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Branching problems for semisimple Lie groups and reproducing kernels

Règles de branchement pour les groupes de Lie semi-simples et les noyaux reproduisants

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A B S T R A C T

For a semisimple Lie group G satisfying the equal rank condition, the most basic family of unitary irreducible representations is the discrete series found by Harish-Chandra. In this paper, we study some of the branching laws for these when restricted to a subgroup H of the same type by combining the classical results with the recent work of T. Kobayashi. We analyze aspects of having differential operators being symmetry-breaking operators; in particular, we prove in the so-called admissible case that every symmetry breaking (H-map) operator is a differential operator. We prove discrete decomposability under Harish-Chandra's condition of cusp form on the reproducing kernels. Our techniques are based on realizing discrete series representations as kernels of elliptic invariant differential operators.

R É S U M É

Pour un groupe de Lie semi-simple G satisfaisant la condition de rang, la famille de représentations irréductibles unitaires la plus fondamentale est la série discrète trouvée par Harish-Chandra. Dans cet article, nous étudions quelques règles de branchement pour ces séries restreintes à un sous-groupe H de G du même type, en combinant les résultats classiques avec des travaux récents de T. Kobayashi. Nous analysons des cas où des opérateurs de brisure de symétrie sont des opérateurs différentiels ; en particulier, nous prouvons dans le cas dit admissible que tout opérateur de brisure de symétries H-équivariant est un opérateur différentiel. Nous prouvons la propriété de décomposabilité discrète sous la condition de cuspidalité de Harish-Chandra sur les noyaux reproduisants.

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1. Discrete series and reproducing kernel

Let G be an arbitrary, matrix, connected semisimple Lie group. Henceforth we fix a maximal compact subgroup K for G and a maximal torus T for K. Harish-Chandra showed that G admits square integrable irreducible representations if and only if T is a Cartan subgroup of G. For this note, we always assume that T is a Cartan subgroup of G. Under these hypothesis, Harish-Chandra showed that the set of equivalence classes of irreducible square integrable representations is parameterized by a subset of a lattice in $i\mathfrak{t}_{\mathbb{R}}^{\star}$. In order to state our results, we need to make explicit this parametrization and set up some notation. As usual, the Lie algebra of a Lie group is denoted by the corresponding lower-case Gothic letter followed by the subindex $\mathbb{R}$. The complexification of the Lie algebra of a Lie group is denoted by the corresponding Gothic letter without any subscript. $V^{\star}$ denotes the dual space to a vector space V. Let θ be the Cartan involution that corresponds to the subgroup K; the associated Cartan decomposition is denoted by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let $\Phi(\mathfrak{g}, \mathfrak{t})$ denote the root system attached to the Cartan subalgebra $\mathfrak{t}$. Hence, the set of roots $\Phi(\mathfrak{g}, \mathfrak{t}) = \Phi_{\mathrm{c}} \cup \Phi_{\mathrm{n}} = \Phi_{\mathrm{c}}(\mathfrak{g}, \mathfrak{t}) \cup \Phi_{\mathrm{n}}(\mathfrak{g}, \mathfrak{t})$ is equal to the union the set of compact roots and the set of noncompact roots. From now on, we fix a system of positive roots Δ for Φ_{c}. Henceforth, either the highest weight or the infinitesimal character of an irreducible representation of K is dominant with respect to Δ. The Killing form gives rise to an inner product $(\ldots, \ldots)$ in $i\mathfrak{t}_{\mathbb{R}}^{\star}$. As usual, let $\rho = \rho_G$ denote half of the sum of the roots for some system of positive roots for $\Phi(\mathfrak{g}, \mathfrak{t})$. *A Harish-Chandra parameter* for G is $\lambda \in i\mathfrak{t}_{\mathbb{R}}^{\star}$ such that $(\lambda, \alpha) \neq 0$, for every $\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})$, and such that $\lambda + \rho$ lifts to a character of T. With each Harish-Chandra parameter λ, Harish-Chandra associates a unique irreducible square integrable representation $(\pi_{\lambda}^{G}, V_{\lambda}^{G})$ of G of infinitesimal character λ. Moreover, Harish-Chandra showed that the map $\lambda \to (\pi_{\lambda}^{G}, V_{\lambda}^{G})$ is a bijection from the set of Harish-Chandra parameters dominant with respect to Δ onto the set of equivalence classes of irreducible square integrable representations for G. For short, we will refer to an irreducible square integrable representation as a discrete series representation.

Let $(\tau, W) := (\pi_{\lambda+\rho_{\mathrm{n}}}^{K}, V_{\lambda+\rho_{\mathrm{n}}}^{K})$ denote the lowest K-type of $\pi_{\lambda} := \pi_{\lambda}^{G}$. The highest weight of $(\pi_{\lambda+\rho_{\mathrm{n}}}^{K}, V_{\lambda+\rho_{\mathrm{n}}}^{K})$ is $\lambda + \rho_{\mathrm{n}} - \rho_{\mathrm{c}}$ [4]. We recall a Theorem of Vogan's thesis, which states that (τ, W) determines $(\pi_{\lambda}, V_{\lambda}^{G})$ up to unitary equivalence. We recall the set of square integrable sections of the vector bundle determined by the principal bundle $K \to G \to G/K$ and the representation (τ, W) of K is isomorphic to the space

$$L^2(G \times_{\tau} W) := \{f \in L^2(G) \otimes W : \\ f(gk) = \tau(k)^{-1} f(g), g \in G, k \in K\}.$$

Here, the action of G is by left translation $L_x, x \in G$. The inner product on $L^2(G) \otimes W$ is given by

$$(f, g)_{V_{\lambda}} = \int_G (f(x), g(x))_W \, \mathrm{d}x$$

where $(\ldots, \ldots)_W$ is a K-invariant inner product on W. Subsequently, L_D (resp. R_D) denotes the left infinitesimal (resp. right infinitesimal) action on functions from G of an element D in universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ for the Lie algebra $\mathfrak{g}$. As usual, Ω_G denotes the Casimir operator for $\mathfrak{g}$. Following Hotta, Enright-Wallach [21], we realize $V_{\lambda} := V_{\lambda}^{G}$ as the space

$$H^2(G, \tau) = \{f \in L^2(G) \otimes W : f(gk) = \tau(k)^{-1} f(g) \\ g \in G, k \in K, R_{\Omega_G} f = [(\lambda, \lambda) - (\rho, \rho)] f\}.$$

We also recall, $R_{\Omega_G} = L_{\Omega_G}$ is an elliptic G-invariant operator on the vector bundle $W \to G \times_{\tau} W \to G/K$ and hence, $H^2(G, \tau)$ consists of smooth sections, moreover the point evaluation e_x defined by $H^2(G, \tau) \ni f \mapsto f(x) \in W$ is continuous for each $x \in G$. Therefore, the orthogonal projector P_{λ} onto $H^2(G, \tau)$ is an integral map (integral operator, kernel map) represented by the smooth reproducing kernel

$$K_{\lambda} : G \times G \to End_{\mathbb{C}}(W) \tag{1.1}$$

which satisfies $K_{\lambda}(\cdot, x)^{\star} w$ belongs to $H^2(G, \tau)$ for each $x \in G$, $w \in W$ and

$$(P_{\lambda}(f)(x), w)_W = \int_G (f(y), K_{\lambda}(y, x)^{\star} w)_W \, \mathrm{d}y, \; x \in G, w \in W, f \in L_2(G \times_{\tau} W).$$

It can be shown that $K_{\lambda}(y, x) = \Phi_0(x^{-1} y)$, where Φ_0 is a constant times the spherical function associated with the K-type W. It readily follows that $K_{\lambda}(zy, zx) = K_{\lambda}(y, x)$, $x, y, z \in G$ and the function $y \mapsto K_{\lambda}(y, e)^{\star} w$ is K-finite. Since $L_x(K_{\lambda}(\cdot, e)^{\star} w) = K_{\lambda}(\cdot, x)^{\star} w$, we have that $K_{\lambda}(\cdot, x)$ is a smooth vector in $H^2(G, \tau)$. For a closed reductive subgroup H, after conjugation by an inner automorphism of G, we may and will assume that $L := K \cap H$ is a maximal compact subgroup for H. That is, H is θ-stable. In this note, for irreducible square integrable representations $(\pi_{\lambda}, V_{\lambda})$ for G, we would like to analyze its restriction to H. In particular, we study the irreducible H-subrepresentations for π_{λ}. A result is that any irreducible H-subrepresentation of V_{λ} is a square integrable representation for H, for a proof (cf. [12, Cor. 8.7], [6]). Thus, owing to the result of Harish-Chandra on the existence of square integrable representations, from now on we may and will assume that H *admits a compact Cartan subgroup*. After conjugation, we may assume that $U := H \cap T$ is a maximal torus in $L = H \cap K$. Next, we consider a square integrable representation $H^2(H, \sigma) \subset L^2(H \times_\sigma Z)$ of lowest L-type (σ, Z). An aim of this note is to understand the nature of the intertwining operators between the unitary H-representations $H^2(H, \sigma)$ and V_λ^G, the adjoint of such intertwining operators and the consequences of their structure. The details of the proof and some other results on the topic will appear in an article bearing the same title as this note. We would like to call the attention of the reader to our results in Theorem 3.3, Corollary 3.4, Theorem 4.1, and Theorem 2.6.

Notation: $\mathbb{N} = \{1, 2, \dots\}$.

2. Structure of intertwining maps

For this section, besides $G, K, T, (\tau, W), (L, H^2(G, \tau)) = (\pi_\lambda^G, V_\lambda^G), H, L, U$ as in Section 1, we fix (ν, E) a finite-dimensional representation of L and a continuous intertwining linear H-map $T : L^2(H \times_\nu E) \to H^2(G, \tau)$.

Fact 2.1. We show that T is a kernel map.

In fact, for each $x \in G$, $w \in W$ the linear function $L^2(H \times_\nu E) \ni g \mapsto (Tg(x), w)_W$ is continuous. Whence, the Riesz representation Theorem shows that there exists a function

$$K_T : H \times G \to Hom_{\mathbb{C}}(Z, W)$$

such that, for each $x \in G$, $w \in W$, the map $h \mapsto K_T(h, x)^\star(w)$ belongs to $L^2(H \times_\nu E)$ and for $g \in L^2(H \times_\nu E)$, $w \in W$, we have an absolutely convergent integral and the equality

$$(Tg(x), w)_W = \int_H (g(h), K_T(h, x)^\star w)_Z \mathrm{d}h. \tag{2.1}$$

That is, T is the integral map

$$Tg(x) = \int_H K_T(h, x) g(h) \mathrm{d}h, x \in G.$$

We also have $(T(g)(x), w)_W = \int_G (T(g)(y), K_\lambda(y, x)^\star w)_W \mathrm{d}y$. To follow, we make explicit some properties of the kernel of T.

Proposition 2.2. *Let $T : L^2(H \times_\nu E) \to H^2(G, \tau)$ be a continuous intertwining linear H-map. Then, the function K_T satisfies:*

a) $K_T(h, x)^\star w = T^\star(y \mapsto K_\lambda(y, x)^\star w)(h)$;

b) the function $h \mapsto K_T(h, e)^\star w$ is an L-finite vector in $L^2(H \times_\nu E)$;

c) K_T is a smooth map. Further, $K_T(\cdot, x)^\star w$ is a smooth vector;

d) there exists a constant C and finitely many functions $\phi_{a,b} : G \to \mathbb{C}$ such that, for every $x \in G$, $\|K_T(e, x)^\star\|_{Hom(W,Z)} \leq C\|T^\star\| \sum_{a,b} |\phi_{a,b}(x)|$;

e) $\|K_T(\cdot, x)^\star\|_{L^2(H \times_{\tau^\star \otimes \nu} Hom_{\mathbb{C}}(W,E))}$ is a bounded function on G;

f) $K_T(hh_1 s, hxk) = \tau(k^{-1}) K_T(h_1, x) \nu(s)$, $x \in G$, $s \in L$, $h, h_1 \in H$, $k \in K$;

g) if $T^\star$ is a kernel map, with kernel $K_{T^\star} : G \times H \to Hom(W, E)$ and $K_{T^\star}(\cdot, h)^\star z \in H^2(G, \tau)$. Then, $L_D^{(2)} K_T(h, \cdot) = \chi_\lambda(D) K_T(h, \cdot)$ for every D in the center of $\mathcal{U}(\mathfrak{g})$. Here, χ_λ is the infinitesimal character of π_λ.

Note: The functions $\phi_{a,b}$ are defined as follows. We fix a linear basis $\{X_b\}_{1 \leq b \leq N}$ (resp. $\{Y_a\}_{1 \leq a \leq M}$) for the space of elements in $\mathcal{U}(\mathfrak{h})$ of degree less or equal than $\dim \mathfrak{h}$ (resp. for the space of elements in $\mathcal{U}(\mathfrak{g})$ of degree less or equal than $\dim \mathfrak{h}$). Then $\phi_{a,b}$ are defined by $Ad(x^{-1})(X_b) = \sum_{1 \leq a \leq M} \phi_{a,b}(x) Y_a$, $b = 1, \cdots, N$.

Definition 2.3. A representation (π_λ, V_λ) is discretely decomposable over H if there exists an orthogonal family of closed, H-invariant, H-irreducible subspaces of V_λ such that the closure of its algebraic sum is equal to V_λ.

Definition 2.4. A representation (π_λ, V_λ) is H-admissible if the representation is discretely decomposable and the multiplicity of each irreducible factor is finite.

In [15] we find a complete list of triples (G, H, π_λ) such that (G, H) is a symmetric pair and π_λ is an H-admissible representation. For example, for the pair $(SO(2n, 1), SO(2k) \times SO(2n - 2k, 1))$, there is no π_λ with an admissible restriction to H. Whereas, for the pair $(SU(m, n), S(U(m, k) \times U(n - k)))$, there are exactly m Weyl chambers $C_1, \dots, C_m$ in $i\mathfrak{t}_{\mathbb{R}}^\star$ such that π_λ is H-admissible if and only λ belongs to $C_1 \cup \dots \cup C_m$.

Definition 2.5. A unitary representation (π, V) is integrable if some nonzero matrix coefficient is an integrable function.

A theorem of Trombi, Varadarajan, Hecht, and Schmid states that (π_λ, V_λ) is an integrable representation if and only if $|(\lambda, \beta)| > \sum_{\alpha\in\Phi(\mathfrak{g},\mathfrak{t}):(\alpha,\beta)>0}(\alpha, \beta)$ for every noncompact root β.

Examples show that the adjoint $T^\star$ of integral linear map T need not be an integral map, whence we would like to know when $T^\star$ is an integral linear map. Formally, we may write $T^\star f(h) = \int_G K_T(h, x)^\star f(x)\mathrm{d}x$, where the convergence of the integral is in the weak sense. That is, for each $g \in L^2(H \times_\nu E)$, $f \in H^2(G, \tau)$, we have the absolute convergence of the iterated integral

$$(T^\star f, g)_{L^2(H\times_\nu E)} = \int\limits_G \int\limits_H (f(x), K_T(h, x)g(h))_W \,\mathrm{d}h\,\mathrm{d}x.$$

T. Kobayashi has introduced the concept of symmetry-breaking operator. In our setting, a symmetry-breaking operator is a continuous H-map S: $H^2(G, \tau) \to L^2(H \times_\nu E)$. For a symmetry-breaking operator S, the above considerations applied to $T := S^\star$ let us conclude: under our hypothesis, a symmetry-breaking operator is always a weak integral map. The next result gives more information about symmetry-breaking operators.

Theorem 2.6. *Let $S : H^2(G, \tau) \to L^2(H \times_\nu E)$ be a continuous intertwining linear H-map. Then,*

a) if the restriction to H of $(L, H^2(G, \tau))$ is discretely decomposable, then S is an integral map;

b) if $(L, H^2(G, \tau))$ is an integrable representation for G, then S restricted to the subspace of smooth vectors is an integral linear map.

Remark 2.7. We assume that $S : H^2(G, \tau) \to L^2(H \times_\nu E)$ is a continuous symmetry-breaking operator represented by a kernel $K_S : G \times H \to Hom(W, E)$ such that $K_S(\cdot, h)^\star z \in H^2(G, \tau)$, $\forall h \in H$, $z \in E$. This hypothesis implies that $K_S(x, h) = K_{S^\star}(h, x)^\star$; hence, from Proposition 2.2c, we may conclude that:

a) K_S is a smooth map;

b) $K_S(hxk, hh_1s) = \nu(s^{-1})K_S(s, h_1)\tau(k)$, $h, h_1 \in H$, $x \in G$, $s \in L$;

c) the function $G \ni x \mapsto K_S(x, e)^\star z \in W$, $z \in E$ is L-finite.

3. Intertwining operators via differential operators

Let $G, K, H, L, H^2(G, \tau)$, (ν, E) be as usual. In [17], [14], [20], [3], and references therein, these and other authors have constructed H-intertwining maps between holomorphic discrete series by means of differential operators. Some authors also considered the case of intertwining maps between two principal series representations [2], [19]. Motivated by the fact that discrete series can be modeled as function spaces, an aim of this section is to analyze to what extent H-intertwining linear maps agree with the restriction of linear differential operators. In [14] is presented a general conjecture on the subject; we present a partial solution for the particular case of discrete series representations.

For the purpose of this note, *a differential operator* is a linear map $S : C^\infty(G \times_\tau W) \to C^\infty(H \times_\nu E)$ such that there exists finitely many elements $D_b \in \mathcal{U}(\mathfrak{g})$, $\{w_c\}$ basis for W, $d_{a,b,c} \in \mathbb{C}$, $\{z_a\}$ basis for E, and such that we have, for any $f \in C^\infty(G \times_\tau W)$:

$$S(f)(h) = \sum_{a,b,c} d_{a,b,c}\, ([R_{D_b} f](h), w_c)_W\, z_a \;\forall h \in H. \tag{3.1}$$

These family of differential operators include the H-invariant differential operators as it is shown in [17].

Sometimes we will allow the constants $d_{a,b,c}$ to be smooth functions on H.

Example 3.1. Examples of differential operators are the normal derivatives as considered in [10], [18], [21]. For this, we write the Cartan decomposition as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{h} = \mathfrak{l} + \mathfrak{p}'$. We have $\mathfrak{p}' = \mathfrak{p} \cap \mathfrak{h}$. Let $(\mathfrak{p}/\mathfrak{p}')^{(n)}$ denote the n-th symmetric power of the orthogonal with respect to the Killing form of $\mathfrak{p}'$ in $\mathfrak{p}$. We denote by τ_n the natural representation of L in $Hom_{\mathbb{C}}((\mathfrak{p}/\mathfrak{p}')^{(n)}, W)$. Let $\lambda : S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ denote the symmetrization map. Then, for each $D \in (\mathfrak{p}/\mathfrak{p}')^{(n)}$, $f \in H^2(G, \tau)$, $h \in H$, we compute the normal derivative of f in the direction D at the point h, $r_n(f)(D)(h) := R_{\lambda(D)} f(h)$. In [21], it is shown that $r_n(f) \in L^2(H \times_{\tau_n} Hom_{\mathbb{C}}((\mathfrak{p}/\mathfrak{p}')^{(n)}, W))$, and the resulting map

$$r_n : H^2(G, \tau) \to L^2(H \times_{\tau_n} Hom_{\mathbb{C}}((\mathfrak{p}/\mathfrak{p}')^{(n)}, W))$$

is H-equivariant and continuous for L^2-topologies. As before, K_λ is the matrix kernel of P_λ. The map r_n is represented by the matrix kernel

$$K_{r_n} : G \times H \to Hom_{\mathbb{C}}(W, Hom_{\mathbb{C}}((\mathfrak{p}/\mathfrak{p}')^{(n)}, W))$$

given by

$$K_{r_n}(y,h)(w,D) = R^{(2)}_{\lambda(D)}(K_\lambda(y,h)w).$$

Here, the upper index (2) means that we compute the derivative of K_λ on the second variable.

Before we state Theorem 3.3, we derive some properties of symmetry-breaking operators that are restriction of differential operators.

Lemma 3.2. *Let $S : H^2(G,\tau) \to L^2(H \times_\nu E)$ be a not necessarily continuous intertwining H-map such that S is the restriction of a differential operator. Then, S is a kernel map. That is, there exists $K_S : G \times H \to Hom_{\mathbb{C}}(W,E)$ such that $y \mapsto K_S(y,h)^\star z \in H^2(G,\tau)$ for $h \in H$, $z \in E$ and*

a) $(S(f)(h),z)_Z = \int_G (f(y), K_S(y,h)^\star z)_W dy$ for $f \in H^2(G,\tau)$, $z \in E$;

b) $y \mapsto K_S(y,e)^\star z$ is a K-finite vector for π_λ;

c) K_S is a smooth function;

d) S is continuous in L^2-topologies.

Conversely, if $S : H^2(G,\tau) \to L^2(H \times_\nu E)$ is an integral H-map such that $y \mapsto K_S(y,e)^\star z$ is a K-finite vector for π_λ. Then, S is continuous and S is the restriction of a differential operator.

Compare b) with 2.7 c). We differ the proof of the Lemma 3.2 until we state the following theorem.

Theorem 3.3. *Let $G, K, H, L, (\tau, W), H^2(G,\tau), \pi_\lambda, (\nu, E), (\sigma, Z)$ be as in the previous paragraph. Let $S : H^2(G,\tau) \to L^2(H \times_\nu E)$ denote an intertwining linear H-map. If we assume that S is continuous and $res_H(\pi_\lambda)$ is an H-admissible representation, then S is the restriction of a linear differential operator.*

For a converse statement, we have:

i) if we assume for some ν that some nonzero linear intertwining H-map $S : H^2(G,\tau) \to L^2(H \times_\nu E)_{disc}$ is the restriction of a linear differential operator, then $res_H(\pi_\lambda)$ is discretely decomposable;

ii) if we assume for some σ that every nonzero linear intertwining H-map $S : H^2(G,\tau) \to H^2(H,\sigma)$ is the restriction of a linear differential operator, then the multiplicity of $(L, H^2(H,\sigma))$ in $res_H(\pi_\lambda)$ is finite;

iii) if we assume for every σ that every nonzero $S : H^2(G,\tau) \to H^2(H,\sigma)$ is the restriction of a linear differential operator, then $res_H(\pi_\lambda)$ is an H-admissible representation.

An immediate corollary is:

Corollary 3.4. *If π_λ is an H-admissible representation, then, for each (σ, Z), every continuous linear H-map $S : H^2(G,\tau) \to H^2(H,\sigma)$ is the restriction of a differential operator as well as an integral map.*

Example for maps S where the Theorem applies are the normal maps r_n defined in Example 3.1. In particular, we have that $res_H(\pi_\lambda)$ is discretely decomposable if and only if there exists an n such that the image of r_n is contained in $L^2(H \times_{\tau_n} ((\mathfrak{p}/\mathfrak{p}')^{(n)} \otimes W))_{\text{disc}}$.

In [17, Theorem 5.3] as well as in [20, Theorem 3.10.1], a similar result is shown under the hypothesis that both G/K and H/L are Hermitian symmetric spaces, that the inclusion H/L into G/K is holomorphic, and that both representations are holomorphic discrete series.

Proof of Lemma 3.2. We fix $\{z_j, j = 1, \cdots, \dim E\}$, $\{w_i, i = 1, \dots, \dim W\}$ as the respective orthonormal bases for E, W. Our hypothesis provides, for every $f \in H^2(G,\tau)$, $h \in H$, that the following equality holds:

$$S(f)(h) = \sum_{j,b,i} d_{j,b,i}\,([R_{D_b} f](h), w_i)_W\, z_j.$$

In [1] we find a proof that, in the L^2-kernel of an elliptic operator, L^2-convergence implies uniform convergence of the sequence as well as any of its derivatives on compact sets. Since the Casimir operator acting on G/K is an elliptic operator, the result on PDE just quoted applies to $H^2(G,\tau)$. Hence, the equality $(S(f)(h), z_j)_Z = \sum_{b,i} d_{j,b,i}(R_{D_b} f(h), w_i)_W$ yields that the left-hand side determines a continuous linear functional on $H^2(G,\tau)$. Thus, there exists a function $K_S : G \times H \to Hom_{\mathbb{C}}(W,E)$ such that $y \mapsto K_S(y,h)^\star z_j$ belongs to $H^2(G,\tau)$ and a) holds. The hypothesis S is an intertwining map, it yields the equality $K_S(h_1 yk, h_1 hs) = \nu(s^{-1}) K_S(y,h)\tau(k)$, $h_1, h \in H$, $y \in G$, $s \in L$, $k \in K$. The smoothness for K_S follows from the fact that $K_S^\star$ is equal to the map $(y,h) \mapsto h^{-1}y$ followed by the map $x \mapsto K_S(x,e)^\star$ and that $x \mapsto K_S(x,e)^\star$ is an element of $H^2(G,\tau)$. Next, we justify the four equalities in the computation below. The first is due to the expression of S, the second is due to the identity $L_{\check{D}}(f)(e) = R_D f(e)$; hence, we obtain $(S(f)(e), z_j)_Z = \sum_{b,i} d_{j,b,i}(L_{\check{D}_b} f(e), w_i)_W$. The third is due to (1.1); finally, we recall that, for an arbitrary $D \in \mathcal{U}(\mathfrak{g})$, it follows that any smooth vector $f \in V_\lambda^\infty$ is in the domain for L_D;

in particular, $y \mapsto K_\lambda(y,e)^\star w_i$ is in the domain for L_D. These four considerations justify the following equalities for any smooth vector $f \in H^2(G,\tau)$:

$$\int_G (f(y), K_S(y,e)^\star z_j)_W \, \mathrm{d}y = (S(f)(e), z_j)$$

$$= \sum d_{j,b,i} (L_{\check{D}_b} f(e), w_i)_W$$

$$= \sum_{j,b,i} d_{j,b,i} \int_G (L_{\check{D}_b} f(y), K_\lambda(y,e)^\star w_i)_W \, \mathrm{d}y$$

$$= \sum_{b,i} d_{j,b,i} \int_G (f(y), L^{(1)}_{\check{D}_b^\star} K_\lambda(y,e)^\star w_i)_W \, \mathrm{d}y.$$

We observe that the first member and the last member of the above equalities define continuous linear functionals on $H^2(G,\tau)$ and that they agree on the dense subspace of smooth vectors, whence

$$K_S(y,e)^\star z_j = \sum_{b,i} d_{j,b,i} L^{(1)}_{\check{D}_b^\star} K_\lambda(y,e)^\star w_i.$$

Since the right-hand side of the above equality is a K-finite vector for π_λ, we have shown b). To show the continuity of S, we notice that S is defined by the Carleman kernel K_S (for each $h \in H$, $K_S(\cdot,h) \in H^2(G,\tau)$) and, by hypothesis, the domain of the integral operator defined by K_S on $L^2(G \times_\tau W)$ contains $H^2(G,\tau)$. Since a Carleman kernel determines a closed map on its maximal domain, and $H^2(G,\tau)$ is a closed subspace, we have that $S : H^2(G,\tau) \to L^2(H \times_\nu E)$ is a closed linear map with domain $H^2(G,\tau)$; the closed graph Theorem leads to the continuity of S.

To show the converse statement, we explicit the hypotheses on S:

$K_S(hx, hh_1) = K_S(x, h_1)$, $h, h_1 \in H$, $x \in G$; for each $Z \in E$, $y \mapsto K_S(y,e)^\star z$ is a K-finite vector in $H^2(G,\tau)$; the domain of S is equal to $H^2(G,\tau)$. We show that S is a the restriction of a differential operator. In fact, since $K_S(x,h) = K_S(h^{-1}x, e)$ and $K_S(\cdot,e)$ belongs to $H^2(G,\tau)$, we obtain that $K_S(\cdot,h)$ is square integrable and hence S is a Carleman map. As in the direct implication, we obtain that S is continuous. To verify that S is the restriction of a differential map, we fix a nonzero vector $w \in W$. Since $H^2(G,\tau)$ is an irreducible representation, a result of Harish-Chandra shows that the underlying $(\mathfrak{g}, K)$-module for $H^2(G,\tau)$ is $\mathcal{U}(\mathfrak{g})$-irreducible. It readily follows that the function $K_\lambda(\cdot,e)^\star w$ is nonzero (otherwise K_λ would be the null function); therefore, for each z_j, there exists $D_j \in \mathcal{U}(\mathfrak{g})$ such that $K_S(\cdot,e)^\star z_j = L_{D_j} K_\lambda(\cdot,e)^\star w$. For a smooth vector f in $H^2(G,\tau)$,

$$S(f)(h^{-1}) = S(L_h f)(e) = \sum_j (S L_h f(e), z_j) z_j$$

$$= \sum_j \int_G (L_h f(y), K_S(y,e)^\star z_j)_W z_j = \sum_j \int_G (L_h f(y), L_{D_j} K_\lambda(y,e)^\star w)_W z_j$$

$$= \sum_j \int_G (L_{D_j^\star}(L_h f)(y), K_\lambda(y,e)^\star w)_W dy = \sum_j (L_{D_j^\star}(L_h f)(e), w)_W z_j$$

$$= \sum_j ((R_{\check{D}_j^\star})(L_{h^{-1}} f)(e), w)_W z_j = \sum_j ((R_{\check{D}_j^\star})(f)(h^{-1}), w)_W z_j.$$

Thus, after we have fixed a linear basis $\{R_i\}$ for $\mathcal{U}(\mathfrak{g})$, for a smooth vector f we have

$$S(f)(h) = \sum_{j,i} d_{i,j} ([R_{R_i}(f)](h), w) z_j.$$

Owing to the result on PDEs quoted in the direct proof, the right-hand side defines a continuous linear transformation from $H^2(G,\tau)$ into $C^\infty(H \times_\nu E)$. We claim that this forces $S(f)$ to be continuous for every f. In fact, each f in $H^2(G,\tau)$ is the limit of a sequence f_n of smooth vectors, whence the first and last members of the above equalities agree on each f_n, if necessary going to a subsequence; the Riez–Fischer Theorem yields that the left-hand side pointwise converges (a.e.) to $S(f)$. Thus, $S(f)$ agree up to a set of measure zero with a smooth function. Moreover, this argument yields that $S(f)$ is equal to the right-hand side for any f. Thus, we have shown that S is a differential operator. This concludes the proof of Lemma 3.2. □

Proof of Theorem 3.3. For the vector spaces E, W, we fix the respective orthonormal bases $\{z_j\}$, $\{w_i\}$. Since $res_H(\pi_\lambda)$ is discretely decomposable, Theorem 2.6 shows that S is an integral map. For each j, the identity $K_S(l^{-1}y,e)^\star z_j = K_S(y,e)^\star \nu(l) z_j$,

$l \in L$, $y \in G$ yields $x \mapsto K_S(x,e)^\star z_j$ is an L-finite vector. The hypothesis that $res_H(\pi_\lambda)$ is an admissible representation allows us to apply [11, Prop. 1.6]. In this way, we obtain that the subspace of L-finite vectors in $H^2(G,\tau)$ is equal to the subspace of K-finite vectors. Whence $x \mapsto K_S(x,e)^\star z_j$ is a K-finite vector. By hypothesis, $H^2(G,\tau)_{K\text{-fin}}$ is an irreducible representation under the action of $\mathcal{U}(\mathfrak{g})$ and the function $y \mapsto K_\lambda(y,e)^\star w_i$ is nonzero and K-finite, hence, for each i, j there exists $C_{j,i} \in \mathcal{U}(\mathfrak{g})$ such that $[L^{(1)}_{C_{j,i}} K_\lambda](y,e)^\star w_i = K_S(y,e)^\star z_j$, for all $y \in G$. Therefore, since $x \mapsto K_S(x,e)^\star z_j$ is a smooth vector for G, for $f \in V_\lambda^\infty$ we justify, as in the proof of Lemma 3.2, the fourth and the sixth equalities in the following computation; the fifth equality is due to (1.1),

$$\begin{aligned} S(f)(e) &= \sum_j (S(f)(e), z_j)_E z_j = \sum_j \int_G (f(y), K_S(y,e)^\star z_j)_W \,\mathrm{d}y z_j \\ &= \sum_j \int_G (f(y), [L^{(1)}_{C_{j,i}} K_\lambda](y,e)^\star w_i)_W \,\mathrm{d}y z_j \\ &= \sum_j \int_G (L_{C^\star_{j,i}} f(y), K_\lambda(y,e)^\star w_i)_W \,\mathrm{d}y z_j \\ &= \sum_j ([L_{C^\star_{j,i}} f](e), w_i)_W z_j = \sum_j ([R_{\check{C}^\star_{j,i}} f](e), w_i)_W z_j \end{aligned}$$

For $h \in H$, we apply the previous equality to $f := L_{h^{-1}} f$ and, since S intertwines the action of H, we obtain

$$\begin{aligned} S(f)(h) &= S(L_{h^{-1}} f)(e) \\ &= \sum_j ([R_{\check{C}^\star_{j,i}}(L_{h^{-1}} f)](e), w_i)_W z_j = \sum_j ([R_{\check{C}^\star_{j,i}}(f)](h), w_i)_W z_j. \end{aligned}$$

After we set $D_{j,i} := \check{C}^\star_{j,i}$, and recalling definition (3.1), we conclude that the fact that S is restricted to the subspace of smooth vectors agrees with the restriction of a differential operator. In order to show the equality for a general $f \in H^2(G,\tau)$, we argue as follows: there exists a sequence f_r of elements in V_λ^∞ that converges in L^2-norm to f. Due to the fact that the Casimir operator is elliptic on G/K, the sequence f_r as well as any derivatives of the sequence converge uniformly on compact subsets. Moreover, owing to Harish-Chandra's Plancherel Theorem, $L^2(H \times_\nu E)_{\mathrm{disc}}$ is a finite sum of square integrable irreducible representations. More precisely, Harish-Chandra $L^2(H \times_\nu E)_{\mathrm{disc}}$ is a finite sum of eigenspaces for the Casimir operator for $\mathfrak{h}$. We know that the Casimir operator acts as an elliptic differential operator on $L^2(H \times_\nu E)$, whence we have that the point evaluation is a continuous linear functional on $L^2(H \times_\nu E)_{\mathrm{disc}}$ [1]. Finally, the hypothesis on $res_H(\pi_\lambda)$ gives that the image of S is contained in $L^2(H \times_\nu E)_{\mathrm{disc}}$. Therefore, we have justified the steps in

$$\begin{aligned} S(f)(h) &= \lim_r S(f_r)(h) = \lim_r \sum_j ([R_{D_{j,i}} f_r](h), w_i)_W z_j \\ &= \sum_j ([R_{D_{j,i}} \lim_r f_r](h), w_i)_W z_j = \sum_j ([R_{D_{j,i}} f](h), w_i)_W z_j. \end{aligned}$$

Whence we have shown the first statement in Theorem 3.3.

Henceforth, $\mathfrak{z}(\mathcal{U}(\mathfrak{s}))$ denotes the center of the enveloping algebra of $\mathfrak{s}$.

To follow, we assume that, for some σ and some nonzero intertwining, H-map $S : H^2(G,\tau) \to L^2(H \times_\nu E)_{\mathrm{disc}}$ is the restriction of a linear differential operator; we show that $res_H(\pi_\lambda)$ is discretely decomposable.

In fact, the hypothesis allows us to apply Lemma 3.2. In consequence, $y \mapsto K_S(y,e)^\star z_j$ is a K-finite vector in $H^2(G,\tau)$. We claim that $K_S(\cdot,e)^\star z_j$ is $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$-finite. In fact, Harish-Chandra's Plancherel Theorem shows that $L^2(H \times_\nu E)_{\mathrm{disc}}$ is equal to a finite sum of irreducible discrete series for H. Thus, for $f \in V_\lambda^\infty$, $D \in \mathfrak{z}(\mathcal{U}(\mathfrak{h}))$ whenever the image of S is contained in an irreducible subspace, we have the equalities

$$\begin{aligned} \int_G (f(y), L^{(1)}_{D^\star} K_S(y,e)^\star z)_W \,\mathrm{d}y &= \int_G (L_D f(y), K_S(y,e)^\star z)_W \,\mathrm{d}y \\ &= (S(L_D f)(e), z)_Z = \chi_\mu(D)(S(f)(e), z)_Z \\ &= \chi_\mu(D) \int_G (f(y), K_S(y,e)^\star z)_W \,\mathrm{d}y \end{aligned}$$

The third equality holds because, by hypothesis $S(f)$, is an eigenfunction for $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$. Therefore, the first and last members of the above equalities determine continuous linear functionals on $H^2(G,\tau)$ that agree in the dense subspace of smooth vectors. Whence $y \mapsto K_S(y,e)^\star z_j$ is an eigenfunction for $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$. The general case readily follows from a similar computation. Thus, the hypothesis that S is nonzero gives us an $z \in E$ such that $\mathcal{U}(\mathfrak{h})K_S(\cdot,e)^\star z$ is a $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$-finite and nonzero $\mathcal{U}(\mathfrak{h})$-submodule of $V_{K\text{-fin}}$. We quote a result of Harish-Chandra: a $\mathcal{U}(\mathfrak{h})$-finitely generated, $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$-finite, $(\mathfrak{h}, L)$-module has a finite composition series. For a proof, see [22, Corollary 3.4.7 and Theorem 4.2.1]. Therefore, the subspace $\mathcal{U}(\mathfrak{h})K_S(\cdot,e)^\star z$ contains an irreducible $\mathcal{U}(\mathfrak{h})$-submodule. Next, in [11, Lemma 1.5] we find a proof of: If $(\mathfrak{g}, K)$-module contains an irreducible $(\mathfrak{h}, L)$-submodule, then the $(\mathfrak{g}, K)$-module is $\mathfrak{h}$-algebraically decomposable. Thus, $res_H(\pi_\lambda)$ is algebraically discretely decomposable. The fact that π_λ is unitary yields that it is discrete decomposable [13, Theorem 4.2.6]. Whence we have shown i).

We now assume for some σ and every intertwining linear H-map that $S : H^2(G,\tau) \to H^2(H,\sigma)$ is the restriction of a linear differential operator. We show that the multiplicity of $(L, H^2(H,\sigma))$ in $res_H(\pi_\lambda)$ is finite.

Henceforth, $H^2(G,\tau)[V_\mu^H]$ denotes the isotypic component for V_μ^H. That is, the closure of the sum of the H-equivariant linear subspaces such that π_λ is restricted to the subspace gives rise to a representation equivalent to V_μ^H.

The first claim in Theorem 3.3 yields that $res_H(\pi_\lambda)$ is discrete decomposable. Let us assume that the multiplicity of $H^2(H,\sigma)$ in $res_H(\pi_\lambda)$ is infinite. Thus, there exists $T_1, T_2, \dots$ such that $T_j : H^2(H,\sigma) \to H^2(G,\tau)[V_\mu^H]$ are isometric immersion intertwining linear maps such that, for $r \neq s$, the image of T_r is orthogonal to the image of T_s and the algebraic sum of the subspaces $T_r(H^2(H,\sigma))$ $r = 1,2,\dots$ is dense in $H^2(G,\tau)[V_\mu^H]$. Let $\iota : Z \to V_\mu^H[Z]$ be the equivariant immersion adjoint to the evaluation at the identity. We fix a norm-one vector $g_0 := \iota(z_0) \in V_\mu^H[Z]$. There are two possibilities: for some r, the function $K_{T_r}(\cdot,e)^\star z_0$ is not a K-finite vector, or else for every r the function $K_{T_r}(\cdot,e)^\star z_0$ is a K-finite vector. To follow, we analyze the second case; for this, we define $v_n := T_n(g_0)$ and we choose a sequence of nonzero positive real numbers $(a_n)_n$ such that $v_0 := \sum_n a_n v_n$ is not the zero vector. Due to the orthogonality for the image of the T_r and the choice of the sequence, v_0 is not a K-finite vector. Since the stabilizer of v_0 in H is equal to the stabilizer of g_0 on H, the correspondence $T : V_\mu^H \to H^2(G,\tau)$ defined by $T(h.g_0) = \frac{1}{\|v_0\|} h{\cdot}v_0$ extends to an isometric immersion. We claim that $S = T^\star$ is not the restriction of a linear differential operator. For this, we show that $S(\frac{v_0}{\|v_0\|}) = g_0$ and $K_S(\cdot,e)^\star z_0 = \frac{v_0}{\|v_0\|}$. On the one hand, we have that the following system of equations

$$(S(f)(e), z_0)_Z = \int_G (f(y), K_S(y,e)^\star z_0)_W \, dy \;\; \forall f \in H^2(G,\tau)$$

determines the function $K_S(\cdot,e)^\star z_0$. On the other hand, for arbitrary $f \in H^2(G,\tau)$, we have:

$$\int_G (f(y), \frac{v_0(y)}{\|v_0\|})_W \, dy = (f, T(g_0))_{H^2(G,\tau)} = (T^\star f, g_0)_{H^2(H,\sigma)}$$
$$= (S(f), \iota(z_0))_{H^2(H,\sigma)} = (\iota^\star(Sf), z_0)_Z = (S(f)(e), z_0)_Z.$$

Thus, we have shown the equality $K_S(\cdot,e)^\star z_0 = \frac{v_0}{\|v_0\|}$. Therefore, if S were a differential operator, the fact v_0 is not a K-finite vector would yield a contradiction with Lemma 3.2b. In the first case, a similar argument yields that $S := T_r^\star$ is not the restriction of a linear differential operator. This concludes the proof of ii) in Theorem 3.3. The statements iii), iv) are obvious. □

Remark 3.5. The hypothesis on the image of S is quite essential. Examples and counterexamples are provided by r, r_1, r_2...

Example 3.6. For H/L a real form for the Hermitian symmetric space G/K and a holomorphic discrete series $H^2(G,\tau)$ for G, any nonzero intertwining linear H-map $S : H^2(G,\tau) \to H^2(H,\sigma)$ is never the restriction of a differential operator. Indeed, the statement holds because under our hypothesis $res_H(\pi_\lambda)$ is not discretely decomposable [9].

3.1. Extension of an intertwining map to maximal globalization

A conjecture of Toshiyuki Kobayashi [14] predicts that, under certain hypothesis, each continuous intertwining linear operator between two maximal globalizations of Zuckerman modules, realized via Dolbeault cohomology, are given by restriction of a holomorphic differential operator. In this subsection, we show an analogous statement for the maximal globalization provided by the kernel of a Schmid operator.

The symbols $G, K, (\tau, W), H^2(G,\tau), H, L, (\sigma, Z), H^2(H,\sigma)$ are as usual. Let

$$D_G : C^\infty(G \times_\tau W) \to C^\infty(G \times_{\tau_1} W_1)$$

be the Schmid operator [23]. Similarly, we have a Schmid operator $D_H : C^\infty(H \times_\sigma Z) \to C^\infty(H \times_{\sigma_1} Z_1)$. Since D_G is an elliptic operator, $Ker(D_G)$ is a closed subspace of the space of smooth sections. Thus, $Ker(D_G)$ becomes a smooth Fréchet representation for G. Among the properties of the kernel of the operator D_G are: $H^2(G,\tau)$ is a linear subspace of $Ker(D_G)$, the inclusion map $H^2(G,\tau)$ into $Ker(D_G)$ is continuous, the subspace of K-finite vectors in $Ker(D_G)$ is equal to the subspace of K-finite vectors for $H^2(G,\tau)$, $Ker(D_G)$ is a maximal globalization for the underlying Harish-Chandra module for $(\pi_\lambda, H^2(G,\tau))$. A similar statement holds for D_H. Now, we are ready to state the corresponding result.

Theorem 3.7. *We assume that $res_H(\pi_\lambda)$ is an H-admissible representation. Then, the following two statements holds:*

a) any continuous, H-intertwining linear map $S : Ker(D_G) \to Ker(D_H)$ is the restriction of a differential operator;

b) any continuous H-intertwining linear map $S : H^2(G,\tau) \to H^2(H,\sigma)$ extends to a continuous intertwining operator from $Ker(D_G)$ to $Ker(D_H)$.

Nakahama in [20, Theorem 3.6] has shown a similar result under the hypothesis that both G/K, H/L are Hermitian symmetric spaces, the inclusion H/L into G/K is holomorphic, and both representations are holomorphic discrete series.

4. Criteria for discretely decomposable restriction

As in previous sections, we keep the hypothesis and notation of Section 1. The objects are $G, K, (\tau, W), H^2(G,\tau), H, L$. We recall that the orthogonal projector P_λ onto $H^2(G,\tau)$ (1.1) is given by a smooth matrix kernel $K_\lambda(y,x) = K_\lambda(x^{-1}y, e) = \Phi_0(x^{-1}y)$, here Φ_0 is the spherical function associated with the lowest K-type (τ, W) of π_λ^G. In [7] Harish-Chandra showed that Φ_0 (hence $tr(\Phi_0)$) is a tempered function for the definition of Harish-Chandra; for another proof we refer the reader to [22, 8.5.1]. In [8], we find a proof that the tempered functions on G restricted to H are tempered functions. A tempered function is called a *cusp form* if the integral along the unipotent radical of any proper parabolic subgroup of G of any left translate of the function is equal to zero [22, 7.2.2]. Let $r_n : H^2(G,\tau) \to L^2(H \times_{\tau_n} (\mathfrak{p}/\mathfrak{p}')^{(n)} \otimes W)$ be as in Example 3.1. The notation $r_n(\Phi_0^\star)$ means the family of functions $r_n(K_\lambda(\cdot, e)^\star w) = r_n(\Phi_0^\star(\cdot)w)$, $w \in W$. The purpose of this section is to show the following Theorem.

Theorem 4.1. *Let π_λ^G be a discrete series for G, let Φ_0 be its lowest K-type spherical function. Then, $r_n(\Phi_0^\star)$ is a cusp form on H, for every $n = 0, 1, \ldots,$ if and only if π_λ^G restricted to H is discretely decomposable. In turn, this is equivalent to: for each $y \in G$, the restriction of $K_\lambda(\cdot, y)$ to H is a cusp form.*

Remark 4.2. In [12, Fact 4.3], for a symmetric pair (G, H), T. Kobayashi shows a necessary and sufficient condition so that π_λ restricted to H is an admissible representation. Moreover, in [16, Theorem 2.8] it is shown that, for a symmetric pair (G, H), π_λ restricted to H is algebraically discretely decomposable if and only if $res_H(\pi_\lambda)$ is H-admissible. Whence, coupling the previously quoted result of T. Kobayashi with Theorem 4.1 and Proposition 4.3, we may state: for a symmetric pair (G, H), the restriction of π_λ to H is admissible if and only if, for every $n = 0, 1, \ldots, r_n(\Phi_0^\star)$ is a cusp form if and only if $\Phi_0^\star$ is left $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$-finite.

For a symmetric pair (G, H), another criteria for H-admissibility has been obtained by [8]. For this, they write $H = K_0 \times H_1$, with K_0 a compact subgroup and H_1 a noncompact subgroup. Let $H^{\sigma\theta}$ be the dual subgroup to H. Then $H^{\sigma\theta} = K_0 \times H_2$. Let M_i denote the centralizer in $L \cap H_1 = L \cap H_2$ of respective Cartan subspaces. Harris, He, and Olafsson show that, if $M_1 M_2 = L \cap H_1$, then the representation π_λ restricted to H is admissible and $\dim Hom_H(\pi_\mu^H, res_H(\pi_\lambda))$ is computed via a formula that involves r_n, the Harish-Chandra character, the lowest L-type for π_μ^H, and the limit of a sequence.

For any pair (G, H) and π_λ that satisfies Condition C, in [5], it is shown that π_λ is a H-admissible representation, and a "Blattner-Kostant"-type formula for $\dim Hom_H(\pi_\mu^H, res_H(\pi_\lambda))$. For symmetric pairs, condition C is equivalent to H-admissibility.

In order to show 4.1 we first show Proposition 4.3.

Proposition 4.3. *We let G, H, π_λ^G, K_λ, Φ_0 be as usual. Then, π_λ^G restricted to H is a discretely decomposable representation for H if and only if the function $y \mapsto K_\lambda(y, e)^\star = \check{\Phi}_0(y)$ is left $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$-finite.*

Proof. For the direct implication, we proceed as follows: the hypothesis that π_λ is discretely decomposable allows us to write V_λ as the Hilbert sum of the H-isotypic components; hence, there exists a family $(P_i)_{i \in \mathbb{Z}_{\geq 0}}$ of orthogonal projectors on V_λ that are H-equivariant, so that we have the orthogonal direct sum decomposition $V_\lambda = \oplus_i P_i(V_\lambda)$, and, for every i, $P_i(V_\lambda)$ is equal to the isotypic component of an irreducible H-module. Next, we fix $w \in W$, we recall that the function $y \mapsto K_\lambda(y, e)^\star(w) =: k_w(y)$ is a K-finite element of $H^2(G,\tau)$ and k_w belongs to $H^2(G,\tau)[W] \equiv W$. After that, we decompose $H^2(G,\tau)[W]$ as a sum of irreducible L-submodules, we write $k_w = f_1 + \cdots + f_s$, where f_j is such that the linear subspace spanned by $\pi_\lambda(L) f_j$ is an irreducible L-submodule of $H^2(G,\tau)[W]$. To continue, we set $f_1 := f_j$. A result of Harish-Chandra [7, Lemma 70] states that an irreducible representation of L is the L-type of at most finitely many discrete series representations for H. Thus, the representation of L in the subspace spanned by $\pi(L) f_1$ is an L-type of at most finitely many discrete series representations for H. Therefore, $P_i(f_1) = 0$ for all but finitely many indices i. Let us say $P_i(f_1) \neq 0$ for $i = 1, \ldots, N$. Since f_1 is a K-finite vector in V_λ, we have that f_1 is a smooth vector for π_λ; therefore,

$P_i(f_1)$ are H-smooth vectors in $P_i(V_\lambda)$. Owing to the fact that $P_i(V_\lambda)$ is an isotypic representation, we have that $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$ applied to $P_i(f_1)$ is contained in the one-dimensional vector subspace spanned by $P_i(f_1)$. Hence, f_1 is a finite sum of $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$-finite vectors. Thus, k_w is a $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$-finite vector. W is a finite dimensional vector space, let us conclude that the map $K_\lambda(\cdot,e)^\star = \Phi_0(\cdot)^\star$ is left $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$-finite.

For the converse statement, owing to our hypothesis, for each $w \in W$, we have that k_w is $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$-finite element of $H^2(G,\tau)_{K\text{-fin}}$. A result of Harish-Chandra [22, Corollary 3.4.7 and Theorem 4.2.1] asserts that a $\mathcal{U}(\mathfrak{h})$-finitely generated, $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$-finite, $(\mathfrak{h}, L)$-module has a finite composition series, whence we conclude that the representation of $\mathcal{U}(\mathfrak{h})$ in $\mathcal{U}(\mathfrak{h})k_w$ has a finite composition series. Thus, $H^2(G,\tau)_{K\text{-fin}}$ contains an irreducible sub-representation for $\mathcal{U}(\mathfrak{h})$. Whence [11, Lemma 1.5] yields that

$(H^2(G,\tau)_\lambda)_{K\text{-fin}}$ is infinitesimally discretely decomposable as $\mathfrak{h}$-module. Finally, since π_λ is unitary, in [13, Theorem 4.2.6], we find a proof that an algebraically (infinitesimally) discretely decomposable unitary representation is Hilbert discrete decomposable, hence π_λ is discretely decomposable. □

Corollary 4.4. *We assume (π_λ, V_λ) is a Hilbert discretely decomposable representation for H. Then, (π_λ, V_λ) is algebraically discretely decomposable. That is, $(V_\lambda)_{K\text{-fin}}$ can be expressed as a direct sum of $\mathcal{U}(\mathfrak{h})$-irreducible subspaces.*

The Corollary follows because, as in the proof of the direct implication, we obtain each $k_w, w \in W$ is $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$-finite, whence the proof for the converse statement yields that V_λ is algebraically discretely decomposable.

Now, we are ready to show Theorem 4.1.

Proof of Theorem 4.1. For the direct implication, the hypothesis is π_λ restricted to H is discretely decomposable. Thus, Proposition 4.3 yields that $K_\lambda(\cdot,e)^\star w$ is $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$-finite. Since [21] r_n is a continuous intertwining map for H and $K_\lambda(\cdot,e)^\star w$ is a tempered function, a result of [8] previously quoted let us conclude: $r_n(K_\lambda(\cdot,e)^\star w)$ is a tempered, $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$-finite function on H. A result of Harish-Chandra, [7][22, 7.2.2] implies $r_n(K_\lambda(\cdot,e)^\star w)$ is a cusp form. For the converse statement, the results of Harish-Chandra assert that $L^2(H \times_\cdot (\mathfrak{p}/\mathfrak{p}')^{(n)} \otimes W)_{\mathrm{disc}}$ is a finite sum of discrete series representations for H and that the space of cusp forms in $L^2(H \times_\cdot (\mathfrak{p}/\mathfrak{p}')^{(n)} \otimes W)$ is contained in $L^2(H \times_\cdot (\mathfrak{p}/\mathfrak{p}')^{(n)} \otimes W)_{\mathrm{disc}}$. Therefore, owing to the hypothesis, for every n, $r_n(K_\lambda(\cdot,e)^\star w)$ belongs to $L^2(H \times_\cdot (\mathfrak{p}/\mathfrak{p}')^{(n)} \otimes W)_{\mathrm{disc}}$. The L^2-continuity of r_n, yields that $r_n(closure(\pi_\lambda(H)k_w))$ is contained in a finite sum of discrete representations. Whence $\oplus_n r_n$ maps continuously the closure of $\pi_\lambda(H)k_w$ into a discrete Hilbert sum of irreducible representations. Besides, the map $\oplus_n r_n$ is injective (the elements of $H^2(G,\tau)$ are real analytic functions). Hence, the closure of $\pi_\lambda(H)k_w$ is a discrete Hilbert sum of discrete series representations. We now proceed as in the direct proof of Proposition 4.3 and obtain k_w is a left $\mathfrak{z}(\mathcal{U}(\mathfrak{h}))$-finite function. Whence Proposition 4.3 let us conclude that $res_H(\pi_\lambda)$ is Hilbert discretely decomposable. The second equivalence follows from a simple computation. □

Remark 4.5. A simple application of Theorem 4.1 yields that the tensor product representation of G in $H^2(G,\tau) \boxtimes H^2(G,\tau)^\star$ is never discretely decomposable, because the lowest K-type trace spherical function for this particular tensor product is $\phi_0(x)\overline{\phi_0(y)}$; hence, restricted to G, it is not a cusp form.

5. Projectors via reproducing kernel

Let $G, H, (\tau, W)$, $H^2(G,\tau)$ be as usual. Let (π^H_μ, V^H_μ) denote an irreducible square integrable representation for H. By definition, the *isotypic component*, $H^2(G,\tau)[V^H_\mu]$, for V^H_μ is the closure of the sum of the totality of closed H-invariant subspaces in $H^2(G,\tau)$ such that the resulting representation of H on the subspace is equivalent to (π^H_μ, V^H_μ). Since a closed subspace of a reproducing kernel space is a reproducing kernel space and $H^2(G,\tau)$ is a reproducing kernel space, we obtain that $H^2(G,\tau)[V^H_\mu]$ is a reproducing kernel space. Whence the orthogonal projector $P_{\lambda,\mu}$ onto the isotypic component $H^2(G,\tau)[V^H_\mu]$ is an integral map represented by a matrix kernel $K_{\lambda,\mu}$. Next, under the hypothesis that π_λ is H-admissible, we present an expression for the matrix kernel $K_{\lambda,\mu}$. We are quite convinced that the formula is true under a more general hypothesis. The proposed formula is as follows.

Proposition 5.1. *We assume that the restriction to H of π_λ is an H-admissible representation. Then, $P_{\lambda,\mu}$ is equal to the integral operator given by the kernel*

$$K_{\lambda,\mu}(y,x) = d_\mu \Theta_{\pi^H_\mu}(h \mapsto K_\lambda(h^{-1}y,x)) = d_\mu \Theta_{(\pi^H_\mu)^\star}(h \mapsto K_\lambda(hy,x)).$$

The expression on the right is well defined because [8] has shown that the restriction of tempered functions to H yields tempered functions. For H compact, the verification of 5.1 is straightforward.

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