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Distribution of martingales with bounded square functions

La distribution des martingales dont les fonctions carrées sont bornées

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A B S T R A C T

We study the terminate distribution of a martingale whose square function is bounded. We obtain sharp estimates for the exponential and p-moments, as well as for the distribution function itself. The proofs are based on the elaboration of the Burkholder method and on the investigation of certain locally concave functions.

R É S U M É

Nous étudions la distribution terminée d'une martingale dont la fonction carrée est bornée. Nous obtenons les estimations les meilleures possibles pour les p-moments et les moments exponentiels. Un développement de la méthode de Burkholder et les études sur des fonctions localement infléchies servent de base aux démonstrations.

1. Square functions and BMO

Let (Ω, Σ, P) be the standard probability space, let $\mathcal{F} = \{\mathcal{F}_n\}_{n\geqslant 0}$ be a discrete time filtration of finite algebras on it. Assume $\mathcal{F}_0 = \{\Omega, \varnothing\}$. Consider a real-valued martingale $\varphi = \{\varphi_n\}_n$ adapted to $\mathcal{F}$ and define its square function

$$S\varphi = \Big(\sum_{n=0}^{\infty}(\varphi_{n+1} - \varphi_n)^2\Big)^{\frac{1}{2}}.$$

How large can φ be if $S\varphi$ is uniformly bounded? In the case where $\mathcal{F}$ is a uniform dyadic filtration, the famous Chang–Wilson–Wolff inequality (established in [2]) says that the distribution of φ is sub-Gaussian:

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$$P(\varphi - \varphi_0 \geqslant \lambda) \leqslant \mathrm{e}^{-\frac{\lambda^2}{2\|S\varphi\|_{L_\infty}^2}}.$$

In a recent paper [5], Ivanisvili and Treil generalized this result to the case where the filtration $\mathcal{F}$ has bounded distortion α, which means that each atom in $\mathcal{F}_n$ has at least α times the mass of its parential atom. In this case,

$$P(\varphi - \varphi_0 \geqslant \lambda) \leqslant \mathrm{e}^{-\frac{\alpha\lambda^2}{\|S\varphi\|_{L_\infty}^2}}.$$

We see that the distribution function of an arbitrary martingale φ whose square function is bounded may no longer be sub-Gaussian. As we will see, this is indeed the case. We start from a simple observation that $\varphi \in \mathrm{BMO}^{\mathrm{m}}$. The latter space called the space of martingales of bounded mean oscillation can be defined as follows

$$\|\varphi\|_{\mathrm{BMO}^{\mathrm{m}}}^2 = \sup\Big(\Big\{\mathbb{E}\left((\varphi_\infty - \varphi_\tau)^2 \mid \mathcal{F}_\tau\right)\,\Big|\, \tau \text{ is a stopping time}\Big\}\Big).$$

By φ_∞ we denote the limit value of φ. In fact, a simple Hilbert-space computation shows that $\|\varphi\|_{\mathrm{BMO}^{\mathrm{m}}} \leqslant \|S\varphi\|_{L_\infty}$. A more delicate estimate is true.

Theorem 1.1. *The inequality $\|\varphi_\infty^*\|_{\mathrm{BMO}([0,1])} \leqslant \|S\varphi\|_{L_\infty}$ holds true and is sharp.*

By ξ^* we denote the non-decreasing rearrangement (the inverse function to the distribution function of ξ) of a variable ξ:

$$\xi^*(t) = \inf\{\alpha \mid P(\xi > \alpha) \leqslant t\}.$$

The BMO space on the unit interval is defined by formula

$$\|\psi\|_{\mathrm{BMO}([0,1])}^2 = \sup_{\substack{J \text{ is a} \\ \text{subinterval of } [0,1]}} \Big\{\frac{1}{|J|}\int_J \Big(\psi(x) - \frac{1}{|J|}\int_J \psi\Big)^2 \mathrm{d}x\Big\}.$$

Even though Theorem 1.1 says that there is a certain relationship between martingales φ whose square function is bounded and functions ψ on the unit interval that belong to the BMO space, we warn the reader against the identification of these classes of objects, which have different nature and origin. For the sake of clarity, here and in what follows let φ denote a martingale, and let ψ denote a function on the unit interval.

We also note that the estimate $\|\varphi_\infty^*\|_{\mathrm{BMO}([0,1])} \leqslant \|\varphi\|_{\mathrm{BMO}^{\mathrm{m}}}$ is not true in general for discrete time filtrations (i.e. when a martingale is allowed to have jumps).

2. Bellman functions

Though the inequality in Theorem 1.1 is sharp, the set of distributions of functions in the unit ball of BMO([0, 1]) is richer than the set of distributions of φ_∞ such that $S\varphi \leqslant 1$. We want to measure the difference between the two sets. Let f be a measurable non-negative function on the line. We are going to maximize $\mathbb{E}\, f(\varphi_\infty)$ with the condition $\|S\varphi\|_{L_\infty} \leqslant 1$ imposed on φ. For example, the choice $f(t) = |t|^p$ leads to the computation of the optimal value c_p such that

$$\|\varphi_\infty - \varphi_0\|_{L_p} \leqslant c_p\|S\varphi\|_{L_\infty} \tag{1}$$

holds true. The function $f(t) = \mathrm{e}^{\mu t}$ allows us to estimate $\mathbb{E}\,\mathrm{e}^{\mu(\varphi_\infty - \varphi_0)}$, and the choice $f(t) = \chi_{[\mu,\infty)}(t)$ leads to classical weak-type or tail estimates.

To solve the posed extremal problem, consider the Bellman function $\boldsymbol{B}$:

$$\boldsymbol{B}(x, y, z) = \sup\Big(\Big\{\mathbb{E}\, H_f(\varphi, S\varphi^2 + z^2)\,\Big|\, \mathbb{E}\,\varphi = x, \mathbb{E}\,\varphi_\infty^2 = y\Big\}\Big), \tag{2}$$

where

$$H_f(s,t) = \begin{cases} -\infty, & t \notin [0,1]; \\ f(s), & t \in [0,1]. \end{cases}$$

The specification of the quantities $\mathbb{E}\,\varphi$ and $\mathbb{E}\,\varphi^2$ allows us to track the dynamics of the martingale. The idea of using Bellman functions in the context of Chang–Wilson–Wolf-type inequalities is quite old, see, e.g., [9].

Lemma 2.1.

(i) The function $\boldsymbol{B}$ *is non-negative on the set*

$$\Omega = \left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 \leqslant y \leqslant 1 - z^2 + x^2\right\}$$

and equals $-\infty$ *outside it.*

(ii) The function $\boldsymbol{B}$ *satisfies the boundary condition* $\boldsymbol{B}(x, x^2, z) = f(x)$ *when* $z \in [0, 1]$.

(iii) The function $\boldsymbol{B}$ *satisfies the inequality*

$$\boldsymbol{B}(x, y, z) \geqslant \sum_{j=1}^{n} \alpha_j \boldsymbol{B}(x_j, y_j, z_j),$$

whenever

$$\begin{gathered} \sum_{j=1}^{n} \alpha_j = 1,\ \alpha_j \in [0, 1]; \\ \sum_{j=1}^{n} \alpha_j x_j = x; \quad \sum_{j=1}^{n} \alpha_j y_j = y; \\ \forall j \quad z_j^2 = z^2 + (x_j - x)^2. \end{gathered} \tag{3}$$

(iv) Finally, the function $\boldsymbol{B}$ *is pointwise minimal among the functions having these three properties.*

This lemma is standard, such type lemmas form the core of the Burkholder method (see [1], [6]). Consider the Bellman function $b_\varepsilon \colon \omega_\varepsilon \to \mathbb{R}$,

$$\begin{gathered} b_\varepsilon(x, y) = \sup\left\{ \int_0^1 f(\psi) \,\middle|\, \int_0^1 \psi = x,\ \int_0^1 \psi^2 = y,\ \|\psi\|_{\mathrm{BMO}([0,1])} \leqslant \varepsilon \right\}, \\ \omega_\varepsilon = \{(x, y) \in \mathbb{R}^2 \mid x^2 \leqslant y \leqslant x^2 + \varepsilon^2\}. \end{gathered} \tag{4}$$

By the main theorem of [10], the function b_ε may be described as the minimal function among locally concave functions on the domain ω_ε that satisfy the boundary condition $b_\varepsilon(x, x^2) = f(x)$. Note that the function b_ε may be computed explicitly for any sufficiently regular boundary data f, see [3] and [4]. We start with a simple observation that is the main result of this note.

Lemma 2.2. *For any triple* $(x, y, z) \in \Omega$*, the inequality* $\boldsymbol{B}(x, y, z) \leqslant b_{\sqrt{1-z^2}}(x, y)$ *is true.*

Proof. Note that if the point (x, y, z) is split into the points (x_j, y_j, z_j) inside Ω according to the rule (3), then the convex hull of the points (x_j, y_j) lie in the parabolic strip $\omega_{\sqrt{1-z^2}}$. In fact, all these points lie below the tangent at $(x, x^2 + 1 - z^2)$ to the upper boundary of the said domain. To prove this, take some points $(x_j, y_j, z_j) \in \Omega$, $j = 1, 2, \ldots, n$, and (x, y, z) that satisfy (3). Without loss of generality, we may assume $x = 0$. Then, for any j,

$$y_j \leqslant 1 - z_j^2 + x_j^2,$$

simply because $(x_j, y_j, z_j) \in \Omega$. Therefore, by the last rule in (3) and the assumption $x = 0$,

$$y_j \leqslant 1 - z_j^2 + x_j^2 = 1 - z^2,$$

which exactly means that (x_j, y_j) lies below the tangent to the parabola $y = x^2 + 1 - z^2$ at the point $(0, 1 - z^2)$.

With this principle at hand, we write the chain of inequalities:

$$b_{\sqrt{1-z^2}}(x, y) \geqslant \sum_{j=1}^{n} \alpha_j b_{\sqrt{1-z^2}}(x_+, y_+) \geqslant \sum_{j=1}^{n} \alpha_j b_{\sqrt{1-z_j^2}}(x_+, y_+). \tag{5}$$

The first inequality in the chain follows from the local concavity of b_ε and the geometric statement proved in the previous paragraph. The second inequality is a consequence of the fact that b_ε is an increasing function of ε (we maximize over a larger set in (4)).

So, the function $(x, y, z) \mapsto b_{\sqrt{1-z^2}}(x, y)$ satisfies the first three requirements of Lemma 2.1. Since $\boldsymbol{B}$ is the minimal function (by the fourth statement in Lemma 2.1), $\boldsymbol{B}(x, y, z) \leqslant b_{\sqrt{1-z^2}}(x, y)$. $\square$

Proof of Theorem 1.1. We recall the notion of an ω_1-martingale introduced in [10]. An $\mathbb{R}^2$-valued martingale $M = \{M_n\}_{n\in\mathbb{N}}$ adapted to $\mathcal{F}$ is called an ω_1-martingale provided it satisfies two properties. The first one is that M has an L_1 limit M_∞, i.e. $M_n \to M_\infty$ in L_1 as $n \to \infty$, which attains values inside the lower boundary of ω_1, that is, the parabola $\{(x, x^2) \mid x \in \mathbb{R}\}$ only. The second one is that M is prohibited to jump over the upper boundary of ω_1, i.e., for any atom $a \in \mathcal{F}_n$, the convex hull of the set $\{M_{n+1}(z) \mid z \in a\}$ belongs to ω_1 entirely.

Assume that $S\varphi \leqslant 1$. Let us show that in this case the $\mathbb{R}^2$-valued martingale $M_n = (\varphi_n, \mathbb{E}(\varphi^2 \mid \mathcal{F}_n))$ is an ω_1-martingale. The first property may be justified by the martingale convergence theorem since $\varphi \in L_2$. To verify the second property, we consider an $\mathbb{R}^3$-valued process $\mu_n = (\varphi_n, \mathbb{E}(\varphi^2 \mid \mathcal{F}_n), S\varphi_n)$, here

$$S\varphi_n = \Big(\sum_{m<n} (\varphi_{m+1} - \varphi_m)^2\Big)^{\frac{1}{2}}.$$

Let $a \in \mathcal{F}$ be an atom. Then, the points $(x, y, z) = \mu_n(a)$ and $(x_j, y_j, z_j) = \mu_{n+1}(a_j)$, where the a_j are all the children of a, satisfy (3). Thus, by the geometric observation in the proof of Lemma 2.2, the convex hull of the points $M_{n+1}(a_j)$ lies inside ω_1. Therefore, M is an ω_1 martingale.

The random variable M_∞ is vector-valued. Let M^1_∞ be its first coordinate. We recall Theorem 3.4 from [10], which says that $\|(M^1_\infty)^*\|_{\mathrm{BMO}([0,1])} \leqslant 1$ whenever M is an ω_1-martingale. We notice that M^1_∞ coincides with φ_∞ and finally obtain the inequality

$$\|\varphi^*_\infty\|_{\mathrm{BMO}([0,1])} \leqslant \|S\varphi\|_{L_\infty}.$$

The sharpness of this inequality is obtained by considering the martingale φ such that $\varphi_0 = 0$ and φ_1 is ± 1 with equal probability. □

It appears that for some choices f, the inequality of Lemma 2.2 turns into equality.

Theorem 2.3. *Assume that f''' either does not change its sign on the whole line, or changes it once from $+$ to $-$. Then, $\boldsymbol{B}(x, y, z) = b_{\sqrt{1-z^2}}(x, y)$ for all $(x, y, z) \in \Omega$.*

The proof of this theorem is more complicated. We will present neither the details nor the main idea of the construction, but rather give a plan of the proof. In view of Lemma 2.2, we need to construct a martingale φ such that

$$\varphi_0 = x,\ \mathbb{E}\,\varphi^2_\infty = y,\ S\varphi \leqslant \sqrt{1-z^2}, \quad \mathbb{E}\, f(\varphi_\infty) = b_{\sqrt{1-z^2}}(x, y). \tag{6}$$

To do this, we recall Theorem 2.21 in [10], which says that

$$b_\varepsilon(x, y) = \sup\Big\{\mathbb{E}\, f(M^1_\infty) \,\Big|\, M_0 = (x, y), \quad M \text{ is an } \omega_\varepsilon\text{-martingale}\Big\}. \tag{7}$$

In the proof of Theorem 1.1, we showed that $(\varphi_n, \mathbb{E}(\varphi^2 \mid \mathcal{F}_n))$ is an ω_1-martingale provided $S\varphi \leqslant 1$. However, not every ω_1-martingale may be represented in this form (in the proof of Lemma 2.2, we show that the convex hull of all the points (x_j, y_j) lies under a certain tangent to the upper parabola; this might not be the case for an arbitrary ω_1-martingale). We use the theory from [4] (in fact, here we may use a simpler version [3]), which, in particular, describes the optimal martingales in formula (7). In the case where f''' either does not change its sign on the whole line, or changes it once from $+$ to $-$, these optimal martingales split either along the tangent to the upper parabola, or along a chord connecting two points on the lower boundary. In both cases, the points (x_j, y_j) lie below the corresponding tangent, and one is able to construct φ such that

$$M_n = (\varphi_n, \mathbb{E}(\varphi^2 \mid \mathcal{F}_n)), \quad S\varphi \leqslant \sqrt{1-z^2},$$

where M is the optimal martingale in (7). This martingale φ will satisfy (6).

Corollary 2.4. *The optimal constant c_p in the inequality (1) equals 1 when $1 \leqslant p \leqslant 2$.*

The corresponding Bellman function b_ε was computed in [8].

Corollary 2.5. *The optimal constant $C(\varepsilon)$ in the inequality*

$$\mathbb{E}\, \mathrm{e}^{\varphi-\varphi_0} \leqslant C(\varepsilon), \quad S\varphi \leqslant \varepsilon,$$

equals $\frac{\mathrm{e}^{-\varepsilon}}{1-\varepsilon}$.

This result follows from Theorem 2.3 applied to the case $f(t)=\mathrm{e}^t$. In fact, the corresponding Bellman function b_ε was computed in [7]:

$$b_\varepsilon(x,y)=\frac{1-\sqrt{x^2+\varepsilon^2-y}}{1-\varepsilon}\mathrm{e}^{x-\varepsilon+\sqrt{x^2+\varepsilon^2-y}},\quad f(t)=\mathrm{e}^t.$$

On the other hand, sometimes the inequality in Lemma 2.2 is strict on a part of Ω. An example is given by $f(t)=\chi_{[0,\infty)}(t)$. In particular, it shows that the set of distributions of functions in the unit ball of $\mathrm{BMO}([0,1])$ is larger than the set of distributions of φ_∞ such that $S\varphi\leqslant 1$.

The function b_ε for the case $f(t)=\chi_{[0,\infty)}(t)$ was computed in [11]. The domain ω_ε is split into four parts:

$$D_1^\varepsilon=\{(x,y)\in\omega_\varepsilon \mid y\geqslant 2\varepsilon x,\ x\geqslant\varepsilon\}\cup\{(x,y)\in\omega_\varepsilon \mid y\leqslant 2\varepsilon x\};$$
$$D_2^\varepsilon=\{(x,y)\in\omega_\varepsilon \mid |x|\leqslant\varepsilon,\ y\geqslant 2\varepsilon|x|\};$$
$$D_3^\varepsilon=\{(x,y)\in\omega_\varepsilon \mid y\leqslant -2\varepsilon x\};$$
$$D_4^\varepsilon=\{(x,y)\in\omega_\varepsilon \mid x\leqslant -\varepsilon,\ y\geqslant -2\varepsilon x\},$$

and the function b_ε is given by cases:

$$b_\varepsilon(x,y)=\begin{cases}1, & (x,y)\in D_1^\varepsilon;\\ 1-\frac{y-2\varepsilon x}{8\varepsilon^2}, & (x,y)\in D_2^\varepsilon;\\ 1-\frac{x^2}{y}, & (x,y)\in D_3^\varepsilon;\\ \frac{e}{2}\left(1-\sqrt{1-\frac{y-x^2}{\varepsilon^2}}\right)\mathrm{e}^{\frac{x}{\varepsilon}+\sqrt{1-\frac{y-x^2}{\varepsilon^2}}}, & (x,y)\in D_4^\varepsilon.\end{cases}$$

Theorem 2.6. *Let $f(t)=\chi_{[0,\infty)}(t)$. Whenever $(x,y)\in D_j^{\sqrt{1-z^2}}$ and $j=1,3,4$, we have $\boldsymbol{B}(x,y,z)=b_{\sqrt{1-z^2}}(x,y)$. However, for some points $(x,y)\in\operatorname{int}D_2^{\sqrt{1-z^2}}$ the strict inequality $\boldsymbol{B}(x,y,z)<b_{\sqrt{1-z^2}}(x,y)$ holds.*

We caution the reader that this result is more interesting for the geometry of Bellman functions itself rather than probabilistic inequalities. In fact, after some computations, we can see that though $\boldsymbol{B}(x,y,z)\neq b_{\sqrt{1-z^2}}(x,y)$, the optimal constants c and d in the inequalities

$$P(\varphi_\infty-\varphi_0\geqslant\lambda)\leqslant c\,\mathrm{e}^{-\frac{\lambda}{\|S\varphi\|_{L^\infty}}}$$

and

$$P(\psi-\psi_0\geqslant\lambda)\leqslant d\,\mathrm{e}^{-\frac{\lambda}{\|\psi\|_{\mathrm{BMO}([0,1])}}}$$

are both equal to $\frac{e}{2}$.

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