C. R. Acad. Sci. Paris, Ser. I 357 (2019) 583–588

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Group theory/Harmonic analysis

L^p—Fourier inversion formula on certain locally compact groups ☆

L^p—Formule d'inversion de Fourier sur certains groupes localement compacts

Wassim Nasserddine

Université libanaise, Faculté des sciences – Section I, 2905-3901 Hadath-Beyrouth, Liban

A R T I C L E I N F O

Article history:
Received 22 February 2019
Accepted after revision 28 June 2019
Available online 5 July 2019

Presented by the Editorial Board

A B S T R A C T

Let G be a second countable locally compact group with type-I left regular representation, $\widehat{G}$ its dual and $\mathcal{K} = (K_\pi)_{\pi\in\widehat{G}}$ a specific measurable field of operators. In this paper, we investigate an inversion formula for $L^p(G)$. Let $1 < p, r \leq 2$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{s} + \frac{1}{r} = 1$, and $\mathcal{F}_p : L^p \cap L^1(G) \longrightarrow L^q(\widehat{G})$ be defined by $\mathcal{F}_p(f)\pi = \pi(f)K_\pi^{\frac{1}{q}}$. The map $\mathcal{F}_p$ extends uniquely to a linear map of $L^p(G)$ into $L^q(\widehat{G})$, denoted by $\mathcal{F}_p$. Let $\bar{\mathcal{F}}_p$ be the transpose of $\mathcal{F}_p$ and $f \in L^p(G)$. We prove that $\bar{f} \in \bar{\mathcal{F}}_r[L^r(\widehat{G})]$ if and only if $\mathcal{F}_p(f)\mathcal{K}^{\frac{1}{p}-\frac{1}{s}} \in L^r(\widehat{G})$, and, if that is the case, we have

$$\bar{f} = \bar{\mathcal{F}}_r(\mathcal{K}^{\frac{1}{p}-\frac{1}{s}}[\mathcal{F}_p(f)]^*).$$

R É S U M É

Soit G un groupe localement compact à base dénombrable dont la représentation régulière gauche est de type I, $\widehat{G}$ son dual et $\mathcal{K} = (K_\pi)_{\pi\in\widehat{G}}$ un champ mesurable spécifique d'opérateurs. Dans cette note, on aborde une formule d'inversion pour $L^p(G)$. Soit $1 < p, r \leq 2$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{s} + \frac{1}{r} = 1$, et $\mathcal{F}_p : L^p \cap L^1(G) \longrightarrow L^q(\widehat{G})$ la fonction définie par $\mathcal{F}_p(f)\pi = \pi(f)K_\pi^{\frac{1}{q}}$. L'application $\mathcal{F}_p$ se prolonge de manière unique en une application linéaire de $L^p(G)$ dans $L^q(\widehat{G})$, notée $\mathcal{F}_p$. Soit $\bar{\mathcal{F}}_p$ la transposée de $\mathcal{F}_p$ et $f \in L^p(G)$. On prouve que $\bar{f} \in \bar{\mathcal{F}}_r[L^r(\widehat{G})]$ si et seulement si $\mathcal{F}_p(f)\mathcal{K}^{\frac{1}{p}-\frac{1}{s}} \in L^r(\widehat{G})$, et, si c'est le cas, on a

$$\bar{f} = \bar{\mathcal{F}}_r(\mathcal{K}^{\frac{1}{p}-\frac{1}{s}}[\mathcal{F}_p(f)]^*).$$

☆ Ce projet a été financé avec le soutien de l'Université libanaise.
E-mail address: wassim.nasserddine@ul.edu.lb.

https://doi.org/10.1016/j.crma.2019.06.011
1631-073X/

1. Introduction

Let G be a locally compact group, $\widehat{G}$ its dual, $1 < p < 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^1 \cap L^p(G)$ and $\mathcal{F}(f)$ be the usual Fourier transform of f. If G is an abelian or separable unimodular type-I group (cf. [3, p. 214]), then the L^p-Fourier transform $\mathcal{F}_p(f)$ of f is defined by $\mathcal{F}_p(f) := \mathcal{F}(f)$, $\mathcal{F}_p(f) \in L^q(\widehat{G})$ and the map $f \longrightarrow \mathcal{F}_p(f)$ extends uniquely to a linear map of $L^p(G)$ into $L^q(\widehat{G})$ with norm ≤ 1. If G is not unimodular, then $\mathcal{F}_p(f)$ could not be defined as $\mathcal{F}_p(f) = \mathcal{F}(f)$ so that the Plancherel formula, the Hausdorff-Young theorem, and other harmonic results hold on G.

Let $G = ax + b$ denote the affine group of a local field. Then G is not unimodular and there is (up to equivalence) a unique infinite-dimensional irreducible unitary representation π of G (cf. [2, p. 209]), which acts in a Hilbert space $\mathcal{H}$. The L^p-Fourier transform $\mathcal{F}_p : L^p \cap L^1(G) \longrightarrow \mathcal{L}_q(\mathcal{H})$ is defined by

$$\mathcal{F}_p(f) = \mathcal{F}(f) \circ \delta_{\frac{1}{q}} = \pi(f) \circ \delta_{\frac{1}{q}},$$

where $\mathcal{L}_q(\mathcal{H})$ is the Banach space of bounded operators A on $\mathcal{H}$ with $\|A\|_q = \{Tr(|A|^q)\}^{\frac{1}{q}} < \infty$, and $\delta_{\frac{1}{q}}$ is a specific unbounded positive self-adjoint operator in $\mathcal{H}$. The map $\mathcal{F}_p$ extends uniquely to a linear map of $L^p(G)$ into $\mathcal{L}_q(\mathcal{H})$, denoted by $\mathcal{F}_p$. Let $1 < p, r \leq 2$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{s} + \frac{1}{r} = 1$, $f \in L^p(G)$ and $\bar{\mathcal{F}}_p$ be the transpose of $\mathcal{F}_p$. Then by the inversion theorem for $L^p(G)$ [2, Th. 2.22, p. 244], $\bar{f} \in \bar{\mathcal{F}}_r[\mathcal{L}_r(\mathcal{H})]$ if and only if $\mathcal{F}_p(f)\delta_{\frac{1}{p}-\frac{1}{s}} \in \mathcal{L}_r(\mathcal{H})$, and, if that is the case, we have

$$\bar{f} = \bar{\mathcal{F}}_r(\delta_{\frac{1}{p}-\frac{1}{s}}[\mathcal{F}_p(f)]^*).$$

Note that the "essential" dual $\widehat{G}_{ess}$ of G, which gives the Plancherel formula, reduces to the single representation $\{\pi\}$.

In this paper, by using the dual object $\widehat{G}$, we shall generalize the L^p-inversion theorem to (nonunimodular) second countable locally compact groups with type-I regular representation. For another approach to the L^p-Fourier transform, for any nonunimodular group, which does not use the dual object $\widehat{G}$ or a subset of $\widehat{G}$, see [7].

2. Notations

Let G be a locally compact group equipped with a left Haar measure ν. Let Δ be the modular function of G, $\mathcal{K}(G)$ the space of continuous functions with compact support and $\mathcal{D}(G)$ the space of regular (Bruhat) functions with compact support on G. For a complex function f on G and $x \in G$, we adopt the notations: $d_\nu x = dx$, $\check{f}(x) = f(x^{-1})$, $\tilde{f}(x) = \overline{f(x^{-1})}$. The Fourier algebra of G is

$$A(G) = \{u = f * \tilde{g} : \ f, g \in L^2(G)\},$$

with the norm $\|u\|_{A(G)} = \inf\{\|f\|_2\|g\|_2 : u = f * \tilde{g}\}$. If G is abelian, the dual group $\widehat{G}$ is the set of all characters on G. If G is not abelian, the dual $\widehat{G}$ of G is the set of (equivalence classes of) irreducible unitary representations of G. For $\pi \in \widehat{G}$, we denote the associated representation space by $\mathcal{H}_\pi$, and if an operator A is bounded on a densely domain, we give the same notation A to its extension. In the following G is second countable with type-I left regular representation. There are a standard measure μ on $\widehat{G}$, called Plancherel measure (see [1, p. 225] or [5, p. 545]), a μ-measurable field $(\pi, \mathcal{H}_\pi)_{\pi \in \widehat{G}}$ of representations of G, and a measurable field $\mathcal{K} = (K_\pi)_{\pi \in \widehat{G}}$ of nonzero positive self-adjoint operators such that, for almost all $\pi \in \widehat{G}$, K_π is semi-invariant with weight Δ^{-1}, the operator $\pi(f)K_\pi^{\frac{1}{2}}$, where $f \in L^1 \cap L^2(G)$, extends to a Hilbert–Schmidt operator on $\mathcal{H}_\pi$ denoted by $\mathcal{P}(f)(\pi)$, and the map $\mathcal{P} : L^1 \cap L^2(G) \longrightarrow L^2(\widehat{G})$ extends uniquely to a unitary map of $L^2(G)$ onto $L^2(\widehat{G})$. In this case, the Plancherel formula, for $f \in L^1 \cap L^2(G)$, is

$$\int_G |f(x)|^2 dx = \int_{\widehat{G}} \|\pi(f) \circ K_\pi^{\frac{1}{2}}\|_2^2 \, d\mu(\pi).$$

Throughout this paper, the Plancherel measure μ and the field $\mathcal{K} = (K_\pi)_{\pi \in \widehat{G}}$ above are fixed so that the conditions (i) and (ii) of [1, Th. 5, p. 225] hold.

3. Results

Recall that for $\pi \in \widehat{G}$ and $f \in L^1(G)$, the operator $\pi(f) := \int_G f(x)\pi(x)\, dx$ is defined in the weak sense, that is,

$$< \pi(f)\xi, \eta >_{\mathcal{H}_\pi} = \int_G f(x) < \pi(x)\xi, \eta >_{\mathcal{H}_\pi} dx,$$

where ξ, $\eta \in \mathcal{H}_\pi$ and $< ., . >_{\mathcal{H}_\pi}$ is the inner product on $\mathcal{H}_\pi$.

Lemma 3.1. *Let $f, g \in \mathcal{D}(G)$. Then*

$$\Delta f * (\Delta^{-1} g)^{\sim} = \Delta(f * \tilde{g}).$$

Lemma 3.2. *Let $f \in \mathcal{D}(G)$. Then, for almost all $\pi \in \widehat{G}$, we have*

$$\pi(\Delta^{\frac{1}{2}} f)(K_\pi)^{\frac{1}{2}} = (K_\pi)^{\frac{1}{2}} \pi(f).$$

Lemma 3.3. *If $f, g \in \mathcal{D}(G)$, then, for almost all $\pi \in \widehat{G}$,*

$$K_\pi \, \pi(f * \tilde{g}) = \pi(\Delta f * (\Delta^{-1} g)^{\sim}) K_\pi.$$

Lemma 3.4. *Let $f, g \in \mathcal{D}(G)$. Then, for almost all $\pi \in \widehat{G}$,*

$$\mathcal{P} f(\pi)(\mathcal{P} g)^*(\pi) = \pi(f)\pi(\tilde{g}) K_\pi.$$

The Fourier transform $\mathcal{F}(u)$ of a function $u \in L^1(G)$ is defined by

$$\mathcal{F}(u)(\pi) := \pi(u) = \int_G u(x)\pi(x)\,\mathrm{d}x.$$

Theorem 3.5. *Let $u \in A(G)$. If $u \in L^1(G)$, then $\mathcal{F}(u) \circ \mathcal{K} \in L^1(\widehat{G})$, where $\mathcal{F}(u)$ is the Fourier transform of u, and we have*

$$u(x) = \int_{\widehat{G}} Tr[\pi(x^{-1})\mathcal{F}(u)(\pi) \circ K_\pi]\,\mathrm{d}_\mu(\pi).$$

Proof. In view of [5, Cor. 3.5, p. 547], there exists a unique $T \in L^1(\widehat{G})$ such that

$$u(x) = \int_{\widehat{G}} Tr[\pi(x^{-1}) T(\pi)]\,\mathrm{d}_\mu(\pi).$$

Now, for all $f, g \in \mathcal{D}(G)$, by first applying [5, Th. 3.3, p. 547] (to the first equality) and then Lemma 3.4, Lemma 3.3 (to the last two equalities), we obtain

$$\begin{aligned}
\int_{\widehat{G}} Tr[\pi(f * \tilde{g}) T(\pi)]\,\mathrm{d}\mu(\pi) &= \int_G (f * \tilde{g})(x) u(x^{-1})\,\mathrm{d}x \\
&= \int_G \int_G f(xy)\overline{g(y)} u(x^{-1})\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_G \int_G f(x^{-1}y)\overline{g(y)} u(x)\,\Delta^{-1}(x)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_G \int_G u(x)\,\Delta f(x^{-1}y)\,\Delta^{-1}(y)\overline{g(y)}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_G (u * \Delta f)(y)\,\overline{\Delta^{-1} g(y)}\,\mathrm{d}y \\
&= < u * \Delta f, \Delta^{-1} g >_{L^2(G)} = < \mathcal{P}(u * \Delta f), \mathcal{P}(\Delta^{-1} g) >_{L^2(\widehat{G})} \\
&= \int_{\widehat{G}} Tr[\mathcal{P}(u * \Delta f)(\pi)\mathcal{P}^*(\Delta^{-1} g)(\pi)\,\mathrm{d}\mu(\pi) \\
&= \int_{\widehat{G}} Tr[\pi(u)\pi(\Delta f * (\Delta^{-1} g)^{\sim}) K_\pi]\,\mathrm{d}\mu(\pi) \\
&= \int_{\widehat{G}} Tr[\pi(u) K_\pi \pi(f * \tilde{g})]\,\mathrm{d}\mu(\pi).
\end{aligned}$$

This implies that $\mathcal{F}(u)\circ\mathcal{K}=T\in L^1(\widehat{G})$, which in turn finishes the proof. □

Let $1\le p\le 2$, $\frac{1}{p}+\frac{1}{q}=1$ and $f\in L^p\cap L^1(G)$. We define the L^p-Fourier transform $\mathcal{F}_p(f)$ of f by $\mathcal{F}_p(f)\pi=\pi(f)K_\pi^{\frac{1}{q}}$. By adapting the proof of [4, Th. 6, p. 582], we obtain the following theorem.

Theorem 3.6. *(Hausdorff–Young) If*

$$1<p<2,\qquad \frac{1}{p}+\frac{1}{q}=1,\qquad f\in L^1\cap L^p(G),$$

then $\mathcal{F}_p(f)\in L^q(\widehat{G})$, and the map $f\longrightarrow\mathcal{F}_p(f)$ extends uniquely to a linear map of $L^p(G)$ into $L^q(\widehat{G})$ with norm ≤ 1.

The spaces $L^p(G)$ and $L^q(G)$ are in duality by

$$< f,g >=\int_G f(x)g(x)\,\mathrm{d}x,$$

where $f\in L^p(G)$ and $g\in L^q(G)$. If $p=2$, $< . >_{L^2(G)}$ indicates the usual inner product on $L^2(G)$. If $p\neq 1$, $\bar{\mathcal{F}}_p$ denotes the transpose of $\mathcal{F}_p$ and if $p=1$, $\bar{\mathcal{F}}_1=\bar{\mathcal{F}}$ denotes the Fourier cotransformation, which is an isometric isomorphism of Banach spaces from $L^1(\widehat{G})$ onto $A(G)$. The map $\bar{\mathcal{F}}$ is defined by

$$\bar{\mathcal{F}}(T)(x)=\int_{\widehat{G}} Tr[T(\pi)\pi(x)]\,\mathrm{d}\mu(\pi),$$

where $T\in L^1(\widehat{G})$ and $x\in G$.

Lemma 3.7. *Let $1<p,r\le 2$, $\frac{1}{p}+\frac{1}{q}=1$, $H\in L^p(\widehat{G})$ and $M\in L^r(\widehat{G})$.*

(i) If $f\in\mathcal{D}(G)$, then, for almost all $\pi\in\widehat{G}$, we have $\mathcal{F}_r f(\pi)(K_\pi)^{\frac{1}{r}-\frac{1}{p}}=\mathcal{F}_p f(\pi)$.

(ii) If, for almost all $\pi\in\widehat{G}$, we have $M(\pi)=(K_\pi)^{\frac{1}{r}-\frac{1}{p}}H(\pi)$, then

$$\bar{\mathcal{F}}_p(H)=\bar{\mathcal{F}}_r(M).$$

Lemma 3.8. *Let $p\in\,]1,2]$. Then, for all f in $\mathcal{D}(G)$, we have:*

$$[\mathcal{F}_p(f)]^*=\mathcal{F}_p(\Delta^{-\frac{1}{p}}f^{\sim}).$$

Lemma 3.9. *Let $1<p\le 2$ and $\frac{1}{p}+\frac{1}{q}=1$. Then*

(i) For every $H\in L^p(\widehat{G})$, we have

$$\bar{\mathcal{F}}_p(H^*)=\Delta^{-\frac{1}{q}}[\bar{\mathcal{F}}_p(H)]^{\sim}.$$

(ii) For all $H\in L^2(\widehat{G})$, we have $\bar{\mathcal{F}}_2(H)=\overline{\mathcal{F}_2^{-1}(H^*)}$.

Theorem 3.10 *(Inversion theorem for $L^p(G)$). Let $1<p,r\le 2$, $\frac{1}{s}+\frac{1}{r}=1$, $f\in L^p(G)$. Then $\bar{f}\in\bar{\mathcal{F}}_r[L^r(\widehat{G})]$ if and only if $\mathcal{F}_p(f)\mathcal{K}^{\frac{1}{p}-\frac{1}{s}}\in L^r(\widehat{G})$, and, if that is the case, we have*

$$\Delta^{-\frac{1}{s}}\check{f}=\bar{\mathcal{F}}_r(\mathcal{F}_p(f)\mathcal{K}^{\frac{1}{p}-\frac{1}{s}}),$$

and

$$\bar{f}=\bar{\mathcal{F}}_r(\mathcal{K}^{\frac{1}{p}-\frac{1}{s}}[\mathcal{F}_p(f)]^*).$$

Proof. For every $g\in\mathcal{K}(G)*\mathcal{K}(G)$, we have

$$\bar{g}=\bar{\mathcal{F}}_p(\mathcal{K}^{\frac{1}{r}-\frac{1}{q}}[\mathcal{F}_r(g)]^*).\qquad(1)$$

In fact,

$$\begin{aligned}\bar{g} = \overline{\mathcal{F}_2^{-1}\mathcal{F}_2(g)} &= \overline{\mathcal{F}_2^{-1}\mathcal{F}(g)\circ\mathcal{K}^{1-\frac{1}{2}}} \\ &= \overline{\mathcal{F}_2^{-1}\mathcal{F}_r(g)\circ\mathcal{K}^{\frac{1}{r}-\frac{1}{2}}} \\ &= \bar{\mathcal{F}}_2(\mathcal{K}^{\frac{1}{r}-\frac{1}{2}}[\mathcal{F}_r(g)]^*) \qquad (2)\\ &= \bar{\mathcal{F}}_p(\mathcal{K}^{\frac{1}{p}-\frac{1}{2}}\mathcal{K}^{\frac{1}{r}-\frac{1}{2}}[\mathcal{F}_r(g)]^*) \qquad (3)\\ &= \bar{\mathcal{F}}_p(\mathcal{K}^{\frac{1}{r}-\frac{1}{q}}[\mathcal{F}_r(g)]^*),\end{aligned}$$

where (2) follows from Lemma 3.9 (ii) and (3) follows from Lemma 3.7 (ii) by setting $H=\mathcal{K}^{\frac{1}{r}-\frac{1}{2}}[\mathcal{F}_r(g)]^*$ and $M=\mathcal{K}^{\frac{1}{p}-\frac{1}{2}}H$.

Now we prove the necessity. Let $f\in L^p(G)$ such that $\bar{f}=\bar{\mathcal{F}}_r(H)$, where $H\in L^r(\widehat{G})$. Then, for all $g\in\mathcal{K}(G)*\mathcal{K}(G)$, we have

$$\begin{aligned}&\int_{\widehat{G}} Tr(\mathcal{F}_r g(\pi)(K_\pi)^{\frac{1}{p}-\frac{1}{s}}[\mathcal{F}_p(f)]^*(\pi))\,\mathrm{d}_\mu(\pi)\\ &=\int_{\widehat{G}}\overline{Tr([\mathcal{F}_p(f)](\pi)(K_\pi)^{\frac{1}{p}-\frac{1}{s}}[\mathcal{F}_r(g)]^*(\pi))}\,\mathrm{d}_\mu(\pi)\\ &=\overline{<f,\bar{\mathcal{F}}_p(\mathcal{K}^{\frac{1}{p}-\frac{1}{s}}[\mathcal{F}_r(g)]^*)>}\\ &=<\bar{f},g> \qquad \text{by}\,(1)\\ &=\int_G g(x)\bar{\mathcal{F}}_r(H)(x)\,\mathrm{d}x\\ &=\int_{\widehat{G}} Tr[\mathcal{F}_r g(\pi)H(\pi)]\,\mathrm{d}_\mu(\pi).\end{aligned}$$

Let ε_x be the Dirac measure at $x\in G$. Then, for every $x\in G$,

$$\begin{aligned}\bar{\mathcal{F}}[\mathcal{F}_r(g)H](x) &= \int_{\widehat{G}} Tr[\pi(x)\mathcal{F}_r g(\pi)H(\pi)]\,\mathrm{d}_\mu(\pi)\\ &=\int_{\widehat{G}} Tr[\mathcal{F}_r(\varepsilon_x * g)(\pi)H(\pi)]\,\mathrm{d}_\mu(\pi)\\ &=\int_{\widehat{G}} Tr[\mathcal{F}_r(\varepsilon_x * g)(\pi)(K_\pi)^{\frac{1}{p}-\frac{1}{s}}[\mathcal{F}_p(f)]^*(\pi)]\,\mathrm{d}_\mu(\pi)\\ &=\bar{\mathcal{F}}(\mathcal{F}_r(g)\mathcal{K}^{\frac{1}{p}-\frac{1}{s}}[\mathcal{F}_p(f)]^*)(x).\end{aligned}$$

Hence, since $\bar{\mathcal{F}}$ is one-to-one, we obtain $\mathcal{F}_r(g)H=\mathcal{F}_r(g)\mathcal{K}^{\frac{1}{p}-\frac{1}{s}}[\mathcal{F}_p(f)]^*$ and thus $H^*[\mathcal{F}_r(g)]^*=\mathcal{F}_p(f)\mathcal{K}^{\frac{1}{p}-\frac{1}{s}}[\mathcal{F}_r(g)]^*$. Therefore $\mathcal{F}_p(f)\mathcal{K}^{\frac{1}{p}-\frac{1}{s}}=H^*\in L^r(\widehat{G})$.

We prove the sufficiency. Let $f\in L^p(G)$ such that $\mathcal{F}_p(f)\mathcal{K}^{\frac{1}{p}-\frac{1}{s}}\in L^r(\widehat{G})$. Then, for all $g\in\mathcal{K}(G)*\mathcal{K}(G)$, we have

$$\begin{aligned}<\bar{f},g>=\overline{<f,\bar{g}>} &= \overline{<f,\bar{\mathcal{F}}_p(\mathcal{K}^{\frac{1}{r}-\frac{1}{q}}[\mathcal{F}_r(g)]^*)>}\\ &=\int_{\widehat{G}}\overline{Tr[\mathcal{F}_p f(\pi)(K_\pi)^{\frac{1}{r}-\frac{1}{q}}[\mathcal{F}_r(g)]^*(\pi)]}\,\mathrm{d}_\mu(\pi)\\ &=\int_{\widehat{G}}\overline{Tr[\mathcal{F}_p f(\pi)(K_\pi)^{\frac{1}{p}-\frac{1}{s}}[\mathcal{F}_r(g)]^*(\pi)]}\,\mathrm{d}_\mu(\pi) \qquad \left(\frac{1}{p}-\frac{1}{s}=\frac{1}{r}-\frac{1}{q}\right)\\ &=\int_{\widehat{G}} Tr([\mathcal{F}_r(g)](K_\pi)^{\frac{1}{p}-\frac{1}{s}}[\mathcal{F}_p f]^*(\pi))\,\mathrm{d}_\mu(\pi)\\ &=<g,\bar{\mathcal{F}}_r(\mathcal{K}^{\frac{1}{p}-\frac{1}{s}}[\mathcal{F}_p(f)]^*)>.\end{aligned}$$

Consequently,

$$\bar{f} = \bar{\mathcal{F}}_r(\mathcal{K}^{\frac{1}{p}-\frac{1}{s}}[\mathcal{F}_p(f)]^*),$$

and thus

$$\begin{aligned}\Delta^{-\frac{1}{s}}(\Delta^{-\frac{1}{s}}\check{f})^{\sim} = \bar{f} &= \bar{\mathcal{F}}_r(\mathcal{K}^{\frac{1}{p}-\frac{1}{s}}[\mathcal{F}_p(f)]^*)\\ &= \Delta^{-\frac{1}{s}}[\bar{\mathcal{F}}_r(\mathcal{F}_p(f)\mathcal{K}^{\frac{1}{p}-\frac{1}{s}})]^{\sim} \qquad \text{by Lemma 3.9 (i)},\end{aligned}$$

which implies that $\Delta^{-\frac{1}{s}}\check{f} = \bar{\mathcal{F}}_r(\mathcal{F}_p(f)\mathcal{K}^{\frac{1}{p}-\frac{1}{s}})$. □

Remark 1. By using some adequate traces that generalize the ordinary traces of operator (cf. [6]), some of these results could be extended to other nonunimodular groups for which the Plancherel formula holds.

Acknowledgements

I am deeply grateful to the reviewer for pointing out an error in the proof of Theorem 3.10 in an earlier version.

References

[1] M. Duflo, C.-C. Moore, On the regular representation of a nonunimodular locally compact group, J. Funct. Anal. 21 (2) (1976) 209–243.
[2] P. Eymard, M. Terp, La transformation de Fourier et son inverse sur le groupe des $ax+b$ d'un corps local, in: Analyse harmonique sur les groupes de Lie II, Séminaire Nancy–Strasbourg (1976–1978), in: Lecture Notes in Mathematics, vol. 739, 1979, pp. 207–248 (in French).
[3] R.L. Lipsman, Non-Abelian Fourier analysis, Bull. Sci. Math. (2) 98 (1974) 209–233.
[4] W. Nasserddine, The Hausdorff–Young theorem for the matricial groups $G_{nm} = ax + b$, Arch. Math. 87 (2006) 578–590.
[5] W. Nasserddine, Une caractérisation de l'algèbre de Fourier pour certains groupes localement compacts, C. R. Acad. Sci. Paris, Ser. I 355 (2017) 543–548.
[6] N. Tatsuuma, Plancherel formula for non-unimodular locally compact groups, J. Math. Kyoto Univ. 12 (1972) 179–261.
[7] M. Terp, L^p-Fourier transformation on non-unimodular locally compact groups, Adv. Oper. Theory 2 (4) (2017) 547–583.