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On left-invariant Einstein metrics that are not geodesic orbit

Sur les métriques d'Einstein invariantes à gauche, qui ne sont pas à orbites géodésiques

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A B S T R A C T

In this article, we prove that compact simple Lie groups SO(n) ($n > 12$) admit at least two left-invariant Einstein metrics that are not geodesic orbit, which gives a positive answer to a problem recently posed by Nikonorov.

R É S U M É

Dans cette Note, nous démontrons que les groupes de Lie simples, compacts, SO(n) ($n > 12$) admettent au moins deux métriques d'Einstein invariantes à gauche, dont des géodésiques maximales ne sont pas des orbites de sous-groupes à un paramètre du groupe d'isométries complet. Ceci répond par l'affirmative à une question récemment posée par Nikonorov.

1. Introduction

Recall that a Riemannian metric on a connected manifold M is said to be a geodesic orbit metric if any maximal geodesic of the metric is the orbit of a one-parameter subgroup of the full group of isometries (in this case, the Riemannian manifold is called a geodesic orbit space). It is well known that any naturally reductive metric must be geodesic orbit, but the converse is not true.

In [1], A. Arvanitoyeorgos, K. Mori, and Y. Sakane constructed non-naturally reductive Einstein metrics on compact Lie groups SO(n) ($n \geq 11$), Sp(n) ($n \geq 3$), E_6, E_7, and E_8. In [3], Z. Chen and K. Liang found three naturally reductive and one non-naturally reductive Einstein metric on the compact Lie group F_4, and I. Chrysikos and Y. Sakane obtained lots of non-naturally reductive Einstein metrics on exceptional Lie groups [4]. Moreover, based on the classification of standard homogeneous Einstein manifolds, Z. Yan and S. Deng found many non-naturally reductive Einstein metrics on compact simple Lie groups [8]. Besides, the authors constructed non-naturally reductive Einstein–Randers metrics on Sp(n) [7].

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However, there are only few examples of left-invariant Einstein metrics that are not geodesic orbit. In [6], Y. Nikonorov proved that there exists a left-invariant Einstein metric on compact simple Lie group G_2 that is not a geodesic orbit metric. The following problem is posed in [6].

Problem 1.1. Is there any other compact simple Lie group admitting a left-invariant Einstein metric that is not geodesic orbit?

In [2], H. Chen, Z. Chen and S. Deng obtained some left-invariant and not geodesic-orbit Einstein metrics on compact simple Lie groups that are arising from three locally symmetric spaces. They proved that the compact simple Lie groups $\mathrm{SU}(n)$ for $n \geq 6$, $\mathrm{SO}(n)$ for $n \geq 7$, $\mathrm{Sp}(n)$ for $n \geq 3$, E_6, E_7, E_8, and F_4 admit left-invariant Einstein metrics that are not geodesic orbit.

In this short article, we construct new metrics that are distinct from the metrics with the same property obtained in [2], and we prove the following.

Theorem 1.1. *The compact simple Lie groups* $\mathrm{SO}(n)$ $(n > 12)$ *admits at least two left-invariant Einstein metrics, which are not geodesic orbit.*

2. Preliminaries

In this section, we will recall some basic facts and the Ricci tensor for reductive homogeneous spaces.

Lemma 2.1. *([5]) Let M be a homogeneous Riemannian manifold and G the identity component of the full group of isometries. Write $M = G/H$, where H is the isotropic subgroup of G at $x \in M$, and suppose the Lie algebra of G has a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, where $\mathfrak{g} = Lie(G)$, $\mathfrak{h} = Lie(H)$, and $\mathfrak{m}$ is the orthogonal complement subspace of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to an $Ad(H)$-invariant inner product on $\mathfrak{g}$. Then M is a geodesic orbit space if and only if, for any $X \in \mathfrak{m}$, there exists $Z \in \mathfrak{h}$ such that $([X+Z, Y]_{\mathfrak{m}}, X) = 0$ for all $Y \in \mathfrak{m}$.*

Let G be a compact simple Lie group, consider the following inner product on the Lie algebra g,

$$\langle , \rangle = u_1(-B)|_{\mathfrak{p}_1} + u_2(-B)|_{\mathfrak{p}_2} + \cdots + u_s(-B)|_{\mathfrak{p}_s}, \tag{2.1}$$

where B is the Killing form of $\mathfrak{g}$, $u_1, \ldots, u_s$ are pairwise distinct, and $u_j > 0$, $j = 1, 2, \ldots, s$. A Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is called adapted for (2.1), if $\mathfrak{k}$ is the direct sum of its ideals $\mathfrak{k} \cap \mathfrak{p}_i$, $i = 1, 2, \cdots, s$, (some of these ideals could be trivial) and the $B-$orthogonal complement to $\mathfrak{k} \cap \mathfrak{p}_i$ in $\mathfrak{p}_i$ is $ad(\mathfrak{k})$-invariant for every $i = 1, 2, \ldots, s$. It is clear that there is a maximal by-inclusion-adapted subalgebra among all subalgebras adapted for (2.1).

Now, we recall a sufficient and necessary condition for a left-invariant Riemannian metric on a compact simple Lie group to be a geodesic orbit metric.

Theorem 2.1. *([6]) The inner product (2.1) generates a geodesic orbit left-invariant Riemannian metric on compact simple Lie group G if and only if there is a maximal by-inclusion-adapted Lie subalgebra $\mathfrak{k}$ such that, for any $X \in \mathfrak{g}$, there exists $W \in \mathfrak{k}$ such that, for any $Y \in \mathfrak{g}$, the equality $([X+W, Y], X) = 0$ holds or, equivalently, $[A(X), X+W] = 0$, where $A : \mathfrak{g} \to \mathfrak{g}$ is a metric endomorphism.*

The following theorem will be useful in the proof of our main theorem.

Theorem 2.2. *([6]) Suppose that the inner product (2.1) generates a geodesic orbit left-invariant Riemannian metric on compact simple Lie group G, $\mathfrak{k}_i = \mathfrak{k} \cap \mathfrak{p}_i$, and that $\mathfrak{n}_i$ is the B-orthogonal complement to $\mathfrak{k}_i$ in $\mathfrak{p}_i$. Then there is a maximal by-inclusion-adapted Lie subalgebra $\mathfrak{k}$ such that one of the following assertions holds:*

(1) *there is no more than one index i such that $\mathfrak{k}_i \neq \mathfrak{p}_i$; in this case (2.1) generates a naturally reductive left-invariant Riemannian metric on G;*
(2) *$rank(\mathfrak{k}) \geq 2$, and $[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_i \oplus \mathfrak{n}_j$ for $i \neq j$;*
(3) *there is only one non-zero $\mathfrak{k}_i = \mathfrak{k} \cap \mathfrak{p}_i$, hence, $\mathfrak{k}_i = \mathfrak{k}$; moreover, $rank(\mathfrak{k}) = 1$ and either $[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_i$ or $[\mathfrak{n}_i, \mathfrak{n}_j] \subset \mathfrak{n}_j$ for $i \neq j$.*

Next, we recall some definitions and fundamental results for a G-invariant Riemannian metric on a reductive homogeneous space, whose isotropy representation is decomposed into the sum of non-equivalent irreducible summands. Let G be a compact semisimple Lie group, K a connected closed subgroup of G, and let $\mathfrak{g}$ and $\mathfrak{k}$ be the corresponding Lie algebras. The Killing form B of $\mathfrak{g}$ is negative definite, so we can define an $Ad(G)$-invariant inner product B on $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a reductive decomposition of $\mathfrak{g}$ with respect to B, such that $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ and $\mathfrak{m} \cong T_0(G/K)$. We assume that $\mathfrak{m}$ admits a decomposition into mutually non-equivalent irreducible $Ad(K)$-modules as follows:

$$\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q. \tag{2.2}$$

Then any G-invariant metric on G/K can be expressed as

$$\langle , \rangle = x_1(-B)|_{\mathfrak{m}_1} + \cdots + x_q(-B)|_{\mathfrak{m}_q}, \tag{2.3}$$

for positive real numbers $(x_1, ..., x_q) \in \mathbb{R}^q_+$.

The Ricci tensor r of a G-invariant Riemannian metric on G/K is of the same form as (2.3), that is

$$r = y_1(-B)|_{\mathfrak{m}_1} + \cdots + y_q(-B)|_{\mathfrak{m}_q}, \tag{2.4}$$

for some real numbers $y_1, ..., y_q$.

Let e_α be a $(-B)$-orthonormal basis adapted to the decomposition of $\mathfrak{m}$, i.e. $e_\alpha \in \mathfrak{m}_i$ for some i, and $\alpha < \beta$ if $i < j$. We put $A^\gamma_{\alpha\beta} = B([e_\alpha, e_\beta], e_\gamma)$ such that $[e_\alpha, e_\beta]_\mathfrak{m} = \sum_\gamma A^\gamma_{\alpha\beta} e_\gamma$, and set $\begin{bmatrix} k \\ ij \end{bmatrix} = \sum (A^\gamma_{\alpha\beta})^2$, where the sum is taken over all indices α, β, γ with $e_\alpha \in \mathfrak{m}_i$, $e_\beta \in \mathfrak{m}_j$, $e_\gamma \in \mathfrak{m}_k$, and $[,]_\mathfrak{m}$ denotes the $\mathfrak{m}$-component. Then the positive numbers $\begin{bmatrix} k \\ ij \end{bmatrix}$ are independent of the B-orthonormal bases chosen for $\mathfrak{m}_i$, $\mathfrak{m}_j$, $\mathfrak{m}_k$, and $\begin{bmatrix} k \\ ij \end{bmatrix} = \begin{bmatrix} k \\ ji \end{bmatrix} = \begin{bmatrix} j \\ ki \end{bmatrix}$, because of the operation law of bracket and Killing form.

3. Non-geodesic orbit Einstein metrics on the compact lie groups SO(n)

For $G = \mathrm{SO}(k_1 + k_2 + k_3 + k_4)$, $K = \mathrm{diag}(\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3) \times \mathrm{SO}(k_4))$, we take into account the diffeomorphism:

$$G/e \cong (G \times \mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3) \times \mathrm{SO}(k_4))/\mathrm{diag}(\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3) \times \mathrm{SO}(k_4)),$$

where $G \times K$ acts on G by $(g, k)y = gyk^{-1}$. We denote $\mathfrak{so}(k_1)$ as $\mathfrak{m}_1$, $\mathfrak{so}(k_2)$ as $\mathfrak{m}_2$, $\mathfrak{so}(k_3)$ as $\mathfrak{m}_3$, $\mathfrak{so}(k_4)$ as $\mathfrak{m}_4$. We denote by $M(p, q)$ the set of all $p \times q$ matrices,

$$\mathfrak{m}_{12} = \{\begin{pmatrix} 0 & A_{12} & 0 & 0 \\ -A'_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} | A_{12} \in M(k_1, k_2)\}, \quad \mathfrak{m}_{13} = \{\begin{pmatrix} 0 & 0 & A_{13} & 0 \\ 0 & 0 & 0 & 0 \\ -A'_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} | A_{13} \in M(k_1, k_3)\},$$

$$\mathfrak{m}_{14} = \{\begin{pmatrix} 0 & 0 & 0 & A_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -A'_{14} & 0 & 0 & 0 \end{pmatrix} | A_{14} \in M(k_1, k_4)\}, \quad \mathfrak{m}_{23} = \{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & A_{23} & 0 \\ 0 & -A'_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} | A_{23} \in M(k_2, k_3)\},$$

$$\mathfrak{m}_{24} = \{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{24} \\ 0 & 0 & 0 & 0 \\ 0 & -A'_{24} & 0 & 0 \end{pmatrix} | A_{24} \in M(k_2, k_4)\}, \quad \mathfrak{m}_{34} = \{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{34} \\ 0 & 0 & -A'_{34} & 0 \end{pmatrix} | A_{34} \in M(k_3, k_4)\},$$

where A'_{ij} denotes the transposed matrix of the matrix $A_{ij}, 1 \le i, j \le 4$. Note that the action of $Ad(k)$ $(k \in K)$ on $\mathfrak{m}$ is given by

$$Ad(k) \begin{pmatrix} 0 & A_{12} & A_{13} & A_{14} \\ -A'_{12} & 0 & A_{23} & A_{24} \\ -A'_{13} & -A'_{23} & 0 & A_{34} \\ -A'_{14} & -A'_{24} & -A'_{34} & 0 \end{pmatrix} = \begin{pmatrix} 0 & h'_1 A_{12} h_2 & h'_1 A_{13} h_3 & h'_1 A_{14} h_4 \\ -h'_2 A'_{12} h_1 & 0 & h'_2 A_{23} h_3 & h'_2 A_{24} h_4 \\ -h'_3 A'_{13} h_1 & -h'_3 A'_{23} h_2 & 0 & h'_3 A_{34} h_4 \\ -h'_4 A'_{14} h_1 & -h'_4 A'_{24} h_2 & -h'_4 A'_{34} h_3 & 0 \end{pmatrix},$$

where $\begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \\ 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_4 \end{pmatrix} \in K$, hence the subspaces $\mathfrak{m}_{12}, \mathfrak{m}_{13}, \mathfrak{m}_{23}, \mathfrak{m}_{24}, \mathfrak{m}_{34}$ are irreducible $Ad(K)$-submodules.

We know that $\mathfrak{g}$ admits a decomposition into mutually non-equivalent irreducible $Ad(K)$-modules as follows:

$$\mathfrak{g} = \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3 + \mathfrak{m}_4 + \mathfrak{m}_{12} + \mathfrak{m}_{13} + \mathfrak{m}_{14} + \mathfrak{m}_{23} + \mathfrak{m}_{24} + \mathfrak{m}_{34} \tag{3.1}$$

and consider left-invariant metrics on G that are determined by the $Ad(\mathrm{SO}(k_1) \times \mathrm{SO}(k_2) \times \mathrm{SO}(k_3) \times \mathrm{SO}(k_4))$-invariant scalar products on $\mathfrak{so}(k_1 + k_2 + k_3 + k_4)$ given by

$$\begin{aligned}\langle , \rangle =& x_1(-B)|_{\mathfrak{so}(k_1)} + x_2(-B)|_{\mathfrak{so}(k_2)} + x_3(-B)|_{\mathfrak{so}(k_3)} + x_4(-B)|_{\mathfrak{so}(k_4)} + x_{12}(-B)|_{\mathfrak{m}_{12}} \\ &+ x_{13}(-B)|_{\mathfrak{m}_{13}} + x_{14}(-B)|_{\mathfrak{m}_{14}} + x_{23}(-B)|_{\mathfrak{m}_{23}} + x_{24}(-B)|_{\mathfrak{m}_{24}} + x_{34}(-B)|_{\mathfrak{m}_{34}}.\end{aligned} \tag{3.2}$$

Proposition 3.1. *The submodules in the decomposition (3.1) satisfy the following bracket relations:*

$$\begin{array}{llll}
[\mathfrak{m}_1, \mathfrak{m}_1] = \mathfrak{m}_1, & [\mathfrak{m}_2, \mathfrak{m}_2] = \mathfrak{m}_2, & [\mathfrak{m}_3, \mathfrak{m}_3] = \mathfrak{m}_3, & [\mathfrak{m}_4, \mathfrak{m}_4] = \mathfrak{m}_4, \\
[\mathfrak{m}_1, \mathfrak{m}_{12}] = \mathfrak{m}_{12}, & [\mathfrak{m}_2, \mathfrak{m}_{12}] = \mathfrak{m}_{12}, & [\mathfrak{m}_3, \mathfrak{m}_{13}] = \mathfrak{m}_{13}, & [\mathfrak{m}_4, \mathfrak{m}_{14}] = \mathfrak{m}_{14}, \\
[\mathfrak{m}_1, \mathfrak{m}_{13}] = \mathfrak{m}_{13}, & [\mathfrak{m}_2, \mathfrak{m}_{23}] = \mathfrak{m}_{23}, & [\mathfrak{m}_3, \mathfrak{m}_{23}] = \mathfrak{m}_{23}, & [\mathfrak{m}_4, \mathfrak{m}_{24}] = \mathfrak{m}_{24}, \\
[\mathfrak{m}_1, \mathfrak{m}_{14}] = \mathfrak{m}_{14}, & [\mathfrak{m}_2, \mathfrak{m}_{24}] = \mathfrak{m}_{24}, & [\mathfrak{m}_3, \mathfrak{m}_{34}] = \mathfrak{m}_{34}, & [\mathfrak{m}_4, \mathfrak{m}_{34}] = \mathfrak{m}_{34}, \\
[\mathfrak{m}_{12}, \mathfrak{m}_{23}] \subset \mathfrak{m}_{13}, & [\mathfrak{m}_{12}, \mathfrak{m}_{24}] \subset \mathfrak{m}_{14}, & [\mathfrak{m}_{13}, \mathfrak{m}_{34}] \subset \mathfrak{m}_{14}, & [\mathfrak{m}_{13}, \mathfrak{m}_{23}] \subset \mathfrak{m}_{12}, \\
[\mathfrak{m}_{14}, \mathfrak{m}_{24}] \subset \mathfrak{m}_{12}, & [\mathfrak{m}_{14}, \mathfrak{m}_{34}] \subset \mathfrak{m}_{13}, & [\mathfrak{m}_{12}, \mathfrak{m}_{13}] \subset \mathfrak{m}_{23}, & [\mathfrak{m}_{23}, \mathfrak{m}_{34}] \subset \mathfrak{m}_{24}, \\
[\mathfrak{m}_{23}, \mathfrak{m}_{24}] \subset \mathfrak{m}_{34}, & [\mathfrak{m}_{24}, \mathfrak{m}_{34}] \subset \mathfrak{m}_{23}, & [\mathfrak{m}_{12}, \mathfrak{m}_{12}] \subset \mathfrak{m}_1 + \mathfrak{m}_2, & [\mathfrak{m}_{13}, \mathfrak{m}_{13}] \subset \mathfrak{m}_1 + \mathfrak{m}_3, \\
[\mathfrak{m}_{14}, \mathfrak{m}_{14}] \subset \mathfrak{m}_1 + \mathfrak{m}_4, & [\mathfrak{m}_{23}, \mathfrak{m}_{23}] \subset \mathfrak{m}_2 + \mathfrak{m}_3, & [\mathfrak{m}_{24}, \mathfrak{m}_{24}] \subset \mathfrak{m}_2 + \mathfrak{m}_4, & [\mathfrak{m}_{34}, \mathfrak{m}_{34}] \subset \mathfrak{m}_3 + \mathfrak{m}_4,
\end{array}$$

and all the other pairs of subspaces not appearing in the above list are all multiply commutative.

From [9], we know that the compact simple Lie group $SO(n)$ $(n > 12)$ admits at least two left-invariant non-naturally reductive Einstein metrics $\rho_i, i = 1, 2$, which both correspond to the coefficients of the metric (3.2) satisfying the conditions

$$x_{12} = x_{13} = x_{14} = 1, x_{24} = x_{34} = x_{23}, x_2 = x_3 = x_4, x_2 \neq x_{23}, x_{23} \neq 1.$$

Moreover, it is easy to see that x_1, x_2, x_{23}, x_{12} are pairwise distinct ([9]).

Set $\mathfrak{p}_1 = \mathfrak{m}_1$, $\mathfrak{p}_2 = \mathfrak{m}_2 + \mathfrak{m}_3 + \mathfrak{m}_4$, $\mathfrak{p}_3 = \mathfrak{m}_{12} + \mathfrak{m}_{13} + \mathfrak{m}_{14}$, $\mathfrak{p}_4 = \mathfrak{m}_{23} + \mathfrak{m}_{24} + \mathfrak{m}_{34}$. Then the metric (3.2) reduces to

$$\langle , \rangle = x_1(-B)(,)|_{\mathfrak{p}_1} + x_2(-B)|_{\mathfrak{p}_2} + x_{12}(-B)|_{\mathfrak{p}_3} + x_{23}(-B)|_{\mathfrak{p}_4}. \tag{3.3}$$

Now we can give the proof of the main result of this paper.

Proof of Theorem 1.1. Let us consider Lie group $SO(n)$ $(n > 12)$ supplied with two left-invariant non-naturally reductive Einstein metrics $\rho_i, i = 1, 2$, generated with the inner product (3.3) (see [9]), now we show that the Riemannian manifolds $(SO(n)\ (n > 12), \rho_i)$ are not geodesic orbit.

Choose any maximal by-inclusion subalgebra $\mathfrak{k}$ adapted to (3.3); by the definition of $\mathfrak{k}_i = \mathfrak{k} \cap \mathfrak{p}_i$, $i = 1, 2, 3, 4$, we have $[\mathfrak{k}_i, \mathfrak{k}_i] \subset \mathfrak{k}_i$ and $[\mathfrak{k}_i, \mathfrak{n}_i] \subset [\mathfrak{k}, \mathfrak{n}_i] \subset \mathfrak{n}_i$, where $\mathfrak{n}_i$ is the orthogonal complement to $\mathfrak{k}_i$ in $\mathfrak{p}_i$. On the other hand, $[\mathfrak{p}_3, \mathfrak{p}_3] \subset \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_4$ and $[\mathfrak{p}_4, \mathfrak{p}_4] \subset \mathfrak{p}_2 \oplus \mathfrak{p}_4$. So $[\mathfrak{k}_3, \mathfrak{p}_3] = 0$, notice $[\mathfrak{k}_3, \mathfrak{k}_3] \subset \mathfrak{k}_3$; it is easy to get $\mathfrak{k}_3 = 0$. From $[\mathfrak{k}_4, \mathfrak{k}_4] \subset \mathfrak{k}_4$ and $[\mathfrak{k}_4, \mathfrak{n}_4] \subset \mathfrak{n}_4$, by Proposition 3.1, it is easy to get $\mathfrak{k}_4 = 0$. Thus, $\mathfrak{k}_3$ and $\mathfrak{k}_4$ are trivial and $\mathfrak{k} \subset \mathfrak{p}_1 \oplus \mathfrak{p}_2$.

Suppose that the inner product (3.3) generates a geodesic orbit left-invariant Riemannian metric, take $X_{13} \in \mathfrak{m}_{13}$ and $X_{23} \in \mathfrak{m}_{23}$ such that $[X_{13}, X_{23}] \neq 0$. By Theorem 2.1, for $X_{13} + X_{23} \in \mathfrak{m}_{13} + \mathfrak{m}_{23}$, there exists $W \in \mathfrak{k} \subset \mathfrak{p}_1 \oplus \mathfrak{p}_2$, such that $[A(X_{13} + X_{23}), X_{13} + X_{23} + W] = 0$, it is easy to see that $A(X_{13} + X_{23}) = x_{12}X_{13} + x_{23}X_{23}$. Then, we have

$$(x_{12} - x_{23})[X_{13}, X_{23}] + [x_{12}X_{13} + x_{23}X_{23}, W] = 0.$$

Since $[\mathfrak{m}_{13}, \mathfrak{m}_{23}] \subset \mathfrak{m}_{12}$, and $\mathfrak{m}_{13}, \mathfrak{m}_{23}$ is $ad(\mathfrak{p}_1 \oplus \mathfrak{p}_2)$-invariant submodules, we have $x_{12} = x_{23}$, which is impossible.

Thus, the Riemannian manifolds $(SO(n)\ (n > 12), \rho_i)$, $i = 1, 2$ are not geodesic orbit. This completes the proof.

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