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Complex analysis

On a family of extremal polynomials

Sur une famille de polynômes extrémaux

Dmitriy Dmitrishin [a], Andrey Smorodin [a], Alex Stokolos [b]

[a] *Odessa National Polytechnic University, 1 Shevchenko Ave., Odessa 65044, Ukraine*
[b] *Georgia Southern University, Statesboro, GA 30460, USA*

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ABSTRACT

For a pair of conjugate trigonometrical polynomials $C(t)=\sum_{j=1}^{N} a_j \cos jt$, $S(t)=\sum_{j=1}^{N} a_j \sin jt$ with real coefficients and normalization $a_1=1$ the following extremal value is found:

$$\sup_{a_2,\ldots,a_N} \min_t \{C(t): S(t)=0\} = -\frac{1}{4}\sec^2\frac{\pi}{N+2}.$$

An application of this result in geometric complex analysis is shown. Several conjectures for a number of extremal problems on classes of polynomials are suggested.

RÉSUMÉ

Pour une paire de polynômes trigonométriques $C(t)=\sum_{j=1}^{N} a_j \cos jt$, $S(t)=\sum_{j=1}^{N} a_j \sin jt$ à coefficients réels avec la normalisation $a_1=1$, on trouve la valeur extrémale

$$\sup_{a_2,\ldots,a_N} \min_t \{C(t): S(t)=0\} = -\frac{1}{4}\sec^2\frac{\pi}{N+2}.$$

Une application en analyse géométrique complexe est montrée. On formule quelques conjectures pour les problèmes extrémaux sur les classes de polynômes.

1. Motivation

Let $C(t)=\sum_{j=1}^{N} a_j \cos jt$ and $S(t)=\sum_{j=1}^{N} a_j \sin jt$ with all a_j real. In [6,8] the following problem was solved:

$$\sup_{a_1+\cdots+a_N=1} \min_t \{C(t): S(t)=0\} = -\tan^2\frac{\pi}{2(N+1)}. \tag{1}$$

E-mail addresses: dmitrishin@opu.ua (D. Dmitrishin), andrey.smorodin@gmail.com (A. Smorodin), astokolos@georgiasouthern.edu (A. Stokolos).

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Its solution led to a new procedure in the problem of chaos stabilization (cf. [6,8]). It also might be interpreted as a problem of geometric complex analysis: the largest interval of the form $(-\mu, 1)$ that can be covered by the inverse image of the complement to a unit disc, $(\mathbb{C} \setminus F_N(\mathbb{D}))^*$, where $z^* = 1/\bar{z}$ and $F_N(z) = a_1 z + \cdots + a_N z^N$ with $F_N(1) = 1$, is $\left(-\cot^2 \frac{\pi}{2(N+1)}, 1\right)$. The extremal polynomials turned out to be the well-known Suffridge polynomials [21].

Below we consider a similar problem just with a different normalization. As a consequence, theorems on covering intervals by polynomial images of $\mathbb{D}$ are obtained. This paper continues the work began in [5].

2. Main result

We now assume that $a_1 = 1$. Let

$$J_N = \sup_{a_2,\ldots,a_N} \min_t \{C(t) : S(t) = 0\}. \tag{2}$$

The following theorem will be proved.

Theorem 1. *We have*

$$J_N = -\frac{1}{4}\sec^2 \frac{\pi}{N+2}.$$

The extremal polynomials are unique, with the coefficients given by the formulas

$$a_j^{(0)} = \frac{1}{U_N'\left(\cos \frac{\pi}{N+2}\right)} U_{N-j+1}'\left(\cos \frac{\pi}{N+2}\right) U_{j-1}\left(\cos \frac{\pi}{N+2}\right), \quad j = 1, \ldots, N, \tag{3}$$

where $U_j(x)$ are Chebyshev polynomials of the second kind and $U_j'(x)$ denotes the derivative of $U_j(x)$.

3. The method

The proof of Theorem 1 consists of two parts. The first one is a reduction:

$$J_N = \sup_{a_2,\ldots,a_N} \{C(\pi) : S(t) \geq 0,\ 0 \leq t \leq \pi\}. \tag{4}$$

This part is nonstandard and quite involved. The second part is to compute (4). It is involved, but ideologically it follows the scheme suggested by L. Fejér [11]. Namely, using the Fejér–Riesz representation of nonnegative trigonometric polynomials, one can reduce finding (4) to the generalized Rayleigh-quotient minimization problem (cf. [20, Section 6I], [16], [17]). Then one can solve the generalized characteristic equation and obtain J_N as the least root of the equation. The next step is to derive the coefficients of the extremal polynomial from the generalized eigenvector that corresponds to that generalized eigenvalue. In spite of a clear idea, it is still a difficult task. As stated in [19], “In general, however, the method (...) is not easily adaptable to obtain explicit results, in particular when N is large.”

4. Outline of the proof

We argue via a sequence of lemmas. A complete proof can be found in [9].

Lemma A. *There exists an extremizer to J_N.*

Define $F_N(z) = z + a_2 z^2 + \cdots + a_N z^N$. Then $J_N = \sup_{a_2,\ldots,a_N} \min_t \{\Re(F_N(e^{it})) : \Im(F_N(e^{it})) = 0\}$.
The following lemma is a core of the first part of the proof. It leads to the reduction (4).

Lemma B. *If the trigonometric polynomial $\Im(F_N(e^{it}))$ vanishes at some point in $(0, \pi)$, then*

$$\min_t \{\Re(F_N(e^{it})) : \Im(F_N(e^{it})) = 0\} < J_N.$$

Denote

$$\begin{aligned} J_N^1 &= \sup_{a_2,\ldots,a_N} \min_t \{\Re(F_N(e^{it})) : \Im(F_N(e^{it})) > 0,\ 0 < t < \pi\} \\ &= \sup_{a_2,\ldots,a_N} \{F_N(-1) : \Im(F_N(e^{it})) > 0,\ 0 < t < \pi\}. \end{aligned}$$

Lemma B implies that $J_N = J_N^1$. Further, let

$$J_N^2 = \sup_{a_2,\dots,a_N} \{F_N(-1) : \Im(F_N(\mathrm{e}^{\mathrm{i}t})) \geq 0\}. \tag{5}$$

Lemma C. *We have*

$$J_N^1 = J_N^2.$$

Proof. It is clear that $J_N^1 \leq J_N^2$. Let $P(z)$ be almost extremal for J_N^2. Mollifying $P(z)$ by ε, i.e. considering $P_\varepsilon(z) = (P(z) + \varepsilon z)/(1+\varepsilon)$, we get $P_\varepsilon(z)$ almost extremal for J_N^1 as well; therefore, $J_N^2 \leq J_N^1$. $\square$

Now, let us define two $N \times N$ matrices

$$A = \begin{pmatrix} 0 & 1/2 & 0 & 0 & \dots \\ 1/2 & 0 & 1/2 & 0 & \dots \\ 0 & 1/2 & 0 & 1/2 & \dots \\ 0 & 0 & 1/2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1/2 & 0 & \dots \\ 0 & 0 & 0 & 1/2 & \dots \\ 1/2 & 0 & 0 & 0 & \dots \\ 0 & 1/2 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

The following lemma is central to the proof of the main result.

Lemma D. *We have*

$$J_N^2 = \max_{d_j} \left\{ -\frac{1 - d^\mathsf{T} A d}{1 - d^\mathsf{T} B d} : d^\mathsf{T} d = 1 \right\} = -\min_{d_j} \left\{ \frac{d^\mathsf{T}(I-A)d}{d^\mathsf{T}(I-B)d} \right\}.$$

Proof. Following the scheme in [19] let us write

$$\sin t + \sum_{j=2}^{N} \alpha_j \sin jt = \beta_0 \sin t \left(1 + 2\sum_{j=1}^{N-1} \beta_j \cos jt\right),$$

where $\beta_0 = 1 + \sum_{j=1}^{\lfloor (N-1)/2 \rfloor} \alpha_{2j+1}$, $\beta_1 = \frac{1}{\beta_0} \sum_{j=1}^{\lfloor N/2 \rfloor} \alpha_{2j}$, $\beta_2 = \frac{1}{\beta_0} \sum_{j=1}^{\lfloor (N-1)/2 \rfloor} \alpha_{2j+1}, \dots, \beta_{N-1} = \frac{1}{\beta_0} \alpha_N$.

Assume now that $F_N^{(0)}(z)$ is an optimal polynomial. Because of (5) and Lemmas B and C,

$$J_N = F_N^{(0)}(-1) = \sup_{a_2,\dots,a_N} \{F_N(-1) : \Im(F_N(\mathrm{e}^{\mathrm{i}t})) \geq 0,\ t \in [0,\pi]\},$$

thus

$$J_N = \sup_{\alpha_j} \left\{ -1 + \alpha_2 - \alpha_3 + \dots : 1 + 2\sum_{j=1}^{N-1} \beta_j \cos jt \geq 0 \right\}$$
$$= \sup_{\beta_j} \left\{ -\frac{1-\beta_1}{1-\beta_2} : 1 + 2\sum_{j=1}^{N-1} \beta_j \cos jt \geq 0 \right\}.$$

By the Fejér–Riesz representation theorem [18, 6.5, problem 41], any nonnegative trigonometric polynomial can be represented as the square of the modulus of some algebraic polynomial on the unit circle,

$$|d_1 + d_2 \mathrm{e}^{\mathrm{i}t} + \dots + d_N \mathrm{e}^{\mathrm{i}(N-1)t}|^2 = 1 + 2\sum_{j=1}^{N-1} \beta_j \cos jt.$$

Consequently, the following relations are valid:

$$d_1^2 + d_2^2 + \dots + d_n^2 = 1, \tag{6}$$
$$d_1 d_2 + d_2 d_3 + \dots + d_{N-1} d_N = \beta_1, \tag{7}$$
$$d_1 d_3 + d_2 d_4 + \dots + d_{N-2} d_N = \beta_2, \tag{8}$$
$$\dots$$
$$d_1 d_N = \beta_{N-1}.$$

Then we can write $J_N = \max_{d_j}\{-\frac{1-d^\intercal Ad}{1-d^\intercal Bd} : d^\intercal d = 1\}$, where $d = (d_1, \ldots, d_N)^\intercal$, and A (resp., B) is the $N \times N$ matrix corresponding to the quadratic form on the left-hand side of (7) (resp., (8)); the superscript $\intercal$ denotes transposition. Consequently,

$$J_N = \max_{d_j}\left\{-\frac{1-\frac{d^\intercal Ad}{d^\intercal d}}{1-\frac{d^\intercal Bd}{d^\intercal d}}\right\} = \max_{d_j}\left\{-\frac{d^\intercal(I-A)d}{d^\intercal(I-B)d}\right\} = -\min_{d_j}\left\{\frac{d^\intercal(I-A)d}{d^\intercal(I-B)d}\right\}. \qquad \square$$

It is clear that the matrices $I - A$ and $I - B$ are positive definite. Thus the problem of finding J_N reduces to a problem on generalized eigenvalues [20, Remark 3, p. 342]). Namely, let $\lambda_1 \le \cdots \le \lambda_N$ be the roots of the equation $\det((I-A) - \lambda(I-B)) = 0$. Then $J_N = -\lambda_1$.

By the positive definiteness of $I - A$ and $I - B$, we have $\lambda_1 > 0$. The corresponding minimum is attained at a generalized eigenvector $\delta^{(0)}$, which is determined from the relation $(I-A)\delta^{(0)} = \lambda_1(I-B)\delta^{(0)}$.

The following lemma might be interesting in itself. It is the most challenging statement in the second part of the proof. The argument in [9] is quite nontrivial.

Lemma E.

$$\det\big(4x^2(I-A) - (I-B)\big) = \frac{1}{2^{N+2}x} U_{N+1}(x) U'_{N+1}(x).$$

Corollary 1. *The roots of the equation* $\det(4x^2(I-A) - (I-B)) = 0$ *are*

$$\{\pm\mu_j\}_{j=1}^{\lfloor (N+1)/2 \rfloor}, \qquad \{\pm\nu_j\}_{j=1}^{N-\lfloor (N+1)/2 \rfloor},$$

where

$$\mu_j = \cos\frac{j\pi}{N+2}, \quad U'_{N+1}(\nu_j) = 0,$$

and they can be ordered as

$$\begin{aligned} 0 < \mu_{(N+1)/2} < \nu_{(N-1)/2} < \cdots < \nu_1 < \mu_1 \quad &\textit{if } N \textit{ is odd},\\ 0 < \nu_{N/2} < \mu_{N/2} < \cdots < \nu_1 < \mu_1 \quad &\textit{if } N \textit{ is even}. \end{aligned}$$

Lemma F. *Let* A *and* B *be the matrices as in Lemma D. The solution to the system of linear equations*

$$\left(4\cos^2\frac{j\pi}{N+2}(I-A) - (I-B)\right)\delta = 0, \quad j = 1, \ldots, \left\lfloor \frac{N+1}{2} \right\rfloor,$$

is the one-parameter family

$$c\delta^{(0)}\left(\cos\frac{j\pi}{N+2}\right),$$

where $c \in \mathbb{R}$ *and*

$$\delta^{(0)}(x) = (U_0(x)U_1(x), \ldots, U_{N-1}(x)U_N(x))^\intercal.$$

Corollary 2.

$$J_N^2 = -\lambda_1 = -\frac{1}{4\mu_1^2} = -\frac{1}{4}\sec^2\frac{\pi}{N+2},$$

$$\delta^{(0)} = \left(U_0\left(\cos\frac{\pi}{N+2}\right)U_1\left(\cos\frac{\pi}{N+2}\right), \ldots, U_{N-1}\left(\cos\frac{\pi}{N+2}\right)U_N\left(\cos\frac{\pi}{N+2}\right)\right)^\intercal.$$

Thus we have found the coefficients d_j. Now we have to come back to α_j and then to a_j. Luckily enough, we have been able to come up with nicely written a_k's. The following lemma contains some useful trigonometric identities.

Lemma G. *For all $k = 0, 1, \ldots, N-1$,*

$$2\sum_{j=1}^{N-k} \sin\frac{(j+1)\pi}{N+2}\sin\frac{(j+k)\pi}{N+2} = (N-k-1)\sin\frac{k\pi}{N+2}\sin\frac{\pi}{N+2}$$
$$+\frac{1}{2}\frac{\cos\frac{\pi}{N+2}}{\sin\frac{\pi}{N+2}}\left((N-k+3)\sin\frac{(k+1)\pi}{N+2}-(N-k+1)\sin\frac{(k-1)\pi}{N+2}\right).$$

From Lemma G we get the extremal coefficients (3). Since $J_N = J_N^1$, we have proved Theorem 1.

5. Covering problems

The most important result about covering of line segments and discs under a conformal mapping of the unit disc $\mathbb{D} = \{z : |z| < 1\}$ is the well-known Koebe One-Quarter Theorem, stating that the image of $\mathbb{D}$ under a univalent function with standard normalization contains the disc of radius 1/4 around the origin. The sharpness of the constant is witnessed by the function $e^{-i\theta}K(ze^{i\theta})$, where $K(z) = \frac{z}{(1-z)^2}$ is the Koebe function. In [7], the Koebe theorem was generalized to nonunivalent functions, by showing that any simply connected set that contains the image of $\mathbb{D}$ contains the disc of radius 1/4 centered at the origin. The proof uses the Riemann mapping theorem and the Lindelöf principle according to which the minimal simply connected set that contains the image of $\mathbb{D}$ can be conformally mapped onto $\mathbb{D}$ by a function whose derivative at the origin is (in modulus) greater than or equal to one [13]. Some similar theorems have been proved for polynomials [10].

Define the *Koebe polynomial radius* R_N to be the supremum of the radii of discs centered at origin that are covered by $F_N(\mathbb{D})$, where $F_N(z) = z + \sum_{j=2}^N a_j z^j$ is an arbitrary polynomial univalent in $\mathbb{D}$. The standard proof of the Koebe theorem [14, p. 391] is based on the estimate

$$|\gamma| \geq \frac{1}{2+|a_2|} \tag{9}$$

where γ is an exceptional value of $F_N(z)$ in $\mathbb{D}$. Then the Bieberbach estimate $|a_2| \leq 2$ implies the Koebe theorem for univalent functions. It is remarkable that, for polynomials, the estimate of the second coefficient can be improved. A sharp bound for typically real polynomials[1] (a class wider than the univalent polynomials with real coefficients) was found by Rogosinski and Szegö [19]. Namely, $|a_2| \leq 2\cos 2\psi_N$, where $\psi_N = \pi/(N+3)$ if N is odd and ψ_N is the smallest positive root of the equation $(N+4)\sin(N+2)\psi_N + (N+2)\sin(N+4)\psi_N = 0$ if N is even. One can check that in the latter case, $\pi/(N+3) < \psi_N < \pi/(N+2)$. Then (9) implies an estimate on the Koebe polynomial radius:

$$R_N \geq \frac{1}{2+\max|a_2|} \geq \frac{1}{4}\sec^2\frac{\pi}{N+3}. \tag{10}$$

The next theorem estimates from above the largest interval of the real axis that can be covered by the image of a polynomial map.

Theorem 2. *A simply connected set that contains $F_N(\mathbb{D})$ for any polynomial mapping $F_N(z) = z + \sum_{j=2}^N a_j z^j$ contains the interval*

$$\left(-\frac{1}{4}\sec^2\frac{\pi}{N+2}, \frac{1}{4}\sec^2\frac{\pi}{N+2}\right). \tag{11}$$

If F_N is typically real, then the interval (11) is covered by $F_N(\mathbb{D})$.

No larger interval than (11) would do, as shown by the example

$$F_N(z) = z + \sum_{j=2}^N (-1)^{j-1} a_j^{(0)} z^j.$$

Corollary 3. *The Koebe radius R_n^{tr} for typically real polynomials satisfies*

$$R_N^{\text{tr}} \leq \frac{1}{4}\sec^2\frac{\pi}{N+2}. \tag{12}$$

[1] Recall that a function f is *typically real* in $\mathbb{D}$ if it is real at every real point of the disc, and at all other points we have $\Im(f(z))\Im(z) > 0$.

Let us note that in [4,21] Suffridge polynomials were used to approximate the Koebe function. Several extremal properties of Suffridge polynomials were established in [4,6]. In particular, the following is valid.

Theorem 3. *The minimal simply connected set that contains $F(\mathbb{D})$ for any polynomial map $F_N(z) = \sum_{j=1}^{N} a_j z^j$ with $F_N(1) = 1$ contains the interval*

$$\left(-\tan^2 \frac{\pi}{2(N+1)}, 1\right). \tag{13}$$

If F_N is typically real then the interval (13) is covered by $F_N(\mathbb{D})$.

Some remarks. Note that the different normalization in problems (1) and (2) leads not only to a different value but also to different extremal polynomials, which turn out to be new, and to a different interpretation as a problem of geometric complex analysis.

Further, let us compare our problem with the problem of sharp constants for the polynomial Bohr radius. Namely, in 1914 Harold Bohr [2] came up with the following

Theorem *(H. Bohr). Suppose that the power series $\sum a_k z^k$ converges in $\mathbb{D}$, and $|\sum a_k z^k| < 1$. Then $\sum |a_k z^k| < 1$ when $|z| < 1/3$. Moreover, the radius $1/3$ is the best possible.*

In 2004 Zdenka Guadarrama [15] introduced the Bohr radius for polynomials. In 2007, Richard Fournier [12] found that the exact value of the radius is the smallest root of the determinant of a specific Toeplitz-type matrix. To compute that value is still an open problem. Let us note that, in our case, the main problem was similar, just our matrix was 5-diagonal, which allowed us to find the determinant.

Harold Boas and Dmitry Khavinson [1] found that the mutidimensional Bohr radius vanishes with dimension increasing at the rate reciprocal to the root of the dimension. That was quite surprising. They posed the problem of finding the sharp value for a fixed dimension, still open. The asymptotics was found in 2015 by Cheng Chu [3].

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