right)\right)^{-1}\in\Omega^\bullet(Y^g,\mathbb{C}),\\ \mathrm{ch}_{g,K}(\underline{E}):&=\mathrm{Tr}\left[g\exp\left(\frac{\mathrm{i}}{2\pi}R_K^E\right)\right]\in\Omega^\bullet(Y^g,\mathbb{C}),\\ \mathrm{ch}_{g,K}(\underline{L^{1/2}}):&=\exp\left(\frac{\mathrm{i}}{4\pi}R_K^L|_{Y^g}+\frac{\mathrm{i}}{2}\theta_1\right)\in\Omega^\bullet(Y^g,\mathbb{C}),\\ \mathrm{Td}_{g,K}(\nabla^{TY},\nabla^L):&=\widehat{\mathrm{A}}_{g,K}(TY,\nabla^{TY})\,\mathrm{ch}_{g,K}(\underline{L^{1/2}}). \end{aligned}\tag{8}$$

If $K=0$, then $\widehat{\mathrm{A}}_{g,K}(TY,\nabla^{TY})$, $\mathrm{ch}_{g,K}(\underline{E})$, $\mathrm{ch}_{g,K}(\underline{L^{1/2}})$ and $\mathrm{Td}_{g,K}(\nabla^{TY},\nabla^L)$ are just the equivariant characteristic forms $\widehat{\mathrm{A}}_g(TY,\nabla^{TY})$, $\mathrm{ch}_g(\underline{E})$, $\mathrm{ch}_g(\underline{L^{1/2}})$ and $\mathrm{Td}_g(\nabla^{TY},\nabla^L)$. When $g=e$, we will write $\mathrm{ch}(\underline{E})$ instead of $\mathrm{ch}_g(\underline{E})$.

Let d be the exterior differential operator. Set

$$\mathrm{d}_K=\mathrm{d}-2\mathrm{i}\pi i_{K^Y},\tag{9}$$

where $i_{\cdot}$ is the interior product on forms. For $K\in\mathfrak{z}(g)$, $\widehat{\mathrm{A}}_{g,K}(TY,\nabla^{TY})$, $\mathrm{ch}_{g,K}(\underline{E})$ and $\mathrm{ch}_{g,K}(\underline{L^{1/2}})$ are d_K-closed [5, Theorem 7.7].

Let $\vartheta_K\in T^*Y$ be the 1-form which is dual to K^Y by the metric g^{TY}. For $g\in G$, $K\in\mathfrak{z}(g)$ and $|K|$ small enough, by [22, Proposition 2.2], the following integral

$$\mathcal{M}_{g,K}(\underline{TY},\underline{L},\underline{E})=-\int_0^{+\infty}\left\{\int_{Y^g}\frac{\vartheta_K}{2\mathrm{i}\pi}\exp\left(\frac{v\,\mathrm{d}_K\vartheta_K}{2\mathrm{i}\pi}\right)\mathrm{Td}_{g,K}(\nabla^{TY},\nabla^L)\,\mathrm{ch}_{g,K}(\underline{E})\right\}\mathrm{d}v\tag{10}$$

is well-defined.

Let us explain first how $\mathcal{M}_{g,K}$ appears naturally in the localization formula of the equivariant cohomology from the local index theory point of view. Note that the Berline–Vergne localization formula [6] says that, if $\alpha\in\Omega^\bullet(Y,\mathbb{C})$, $\mathrm{d}_K\alpha=0$, then

$$\int_Y\alpha=\int_{Y^K}\frac{\mathrm{i}^{-\frac12\dim N_{Y^K/Y}}}{\det^{1/2}\left(R_K^{N_{Y^K/Y}}/(2\mathrm{i}\pi)\right)}\alpha,\tag{11}$$

where Y^K is the zero set of K^Y, $N_{Y^K/Y}$ is the normal bundle of Y^K in Y and $R_K^{N_{Y^K/Y}}$ is the associated equivariant curvatures as in (7). In [9, (1.10)], Bismut proved that, for any $v > 0$,

$$\int_Y \alpha = \int_Y \exp\left(\frac{d_K \vartheta_K}{2v i\pi}\right)\alpha, \tag{12}$$

by establishing the following equation

$$\frac{\partial}{\partial v}\left(\exp\left(\frac{d_K \vartheta_K}{2v i\pi}\right)\right) = -\frac{1}{v^2} d_K\left(\frac{\vartheta_K}{2i\pi}\exp\left(\frac{d_K \vartheta_K}{2v i\pi}\right)\right), \tag{13}$$

to show the derivative of the right-hand side of (12) vanishes, then obtained (12) by making $v \to +\infty$. As $v \to 0$, in [9, (1.14)–(1.21)], he showed that the right-hand side of (12) converges to the right-hand side of (11). From this discussion, the current on Y [10, Theorem 1.8],

$$Q_K = -\int_0^\infty \frac{\vartheta_K}{2v i\pi}\exp\left(\frac{d_K \vartheta_K}{2v i\pi}\right)\frac{dv}{v} = \int_0^\infty \frac{\vartheta_K}{2i\pi}\exp\left(\frac{v d_K \vartheta_K}{2i\pi}\right) dv\,, \tag{14}$$

is such that

$$d_K Q_K = 1 - \frac{i^{-\frac{1}{2}\dim N_{Y^K/Y}}\,\delta_{Y^K}}{\det^{1/2}\left(R_K^{N_{Y^K/Y}}/(2i\pi)\right)}. \tag{15}$$

Set

$$D_K = D^Y \otimes E + \frac{1}{4}c(K^Y). \tag{16}$$

In the following definition of the infinitesimal η-invariant, the operator $\sqrt{u}D^Y \otimes E + \frac{c(K^Y)}{4\sqrt{u}}$ was introduced by Bismut [7] in his heat kernel proof of the Kirillov formula for the equivariant index. As observed by Bismut [8, §1d), §3b)] (cf. also [5, §10.7]), its square plus $\mathcal{L}_{K^Y}$ is the square of the Bismut superconnection for a fibration with compact structure group, by replacing K^Y by the curvature of the fibration.

Theorem 1.1. *For $g \in G$, there exists $\beta > 0$ such that, for $K \in \mathfrak{z}(g)$, $|K| < \beta$, the integral*

$$\bar{\eta}_{g,K}(\underline{TY}, \underline{L}, \underline{E}) = \int_0^{+\infty} \mathrm{Tr}\left[g D_{-K/u}\exp\left(-u D_{K/u}^2 - \mathcal{L}_K\right)\right]\frac{du}{2\sqrt{\pi u}} + \frac{1}{2}\mathrm{Tr}|_{\mathrm{Ker}(D^Y\otimes E)}[g e^K] \in \mathbb{C} \tag{17}$$

is well-defined.

For $K_0 \in \mathfrak{z}(g)$, there exists $\beta > 0$ such that, for $t \in \mathbb{R}$, $0 < |t| < \beta$, we have

$$\bar{\eta}_{g,tK_0}(\underline{TY}, \underline{L}, \underline{E}) = \bar{\eta}_{ge^{tK_0}}(\underline{TY}, \underline{L}, \underline{E}) + \mathcal{M}_{g,tK_0}(\underline{TY}, \underline{L}, \underline{E}). \tag{18}$$

Furthermore, for $K_0 \in \mathfrak{z}(g)$, $\bar{\eta}_{g,tK_0}(\underline{TY}, \underline{L}, \underline{E})$ and $t^{(\dim Y^g+1)/2}\mathcal{M}_{g,tK_0}(\underline{TY}, \underline{L}, \underline{E})$ are analytic functions of t, for $t \in \mathbb{R}$, $|t| < \beta$.

In the sequel, $\bar{\eta}_{g,K}(\underline{TY}, \underline{L}, \underline{E})$ will be called the equivariant infinitesimal (reduced) η-invariant and we denote $\bar{\eta}_K(\underline{TY}, \underline{L}, \underline{E}) := \bar{\eta}_{e,K}(\underline{TY}, \underline{L}, \underline{E})$.

Since $\bar{\eta}_{g,tK_0}(\underline{TY}, \underline{L}, \underline{E})$ is an analytic function of t, when $t \to 0$, the singularity of $\bar{\eta}_{ge^{tK_0}}(\underline{TY}, \underline{L}, \underline{E})$ is the same as that of $-\mathcal{M}_{g,tK_0}(\underline{TY}, \underline{L}, \underline{E})$. In [21, Theorem 0.5], Goette obtained (18) as an equality of formal Laurent series in t when $g = e$ and K_0^Y does not vanish.

Theorem 1.1 is the analogue of the comparison formulas for the holomorphic torsions [13, Theorem 5.1] and for the de Rham torsions [14, Theorem 5.1]. The analytic tools in our proof of Theorem 1.1 are inspired by [13], with necessary modifications.

Remark 1.2. In [28], we establish also the family extension of Theorem 1.1 for a fibration $\pi : W \to B$ of compact manifolds with fiber Y by replacing the η-invariants by the η-forms of Bismut–Cheeger [11, Definitions 4.33, 4.93].

Remark 1.3. Recall that the Bismut superconnection [8, Definition 3.2] for a general fibration with fiber Y is the sum of three parts: the Dirac operator along the fiber Y, a unitary connection $\nabla^{\mathbb{E},u}$ on the infinite-dimensional vector bundle of smooth sections of $\mathcal{S}(TY, L) \otimes E$ over Y, and $-\frac{1}{4}c(T^H)$; here T^H is the curvature of the fibration.

Let $P \to B$ be a G-principal bundle. Then the curvature Ω of P is a $\mathfrak{g}$-valued 2-form on B. For $\underline{TY}, \underline{L}, \underline{E}$ in (1), we get naturally a fibration $P \times_G Y \to B$. Let $\widetilde{\eta}(\underline{TY}, \underline{L}, \underline{E})$ be the associated η-forms of Bismut–Cheeger. For this fibration, by Bismut [8, §1d), §3b)], the term $c(T^H)$ in the Bismut superconnection is $c(\Omega)$, and $(\nabla^{\mathbb{E},u})^2 = \mathcal{L}_\Omega$, thus we get [21, Lemma 1.14],

$$\widetilde{\eta}(\underline{TY}, \underline{L}, \underline{E}) + \frac{1}{2}\operatorname{Tr}|_{\operatorname{Ker}(D^Y \otimes E)}[e^{\frac{i}{2\pi}\Omega}] = \bar{\eta}_{\frac{i}{2\pi}\Omega}(\underline{TY}, \underline{L}, \underline{E}). \tag{19}$$

Remark 1.4. Now assume temporarily that Y is the boundary of a G-equivariant spin^c Riemannian manifold Z with the spinor bundle $\mathcal{S}_Z = \mathcal{S}_Z^+ \oplus \mathcal{S}_Z^-$, which has product structure near Y. We also assume that $\underline{E_Z}$ is a G-equivariant Hermitian vector bundle with connection such that near Y it is the pull-back of $\underline{E}$.

Let D_Z be the associated Dirac operator on $S_Z \otimes E_Z$ over Z. Then the index of $D_Z^+ := D_Z|_{\mathscr{C}^\infty(Z, S_Z^+ \otimes E_Z)}$ with respect to the Atiyah–Patodi–Singer (APS) boundary condition is a virtual representation of G. For $g \in G$, its equivariant APS index $\operatorname{Ind}_{\mathrm{APS},g}(D_Z^+)$ can be computed by Donnelly's theorem [18],

$$\operatorname{Ind}_{\mathrm{APS},g}(D_Z^+) = \int_{Z^g} \operatorname{Td}_g(\nabla^{TZ}, \nabla^L)\operatorname{ch}_g(\underline{E_Z}) - \bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E}). \tag{20}$$

By combining (15), (18) and (20), for any $K \in \mathfrak{g}$, there exists $\beta > 0$ such that, for any $-\beta < t < \beta$, we have

$$\operatorname{Ind}_{\mathrm{APS},e^{tK}}(D_Z^+) = \int_Z \operatorname{Td}_{tK}(\nabla^{TZ}, \nabla^L)\operatorname{ch}_{tK}(\underline{E_Z}) - \bar{\eta}_{tK}(\underline{TY}, \underline{L}, \underline{E}). \tag{21}$$

2. Pre-λ-ring structure in differential *K*-theory

Let Y be a compact manifold. Let $\pi : (y,s) \in Y \times \mathbb{R} \to y \in Y$ be the obvious projection. If $\alpha = \alpha_0 + ds \wedge \alpha_1$ with $\alpha_0, \alpha_1 \in \Lambda^\bullet(T^*Y)$, set $\{\alpha\}^{ds} := \alpha_1$.

Let E be a complex vector bundle on Y. Let h^{π^*E} be a metric on π^*E over $Y \times \mathbb{R}$ and let ∇^{π^*E} be a Hermitian connection on (π^*E, h^{π^*E}) such that

$$(E, h^{\pi^*E}|_{Y\times\{j\}}, \nabla^{\pi^*E}|_{Y\times\{j\}}) = (E, h_j^E, \nabla_j^E) =: \underline{E_j} \text{ for } j = 0,1. \tag{22}$$

The Chern–Simons class $\widetilde{\operatorname{ch}}(\underline{E_0}, \underline{E_1}) \in \Omega^{\mathrm{odd}}(Y, \mathbb{R})/\operatorname{Im} d$ is defined by

$$\widetilde{\operatorname{ch}}(\underline{E_0}, \underline{E_1}) = \int_0^1 \{\operatorname{ch}(\underline{\pi^*E})\}^{ds} ds \in \Omega^{\mathrm{odd}}(Y, \mathbb{R})/\operatorname{Im} d. \tag{23}$$

Then, we have

$$d\,\widetilde{\operatorname{ch}}(\underline{E_0}, \underline{E_1}) = \operatorname{ch}(\underline{E_1}) - \operatorname{ch}(\underline{E_0}). \tag{24}$$

Note that the Chern–Simons class depends only on ∇_j^E for $j = 0,1$ (see [31, Theorem B.5.4]).

Definition 2.1. A cycle for the differential K-theory of Y is a pair $(\underline{E}, \phi)$ where $\underline{E}$ is a geometric triple (without the group action) and ϕ is an element in $\Omega^{\mathrm{odd}}(Y, \mathbb{R})/\operatorname{Im} d$. Two cycles $(\underline{E_1}, \phi_1)$ and $(\underline{E_2}, \phi_2)$ are equivalent if there exist a geometric triple $\underline{E_3} = (E_3, h^{E_3}, \nabla^{E_3})$ and a complex vector bundle isomorphism $\Phi : E_1 \oplus E_3 \to E_2 \oplus E_3$ such that

$$\widetilde{\operatorname{ch}}\left(\underline{E_1} \oplus \underline{E_3}, \Phi^*\left(\underline{E_2} \oplus \underline{E_3}\right)\right) = \phi_2 - \phi_1. \tag{25}$$

We define the differential K-group $\widehat{K}^0(Y)$ as the Grothendieck group of equivalent classes of cycles.

For any $[\underline{E}, \phi], [\underline{F}, \psi] \in \widehat{K}^0(Y)$, set

$$[\underline{E}, \phi] \cup [\underline{F}, \psi] = [\underline{E} \otimes \underline{F}, \operatorname{ch}(\underline{E}) \wedge \psi + \phi \wedge \operatorname{ch}(\underline{F}) - d\phi \wedge \psi]. \tag{26}$$

Then $\widehat{K}^0(Y) = \{[\underline{E} - \underline{E_1}, \phi - \phi_1] := [\underline{E}, \phi] - [\underline{E_1}, \phi_1] : (\underline{E}, \phi), (\underline{E_1}, \phi_1) \text{ are cycles as above}\}$ and $\widehat{K}^0(Y)$ is an abelian group. We also verify directly that the product (26) is well-defined, commutative and associative. Thus $(\widehat{K}^0(Y), +, \cup)$ is a commutative ring with unit $1 := [\underline{\mathbb{C}}, 0]$. Here $\underline{\mathbb{C}}$ is the trivial line bundle over Y with the trivial metric and connection.

For a commutative ring R with identity, a pre-λ-ring structure is defined by a countable set of maps $\lambda^n : R \to R$ with $n \in \mathbb{N}$ such that, for all $x, y \in R$,

$$\lambda^0(x)=1, \quad \lambda^1(x)=x, \quad \lambda^n(x+y)=\sum_{j=0}^{n}\lambda^j(x)\lambda^{n-j}(y). \tag{27}$$

Let

$$\lambda_t(x)=\sum_{n\geq 0}\lambda^n(x)t^n, \quad \gamma_t(x)=\sum_{j\geq 0}\gamma^j(x)t^j:=\lambda_{\frac{t}{1-t}}(x). \tag{28}$$

Consider the vector space (cf. [20, §7.3.1])

$$\Gamma(Y):=Z^{\mathrm{even}}(Y,\mathbb{R})\oplus\left(\Omega^{\mathrm{odd}}(Y,\mathbb{R})/\operatorname{Im}\mathrm{d}\right), \tag{29}$$

where $Z^{\mathrm{even}}(Y,\mathbb{R})$ is the set of even degree real closed forms on Y. Let $[\cdot]_{\mathrm{odd}}$ be the component of $\Gamma(Y)$ in $\Omega^{\mathrm{odd}}(Y,\mathbb{R})/\operatorname{Im}\mathrm{d}$. We define a product operation on $\Gamma(Y)$ by the formula

$$(\omega_1,\phi_1)*(\omega_2,\phi_2):=(\omega_1\wedge\omega_2,\omega_1\wedge\phi_2+\phi_1\wedge\omega_2-\mathrm{d}\phi_1\wedge\phi_2). \tag{30}$$

Set $\Omega^{-1}(\cdot)=\{0\}$. Given $k\in\mathbb{N}$, we define the Adams operations $\Psi^k:\Gamma(Y)\to\Gamma(Y)$ (cf. [20, §7.3.1], [33]) by

$$\Psi^k(\alpha,\beta)=(k^l\alpha,k^l\beta), \qquad \text{for } (\alpha,\beta)\in Z^{2l}(Y,\mathbb{R})\oplus(\Omega^{2l-1}(Y,\mathbb{R})/\operatorname{Im}\mathrm{d}). \tag{31}$$

For any $x\in\Gamma(Y)$, put

$$\sum_{n\geq 0}\lambda^n(x)t^n:=\exp\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\Psi^k(x)t^k}{k}\right), \tag{32}$$

where the multiplication in (32) is defined in (30).

Let $\underline{\Lambda^k E}$ be the k-th exterior power of E with the induced metric and connection. Let $\mathrm{ch}(\underline{E})\in Z^{\mathrm{even}}(Y,\mathbb{R})$ be the Chern character form.

Theorem 2.2. *The differential K-group $\widehat{K}^0(Y)$ has a pre-λ-ring structure defined by*

$$\lambda^k([\underline{E},\phi])=[\underline{\Lambda^k E},[\lambda^k(\mathrm{ch}(\underline{E}),\phi)]_{\mathrm{odd}}]. \tag{33}$$

Assume now that Y is connected.

Set $\lambda^k(\underline{E})=\underline{\Lambda^k E}$. Let $\mathrm{rk}E$ be the rank of the complex vector bundle E. Let $\underline{\mathrm{rk}E}$ be the $\mathrm{rk}E$-dimensional trivial complex vector bundle with trivial metric and connection. Then, by (28),

$$\gamma_t(\underline{E}-\underline{\mathrm{rk}E}):=\gamma_t(\underline{E})(1-t)^{\mathrm{rk}E}=\sum_{i=0}^{\mathrm{rk}E}\underline{\Lambda^i E}\cdot t^i(1-t)^{\mathrm{rk}E-i}. \tag{34}$$

Thus,

$$\gamma^k(\underline{E}-\underline{\mathrm{rk}E})=\begin{cases}\sum_{i=0}^{k}(-1)^{k-i}\begin{pmatrix}\mathrm{rk}E-i\\ k-i\end{pmatrix}\underline{\Lambda^i E}, & \text{if } 0\leq k\leq \mathrm{rk}E;\\ 0, & \text{if } k>\mathrm{rk}E.\end{cases} \tag{35}$$

In particular, $\gamma^k(\underline{E}-\underline{\mathrm{rk}E})$ is a finite-dimensional virtual complex vector bundle with the induced metric and connection. The following theorem is part of the differential K-theory version of [1, Proposition 3.1.5], the locally nilpotent property of the γ-filtration in the usual topological K-group of Y.

Theorem 2.3. *There exists $\mathcal{N}_{r,m}>0$ (depending only on r,m) such that for any geometric triple $\underline{E}$ on Y with $r=\mathrm{rk}E$, $m=\dim Y$ and $(n_1,\cdots,n_r)\in\mathbb{N}^r$ such that $\sum_{i=1}^{r}i\cdot n_i>\mathcal{N}_{r,m}$, we have*

$$\prod_{i=1}^{r}\left(\gamma^i([\underline{E}-\underline{\mathrm{rk}E},0])\right)^{n_i}=\left[\prod_{i=1}^{r}\left(\gamma^i(\underline{E}-\underline{\mathrm{rk}E})\right)^{n_i},0\right]=0\in\widehat{K}^0(Y). \tag{36}$$

3. Localization formula for η-invariants

We use the notation of Section 1 and we assume that $G = S^1$. For $g \in S^1$, we define $\widetilde{\mathrm{ch}}_g(\cdots)$ as in (23) by replacing ch by ch_g and choosing a S^1-invariant couple $h^{\pi^*E}, \nabla^{\pi^*E}$.

Definition 3.1. For $g \in S^1$, a cycle for the g-equivariant differential K-theory of Y is a pair $(\underline{E}, \phi)$, where $\underline{E}$ is an equivariant geometric triple over Y, and ϕ is an element in $\Omega^{\mathrm{odd}}(Y^g, \mathbb{C})/\mathrm{Im}\,\mathrm{d}$. Two cycles $(\underline{E}_1, \phi_1)$ and $(\underline{E}_2, \phi_2)$ are equivalent if there exist an S^1-equivariant geometric triple $\underline{E_3} = (E_3, h^{E_3}, \nabla^{E_3})$ and an S^1-equivariant complex vector bundle isomorphism $\Phi : E_1 \oplus E_3 \to E_2 \oplus E_3$ such that

$$\widetilde{\mathrm{ch}}_g\left(\underline{E_1} \oplus \underline{E_3}, \Phi^*\left(\underline{E_2} \oplus \underline{E_3}\right)\right) = \phi_2 - \phi_1. \tag{37}$$

The g-equivariant differential K-group $\widehat{K}_g^0(Y)$ is the Grothendieck group of equivalent classes of cycles.

For any $[\underline{E}, \phi], [\underline{F}, \psi] \in \widehat{K}_g^0(Y)$, set

$$[\underline{E}, \phi] \cup [\underline{F}, \psi] = [\underline{E} \otimes \underline{F}, \mathrm{ch}_g(\underline{E}) \wedge \psi + \phi \wedge \mathrm{ch}_g(\underline{F}) - \mathrm{d}\phi \wedge \psi]. \tag{38}$$

Again the product (38) is well-defined, commutative, and associative. Thus $(\widehat{K}_g^0(Y), +, \cup)$ is a commutative ring with unit $1 := [\underline{\mathbb{C}}, 0]$.

In the following, we will denote by $\underline{'E}$ the corresponding geometric triple when forgetting the group action.

Let Y^{S^1} be the fixed-point set of the circle action on Y. Then each connected component $Y_\alpha^{S^1}$, $\alpha \in \mathfrak{B}$, of Y^{S^1}, is a compact manifold. Unless stated otherwise, we assume that $Y^{S^1} \neq \emptyset$. Let N_α be the normal bundle of $Y_\alpha^{S^1}$ in Y. Then on $Y_\alpha^{S^1}$, we have the splitting

$$N_\alpha = \bigoplus_{v>0} N_{\alpha,v}, \tag{39}$$

and $g \in S^1$ acts on the complex vector bundle $N_{\alpha,v}$ by multiplication by g^v. For any $\alpha \in \mathfrak{B}$, $Y_\alpha^{S^1}$ also has an equivariant spin^c structure with associated equivariant line bundle $L_\alpha = L|_{Y_\alpha^{S^1}} \otimes (\det N_\alpha)^{-1}$ as $w_2(TY_\alpha^{S^1}) = c_1(L_\alpha) \mod(2)$ (cf. [30, (1.47)]). Set

$$r_{\alpha,v} = \mathrm{rk}\, N_{\alpha,v}. \tag{40}$$

Let $P_{k,\pm}\left(\underline{'N_{\alpha,v}^*}\right)$ be the finite-dimensional Hermitian vector bundles on $Y_\alpha^{S^1}$ with metrics and connections such that

$$P_{k,+}\left(\underline{'N_{\alpha,v}^*}\right) - P_{k,-}\left(\underline{'N_{\alpha,v}^*}\right) = {}^k\sum (-1)^{\sum_{i=1}^{r_{\alpha,v}} n_i} \frac{(\sum_{i=1}^{r_{\alpha,v}} n_i)!}{\prod_{i=1}^{r_{\alpha,v}} n_i!} \prod_{i=1}^{r_{\alpha,v}} \left(\gamma^i\left(\underline{'N_{\alpha,v}^*} - \underline{\mathrm{rk}\,'N_{\alpha,v}^*}\right)\right)^{n_i}, \tag{41}$$

where ${}^k\sum$ is a sum over $(n_1, \cdots, n_{r_{\alpha,v}}) \in \mathbb{N}^{r_{\alpha,v}}$, $\sum_{i=1}^{r_{\alpha,v}} i \cdot n_i = k$. Let $m_\alpha = \dim Y_\alpha^{S^1}$. Set

$$\mathcal{N}_0 := \sup_{\alpha,v} \mathcal{N}_{r_{\alpha,v}, m_\alpha}, \quad \text{with } \mathcal{N}_{r_{\alpha,v}, m_\alpha} \text{ as in Theorem 2.3.} \tag{42}$$

By Theorem 2.3, we know that for any $k > \mathcal{N}_0$,

$$\left[P_{k,+}\left(\underline{'N_{\alpha,v}^*}\right) - P_{k,-}\left(\underline{'N_{\alpha,v}^*}\right), 0\right] = 0 \in \widehat{K}^0(Y_\alpha^{S^1}). \tag{43}$$

From (28), (35) and (41), formally, we have

$$\begin{aligned} \lambda_t\left(\underline{'N_{\alpha,v}^*}\right)^{-1} &= (1+t)^{-r_{\alpha,v}}\left(1 + \sum_{i=1}^{r_{\alpha,v}} \gamma^i\left(\underline{'N_{\alpha,v}^*} - \underline{\mathrm{rk}\,'N_{\alpha,v}^*}\right) t^i (1+t)^{-i}\right)^{-1} \\ &= (1+t)^{-r_{\alpha,v}}\left(1 + \sum_{j=1}^{\infty} (-1)^j \Big(\sum_{i=1}^{r_{\alpha,v}} \gamma^i\left(\underline{'N_{\alpha,v}^*} - \underline{\mathrm{rk}\,'N_{\alpha,v}^*}\right) t^i (1+t)^{-i}\Big)^j\right) \\ &= (1+t)^{-r_{\alpha,v}}\left(1 + \sum_{k=1}^{\infty} t^k (1+t)^{-k}\left(P_{k,+}\left(\underline{'N_{\alpha,v}^*}\right) - P_{k,-}\left(\underline{'N_{\alpha,v}^*}\right)\right)\right). \end{aligned} \tag{44}$$

Let $A \subset S^1$ be the finite set defined by

$$A = \{g \in S^1 : Y^{S^1} \neq Y^g\} \subset S^1. \tag{45}$$

Let $R(S^1)$ be the representation ring of S^1. Let $\widehat{K}_g^0(Y)_{I(g)}$ be the localization of $\widehat{K}_g^0(Y)$ at the prime ideal $I(g)$ of $R(S^1)$, which consists of all characters of S^1 vanishing at g.

For $g \in S^1 \backslash A$, $\mathcal{N} \in \mathbb{N}$, from (44), we define

$$\begin{aligned} &\lambda_{-g^{-v}}\left(\underline{{}'N^*_{\alpha,v}}\right)^{-1}_{\mathcal{N}} := \frac{g^{vr_{\alpha,v}}}{(g^v-1)^{r_{\alpha,v}}}\left(1+\sum_{k=1}^{\mathcal{N}}\frac{(-1)^k}{(g^v-1)^k}\left(P_{k,+}\left(\underline{{}'N^*_{\alpha,v}}\right)-P_{k,-}\left(\underline{{}'N^*_{\alpha,v}}\right)\right)\right),\\ &\underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}}} := \bigotimes_{v:\, r_{\alpha,v}\neq 0}\lambda_{-g^{-v}}\left(\underline{{}'N^*_{\alpha,v}}\right)^{-1}_{\mathcal{N}}. \end{aligned} \tag{46}$$

From (43)-(46), for $g \in S^1 \backslash A$, we see that for any $\mathcal{N}, \mathcal{N}' > \mathcal{N}_0$,

$$\left[\underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}}}, 0\right] = \left[\underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}'}}, 0\right] \in \widehat{K}_g^0(Y_\alpha^{S^1})_{I(g)}, \tag{47}$$

and

$$\left[\underline{\lambda_{-1}(N^*_\alpha)}, 0\right] \cup \left[\underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}}}, 0\right] = 1 \in \widehat{K}_g^0(Y_\alpha^{S^1})_{I(g)}. \tag{48}$$

Thus we have the following theorem, which is the differential K-theory version of Atiyah–Segal's result [4, Lemma 2.7] in usual topological K-theory. A version for arithmetic K-group was obtained in [24, Lemma 4.5].

Theorem 3.2. *For $g \in S^1 \backslash A$, $\left[\underline{\lambda_{-1}(N^*_\alpha)}, 0\right]$ is invertible in $\widehat{K}_g^0(Y_\alpha^{S^1})_{I(g)}$ and, for any $\mathcal{N} > \mathcal{N}_0$, we have:*

$$\left[\underline{\lambda_{-1}(N^*_\alpha)}, 0\right]^{-1} = \left[\underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}}}, 0\right] \in \widehat{K}_g^0(Y_\alpha^{S^1})_{I(g)}. \tag{49}$$

For $f \in \mathbb{Z}[x, x^{-1}]$, there exists a finite dimensional representation M_f of S^1 such that its character $\chi_{M_f}(g)$ is $f(g)$ for any $g \in S^1$. Let $\underline{M_f}$ be the vector bundle $Y \times M_f$ on Y with trivial metric and connection and the induced circle action. By identifying $f(g) \cdot \underline{F}$ with $\underline{M_f} \otimes \underline{F}$ for triple $\underline{F}$, there exist equivariant geometric triples $\underline{\mu_{\alpha,\mathcal{N},+}}$ and $\underline{\mu_{\alpha,\mathcal{N},-}}$ on $Y_\alpha^{S^1}$ such that

$$\underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}}} = F(g)^{-1}(\underline{\mu_{\alpha,\mathcal{N},+}} - \underline{\mu_{\alpha,\mathcal{N},-}}) \quad \text{with } F(g) = \prod_{v:\, r_{\alpha,v}\neq 0}(g^v-1)^{r_{\alpha,v}+\mathcal{N}}. \tag{50}$$

For $g \in S^1 \backslash A$, we define

$$\begin{aligned} &\bar{\eta}_g\left(\underline{TY_\alpha^{S^1}}, \underline{L_\alpha}, \underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}}} \otimes \underline{E}|_{Y_\alpha^{S^1}}\right)\\ &\qquad := F(g)^{-1}\cdot\left[\bar{\eta}_g\left(\underline{TY_\alpha^{S^1}}, \underline{L_\alpha}, \underline{\mu_{\alpha,\mathcal{N},+}} \otimes \underline{E}|_{Y_\alpha^{S^1}}\right) - \bar{\eta}_g\left(\underline{TY_\alpha^{S^1}}, \underline{L_\alpha}, \underline{\mu_{\alpha,\mathcal{N},-}} \otimes \underline{E}|_{Y_\alpha^{S^1}}\right)\right]. \end{aligned} \tag{51}$$

Note that from (46) and (50),

$$\mu_{\alpha,\mathcal{N},\pm} = \bigoplus_{k\geq 0}\xi_{\alpha,k,\pm} \in K^0(Y_\alpha^{S^1}) \tag{52}$$

and S^1 acts fiberwise on $\xi_{\alpha,k}$ with weight k. If S^1 acts on L by sending $g \in S^1$ to g^{l_α} ($l_\alpha \in \mathbb{Z}$) on $Y_\alpha^{S^1}$, then by [30, p. 139] and (40),

$$\sum_v v\, r_{\alpha,v} + l_\alpha = 0 \mod (2). \tag{53}$$

By (52) and (53), for $g \in S^1$, we have:

$$\begin{aligned} &\bar{\eta}_g\left(\underline{TY_\alpha^{S^1}}, \underline{L_\alpha}, \underline{\mu_{\alpha,\mathcal{N},+}} \otimes \underline{E}|_{Y_\alpha^{S^1}}\right) - \bar{\eta}_g\left(\underline{TY_\alpha^{S^1}}, \underline{L_\alpha}, \underline{\mu_{\alpha,\mathcal{N},-}} \otimes \underline{E}|_{Y_\alpha^{S^1}}\right)\\ &\qquad = g^{-\frac{1}{2}\sum_v v\, r_{\alpha,v}+\frac{1}{2}l_\alpha}\sum_{k\geq 0,\, v} g^{k+v}\left[\bar{\eta}\left(\underline{TY_\alpha^{S^1}}, \underline{L_\alpha}, \underline{\xi_{\alpha,k,+}} \otimes \underline{E_v}\right) - \bar{\eta}\left(\underline{TY_\alpha^{S^1}}, \underline{L_\alpha}, \underline{\xi_{\alpha,k,-}} \otimes \underline{E_v}\right)\right]. \end{aligned} \tag{54}$$

Here $\underline{E_v}$ is the weight v part of $\underline{E}|_{Y_\alpha^{S^1}}$ for the S^1-action.

The main result of [29] is a localization formula for equivariant (reduced) η-invariants.

Theorem 3.3. *For any $\mathcal{N}, \mathcal{N}' \in \mathbb{N}$ and $\mathcal{N}' > \mathcal{N} > \mathcal{N}_0$, and for $\underline{E} = (E, h^E, \nabla^E)$ on Y, the functions on $S^1 \backslash A$,*

$$P_{\mathcal{N},\mathcal{N}'}(g) := \bar{\eta}_g \left(\underline{TY^{S^1}_\alpha}, \underline{L_\alpha}, \underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}'}} \otimes \underline{E}|_{Y^{S^1}_\alpha} \right) - \bar{\eta}_g \left(\underline{TY^{S^1}_\alpha}, \underline{L_\alpha}, \underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}}} \otimes \underline{E}|_{Y^{S^1}_\alpha} \right) \tag{55}$$

and

$$Q_{\mathcal{N}}(g) := \bar{\eta}_g \left(\underline{TY}, \underline{L}, \underline{E} \right) - \sum_\alpha \bar{\eta}_g \left(\underline{TY^{S^1}_\alpha}, \underline{L_\alpha}, \underline{\lambda_{-1}(N^*_\alpha)^{-1}_{\mathcal{N}}} \otimes \underline{E}|_{Y^{S^1}_\alpha} \right), \tag{56}$$

are restrictions of rational functions on S^1 with integral coefficients that do not have poles on $S^1 \backslash A$.

Remark 3.4. If $Y^{S^1} = \emptyset$, $A = \{g \in S^1 : Y^g \neq \emptyset\}$, $\bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E})$ as a function on $S^1 \backslash A$ is the restriction of a rational function on S^1 with integral coefficients and it has no poles on $S^1 \backslash A$.

4. Localization in differential K-theory and a proof of Theorem 3.3

Let $K^0_{S^1}(Y)$, $K^1_{S^1}(Y)$ be the S^1-equivariant K-group, K^1-group of Y, respectively. By [34, Definitions 2.7 and 2.8], we have the exact sequence

$$0 \to K^1_{S^1}(Y) \xrightarrow{\varsigma} K^0_{S^1}(Y \times \widehat{S^1}) \xrightarrow{i^*} K^0_{S^1}(Y) \to 0, \tag{57}$$

where $\widehat{S^1}$ is a copy of S^1 with trivial S^1-action and there exists $b \in \widehat{S^1}$ such that the map i is given by $i : Y \ni y \to (y, b) \in Y \times \widehat{S^1}$. By (57), $K^1_{S^1}(Y)$ is a $R(S^1)$-module.

For $y \in K^1_{S^1}(Y)$, from (57), we can represent $\varsigma(y)$ as $W - U$, here U is a trivial S^1-equivariant vector bundle on $Y \times \widehat{S^1}$ associated with a finite-dimensional representation M of S^1, and

$$W = Y \times [0, 1] \times M / \sim_F, \tag{58}$$

where we identify $\widehat{S^1}$ with $\mathbb{R}/\mathbb{Z}$, $F \in \mathscr{C}^\infty(Y, \mathrm{Aut}(M))$ is S^1-equivariant, and $\sim_F$ is the gluing map: $(y, 1, m) \sim_F (y, 0, F(y)m)$ for $y \in Y$, $m \in M$. The odd Chern character of y is defined by the formula

$$\mathrm{ch}_g(y) = \int_{S^1} \mathrm{ch}_g(W). \tag{59}$$

For a finite-dimensional representation M of S^1, let χ_M be its character. Then $\phi \mapsto \chi_M(g) \cdot \phi$ makes $\Omega^{\mathrm{odd}}(Y^{S^1}, \mathbb{C})/\mathrm{Im}\,\mathrm{d}$ a $R(S^1)$-module.

The following proposition is the g-equivariant version of the corresponding results in [19, (2.21)] and [17, Proposition 2.24].

Proposition 4.1. *If $g \in S^1 \backslash A$, we have the exact sequence of $R(S^1)$-modules,*

$$K^1_{S^1}(Y) \xrightarrow{\mathrm{ch}_g} \Omega^{\mathrm{odd}}(Y^{S^1}, \mathbb{C})/\mathrm{Im}\,\mathrm{d} \xrightarrow{a} \widehat{K}^0_g(Y) \xrightarrow{\tau} K^0_{S^1}(Y) \longrightarrow 0, \tag{60}$$

where

$$a(\phi) = [0, \phi], \quad \tau([\underline{E}, \phi]) = [E]. \tag{61}$$

Let $\iota : Y^{S^1} \to Y$ be the obvious embedding. Let

$$\begin{aligned} \iota^* &: K^0_{S^1}(Y)_{I(g)} \to K^0_{S^1}(Y^{S^1})_{I(g)}, \quad E \to E|_{Y^{S^1}}, \\ \hat{\iota}^* &: \widehat{K}^0_g(Y)_{I(g)} \to \widehat{K}^0_g(Y^{S^1})_{I(g)}, \quad (\underline{E}, \phi) \to (\underline{E}|_{Y^{S^1}}, \phi), \end{aligned} \tag{62}$$

be the induced homomorphisms.

Since localization preserves exact sequences [2, Proposition 3.3], from Proposition 4.1, we have the commutative diagram of exact sequences of $R(S^1)_{I(g)}$-modules,

$$\begin{tikzcd} K^1_{S^1}(Y)_{I(g)} \arrow[r, "\mathrm{ch}_g"] \arrow[d, "\iota^*"] & \left(\Omega^{\mathrm{odd}}(Y^{S^1}, \mathbb{C})/\mathrm{Im}\,\mathrm{d}\right)_{I(g)} \arrow[r, "a"] \arrow[d, "\mathrm{Id}"] & \widehat{K}^0_g(Y)_{I(g)} \arrow[r, "\tau"] \arrow[d, "\hat{\iota}^*"] & K^0_{S^1}(Y)_{I(g)} \arrow[r] \arrow[d, "\iota^*"] & 0 \\ K^1_{S^1}(Y^{S^1})_{I(g)} \arrow[r, "\mathrm{ch}_g"] & \left(\Omega^{\mathrm{odd}}(Y^{S^1}, \mathbb{C})/\mathrm{Im}\,\mathrm{d}\right)_{I(g)} \arrow[r, "a"] & \widehat{K}^0_g(Y^{S^1})_{I(g)} \arrow[r, "\tau"] & K^0_{S^1}(Y^{S^1})_{I(g)} \arrow[r] & 0. \end{tikzcd} \tag{63}$$

Using localization in topological K-theory [4, Theorem 1.1], ι^* is an isomorphism on $K^1_{S^1}(Y)_{I(g)}$ and $K^0_{S^1}(Y)_{I(g)}$. By the five lemma, $\hat{\iota}^*$ in (62) is an isomorphism.

Thus we have the following localization theorem, which is a differential K-theory version of the classical localization theorem in topological K-theory [4, Theorem 1.1] (cf. also [17, Theorem 3.27]).

Proposition 4.2 *(Localization theorem). For $g \in S^1\backslash A$, the restriction map $\hat{\iota}^* : \widehat{K}^0_g(Y)_{I(g)} \to \widehat{K}^0_g(Y^{S^1})_{I(g)}$ in (62) is a $R(S^1)_{I(g)}$-module isomorphism.*

For $g \in S^1$, set $\mathbb{Q}_g := \{P(g)/Q(g) \in \mathbb{C} : P, Q \in \mathbb{Z}[x], Q(g) \neq 0\} \subset \mathbb{C}$.

For any equivariant geometric triple $\underline{E}$, $\phi \in \Omega^{\mathrm{odd}}(Y^g, \mathbb{C})/\mathrm{Im}\,d$, $\chi \in R(S^1)$ such that $\chi(g) \neq 0$, put

$$\widehat{f_Y}_*\big((\underline{E}, \phi)/\chi\big) := -\chi(g)^{-1} \int_{Y^g} \mathrm{Td}_g(\nabla^{TY}, \nabla^L) \wedge \phi + \chi(g)^{-1} \bar{\eta}_g(\underline{TY}, \underline{L}, \underline{E}). \tag{64}$$

By the variation formula for equivariant η-invariants, $\widehat{f_Y}_*$ defines a push-forward map $\widehat{f_Y}_* : \widehat{K}^0_g(Y)_{I(g)} \to \mathbb{C}/\mathbb{Q}_g$. Note that, for $g = e$, the family version of (64) is [19, Definition 3.12]. In [24, Proposition 4.3], Köhler–Roessler defined an arithmetic K-theory version of (64).

From the fact that $\widehat{f_{Y^{S^1}}}_* : \widehat{K}^0_g(Y^{S^1})_{I(g)} \to \mathbb{C}/\mathbb{Q}_g$ is well-defined and (47), for any $\mathcal{N}, \mathcal{N}' \in \mathbb{N}$ and $\mathcal{N} > \mathcal{N}' > \mathcal{N}_0$, $g \in S^1\backslash A$, we have $P_{\mathcal{N},\mathcal{N}'}(g) \in \mathbb{Q}_g$.

The following result shows that localization commutes with the push-forward map in differential K-theory.

Theorem 4.3. *For $g \in S^1 \setminus A$, the following diagram commutes,*

$$\begin{tikzcd} \widehat{K}^0_g(Y^{S^1})_{I(g)} \arrow[dr, "\widehat{f_{Y^{S^1}}}_*"'] & & \widehat{K}^0_g(Y)_{I(g)} \arrow[ll, "{\left[\underline{\lambda_{-1}(N^*)},0\right]^{-1} \cup \hat{\iota}^*}"'] \arrow[dl, "\widehat{f_Y}_*"] \\ & \mathbb{C}/\mathbb{Q}_g. & \end{tikzcd} \tag{65}$$

Proof. Let $\underline{\mu}$ be an equivariant geometric triple on Y^{S^1}. Then the equivariant version of the Atiyah–Hirzebruch direct image of $\underline{\mu}$ (cf. [27, §3.3]) is the difference of two equivariant geometric triples $\underline{\xi_+} - \underline{\xi_-}$ on Y and

$$\underline{\xi_+}|_{Y^{S^1}} = \underline{\Lambda^{\mathrm{even}}(N^*)} \otimes \underline{\mu} \oplus \underline{F}, \quad \underline{\xi_-}|_{Y^{S^1}} = \underline{\Lambda^{\mathrm{odd}}(N^*)} \otimes \underline{\mu} \oplus \underline{F}. \tag{66}$$

For $g \in S^1\backslash A$, the map

$$[\underline{\mu}, \phi]/\chi \mapsto [\underline{\xi_+}, \mathrm{ch}_g(\underline{\Lambda^{\mathrm{even}}(N^*)}) \wedge \phi]/\chi - [\underline{\xi_-}, \mathrm{ch}_g(\underline{\Lambda^{\mathrm{odd}}(N^*)}) \wedge \phi]/\chi, \tag{67}$$

defines a direct image map

$$\hat{\iota}_* : \widehat{K}^0_g(Y^{S^1})_{I(g)} \to \widehat{K}^0_g(Y)_{I(g)} \tag{68}$$

and

$$\hat{\iota}^* \circ \hat{\iota}_* = \left[\underline{\lambda_{-1}(N^*)}, 0\right] \cup : \widehat{K}^0_g(Y^{S^1})_{I(g)} \xrightarrow{\sim} \widehat{K}^0_g(Y^{S^1})_{I(g)}. \tag{69}$$

By Proposition 4.2 and by (69), $\hat{\iota}_*$ is an isomorphism.

For any $g \in S^1\backslash A$, by using the embedding formula of the equivariant η-invariant [27, Corollary 3.9], which extends the Bismut–Zhang embedding formula [16, Theorem 2.2] to the equivariant case, from (64) and (67), we have the commutative diagram:

$$\begin{tikzcd} \widehat{K}^0_g(Y^{S^1})_{I(g)} \arrow[rr, "\hat{\iota}_*"] \arrow[dr, "\widehat{f_{Y^{S^1}}}_*"'] & & \widehat{K}^0_g(Y)_{I(g)} \arrow[dl, "\widehat{f_Y}_*"] \\ & \mathbb{C}/\mathbb{Q}_g. & \end{tikzcd} \tag{70}$$

Then (65) follows from (69) and (70). □

In particular, from (64) and (65), for any $\mathcal{N} \in \mathbb{N}$, $\mathcal{N} > \mathcal{N}_0$ with $\mathcal{N}_0$ in (42), we have

$$Q_{\mathcal{N}}(g) = \widehat{f_{Y*}}([\underline{E}, 0]) - \widehat{f_{Y^{S^1}}}_* \circ \left[\underline{\lambda_{-1}(N^*)}, 0\right]^{-1} \cup \hat{\iota}^*([\underline{E}, 0]) \in \mathbb{Q}_g. \tag{71}$$

By Theorem 1.1, for $g \in S^1 \backslash A$, $K_0 \in \mathrm{i}\mathbb{R} = \mathrm{Lie}(S^1)$, the Lie algebra of S^1, there exists $\beta > 0$ such that for $|t| \leq \beta$, $Q_{\mathcal{N}}(ge^{tK_0})$ is real analytic in t. Then combining (71), we know that $Q_{\mathcal{N}}$ is a rational function on each connected component of $S^1 \backslash A$ with integral coefficients. Using Theorem 1.1 for $g \in A$, we know that the two rational functions $Q_{\mathcal{N}}$, defined on two sides of $g \in A$, are the same rational function. The argument for $P_{\mathcal{N},\mathcal{N}'}$ is the same.

The proof of Theorem 3.3 is completed.

Remark 4.4. Eqs. (49) and (65) could be viewed as an analogue of Köhler–Roessler's fixed-point formula of Lefschetz type in equivariant arithmetic K-theory (cf. [24], [25]).

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