C. R. Acad. Sci. Paris, Ser. I 357 (2019) 382–388

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Harmonic analysis

Failure of the Hörmander kernel condition for multilinear Calderón–Zygmund operators ☆

Insuffisance de la condition de noyau de Hörmander pour les opérateurs multilinéaires de Calderón–Zygmund

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ARTICLE INFO

Article history:
Received 10 December 2018
Accepted 5 April 2019
Available online 19 April 2019

Presented by Yves Meyer

ABSTRACT

It is well known that the Hörmander smoothness condition $\sup_{y\neq 0}\int_{|x|\geq 2|y|}|K(x-y)-K(x)|\,dx<\infty$ implies weak-type $(1,1)$ estimates for associated L^2-bounded Calderón–Zygmund operators. It has been an open question to know whether Hörmander's condition also suffices to guarantee weak-type $(1,1,1/2)$ estimates for bilinear Calderón–Zygmund operators that are bounded at one point. In this paper, we provide a negative answer to this question.

RÉSUMÉ

Il est bien connu que la condition de lissage de Hörmander $\sup_{y\neq 0}\int_{|x|\geq 2|y|}|K(x-y)-K(x)|\,dx<\infty$ implique des estimations faibles de type $(1,1)$ pour les opérateurs de Calderón–Zygmund L^2-bornés. La question s'est alors posée de savoir si cette condition de Hörmander est également suffisante pour assurer des estimations faibles de type $(1,1,1/2)$ pour les opérateurs bilinéaires de Calderón–Zygmund qui sont bornés en un point. Nous donnons ici une réponse négative à cette question.

1. Introduction

Hörmander's [12] adaptation of the Calderón–Zygmund theorem says that an L^2-bounded convolution operator associated with a kernel K on $\mathbb{R}^d$ satisfying the smoothness condition

☆ The first author was supported by the Simons Foundation (No. 315380). The second author was supported by the NNSF of China (No. 11701583), the Guangdong Natural Science Foundation (No. 2017A030310054) and the Fundamental Research Funds for the Central Universities (No. 17lgpy11).

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https://doi.org/10.1016/j.crma.2019.04.002
1631-073X/

$$\|K\|_H = \sup_{y\neq 0} \int_{|x|\geq 2|y|} |K(x-y) - K(x)|\,\mathrm{d}x < \infty \tag{1}$$

is also bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$. By duality and interpolation, this classical result implies that the operator also admits an L^p-bounded extension for all $p \in (1,\infty)$. Recent interest in multilinear extensions of the Calderón–Zygmund theory has led to the development of multilinear harmonic analysis; see [7, Chapter 7] and [17]. This area was introduced by Coifman and Meyer in their seminal work [3], [4], [5]. A fundamental result in this theory is that, if an m-linear Calderón–Zygmund operator is bounded from $L^2 \times \cdots \times L^2$ to $L^{2/m}$ and its kernel K satisfies an appropriate size condition and a standard Lipschitz smoothness condition on $\mathbb{R}^{md}$, then it is bounded from $L^1 \times \cdots \times L^1$ to $L^{1/m,\infty}$; this result implies strong boundedness for the operator from the product of Lebesgue spaces to another Lebesgue space L^p in the largest range of indices possible, and also implies weak-type boundedness at the endpoints. Boundedness in the region where the target space is L^p with $p > 1$ was first proved by Coifman and Meyer [4], [5], and was extended to the case $p \leq 1$ by Kenig and Stein [13], and independently by Grafakos and Torres [11]. A natural question, inspired by linear theory, is whether this result also holds if the kernel K, which is a function on $\mathbb{R}^{md} \setminus \{0\}$, satisfies only Hörmander's condition (1). This question has been around since 2002 and has attracted some attention. In this note, we provide a negative answer to it. Our argument is mainly inspired by two ingredients related to bilinear rough singular integrals. The first one is a reinforced and quantitative version of the counterexample in [6], while the second one is the $L^2 \times L^2 \to L^1$ boundedness of bilinear rough singular integrals recently obtained in [8] and [9].

Our counterexample is a homogeneous kernel, i.e. a kernel that has the form:

$$K_\Omega(x_1,x_2) = \Omega((x_1,x_2)/|(x_1,x_2)|)|(x_1,x_2)|^{-2d}, \qquad (x_1,x_2) \in \mathbb{R}^{2d}$$

where Ω is integrable on the sphere $\mathbb{S}^{2d-1}$ with vanishing integral. The associated bilinear Calderón–Zygmund operator T_{K_Ω} is then defined as

$$T_{K_\Omega}(f,g)(x) = \mathrm{p.v.} \int_{\mathbb{R}^{2d}} K_\Omega(x-y_1, x-y_2) f(y_1) g(y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2.$$

We prove the following result:

Theorem 1. *Let $1 \leq q < \infty$. There exists an odd function Ω in $L^q(\mathbb{S}^{2d-1})$ such that the associated kernel K_Ω satisfies the Hörmander kernel condition (1), but the associated bilinear Calderón–Zygmund operator T_{K_Ω} does not map $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d) \to L^{p,\infty}(\mathbb{R}^d)$ whenever $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $1 \leq p_1, p_2 \leq \infty$ and $\frac{1}{p} + \frac{2d-1}{q} > 2d$. In particular, this operator is not of weak type $(1,1,\frac{1}{2})$ when $1 \leq q < \frac{2d-1}{2d-2}$.*

If $\Omega \in L^q(\mathbb{S}^{2d-1})$ with $q \geq 2$, then T_{K_Ω} is always $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ bounded, see [8]; this result was later extended to $\frac{4}{3} < q \leq \infty$ in [9]. Thus Theorem 1 yields the following corollary:

Corollary 2. *Let $d \in \{1,2\}$. There exists an odd function Ω on $\mathbb{S}^{2d-1}$ such that K_Ω satisfies Hörmander's condition (1) and the associated operator T_{K_Ω} is bounded from $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$, but is unbounded from $L^{p_1}(\mathbb{R}^d) \times L^{p_2}(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{R}^d)$ whenever $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $1 \leq p_1, p_2 \leq \infty$ and $p < \frac{4}{2d+3}$. In particular, this operator is not of weak type $(1,1,\frac{1}{2})$.*

Remark 1. To obtain, via these techniques, an example of an $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ bounded bilinear Calderón–Zygmund operator whose kernel satisfies Hörmander's condition (1) but which does not satisfy a weak-type $(1,1,\frac{1}{2})$ estimate in an arbitrary dimension d, we would need to know that

$$\|T_{K_\Omega}\|_{L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)\to L^1(\mathbb{R}^d)} \leq C\|\Omega\|_{L^q(\mathbb{S}^{2d-1})} \tag{2}$$

for all $q > 1$; but (2) remains open, as of this writing, for $1 < q \leq \frac{4}{3}$.

Other versions of the Hörmander kernel condition in the multilinear setting are given in [16], [15] and [2]; these conditions are weaker than (1), so our example applies also in that case. Our result should be contrasted with the positive result in [18] concerning a stronger geometric version of condition (1).

Additionally, it was observed in [11] that, if $\Omega \in L^1(\mathbb{R})$ is an odd function, then the boundedness of T_{K_Ω} can be obtained as a consequence of the uniform boundedness of the bilinear Hilbert transforms, see [10], [14]. Thus, in particular, T_{K_Ω} is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ whenever the triple $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ belongs to the hexagon $\mathcal{H}$ defined by the relations $1 < p_1, p_2, p < \infty$, $\frac{1}{p_2} + \frac{1}{p_2} = \frac{1}{p}$ and

$$\left|\frac{1}{p_1} - \frac{1}{p_2}\right| < \frac{1}{2}, \qquad \left|\frac{1}{p_1} - \frac{1}{p'}\right| < \frac{1}{2}, \qquad \left|\frac{1}{p_2} - \frac{1}{p'}\right| < \frac{1}{2},$$

where $p' = \frac{p}{p-1}$. We note that this hexagon contains points $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ with $p > 1$ arbitrarily close to 1. Another corollary of Theorem 1 is the following.

Corollary 3. *There exists an odd function Ω on $\mathbb{S}^1$ such that the kernel K_Ω satisfies the 2-dimensional Hörmander condition (1) and the associated operator T_{K_Ω} is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ whenever $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p}) \in \mathcal{H}$, but does not map $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ if $0 < p < 1$, $1 \le p_1, p_2 \le \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.*

For clarity, we prove the one-dimensional version of Theorem 1 in the next section. The proof in the d-dimensional case is given in Section 3; this contains an additional perturbation argument. We verify that K_Ω satisfies (1) in Section 4. In Section 5, we briefly discuss the multilinear situation. The notations $A \gtrsim B$ and $A \lesssim B$ mean that $A \ge cB$ and $A \le cB$, where c is an inessential constant, while $A \sim B$ means both $A \gtrsim B$ and $A \lesssim B$.

2. Proof of Theorem 1 when $d = 1$

Define points on the circle $\mathbb{S}^1$

$$a_n = \left(\cos\left(\frac{\pi}{4} + \frac{\pi}{2^n}\right), \sin\left(\frac{\pi}{4} + \frac{\pi}{2^n}\right)\right)$$

and define circular arcs I_n^+ with endpoints a_n and a_{n+1} for $n = 10, 11, 12, \dots$ Let I_n^- be the reflection about the origin of I_n^+. We observe that the length ℓ_n of both I_n^+ and I_n^- is approximately 2^{-n}. Consider the function

$$\Omega = \sum_{n=10}^{\infty} h_n \big(\chi_{I_n^+} - \chi_{I_n^-}\big)$$

where $h_n = 2^{n\delta}$ for some $\delta < 1/q$. Note that

$$\|\Omega\|_{L^q(\mathbb{S}^1)} \le c\Big(\sum_{n=10}^{\infty} h_n^q \ell_n\Big)^{\frac{1}{q}} \le c\Big(\sum_{n=10}^{\infty} 2^{n\delta q - n}\Big)^{\frac{1}{q}} < \infty$$

and that Ω is an odd function on $\mathbb{S}^1$.

For $0 < \varepsilon < \frac{1}{100}$, define $f_\varepsilon = (2\varepsilon)^{-\frac{1}{p_1}} \chi_{[-\varepsilon,\varepsilon]}$, $g_\varepsilon = (2\varepsilon)^{-\frac{1}{p_2}} \chi_{[-\varepsilon,\varepsilon]}$; these functions satisfy $\|f_\varepsilon\|_{L^{p_1}} = \|g_\varepsilon\|_{L^{p_2}} = 1$.

Let us fix an $x \in \mathbb{R}$ such that $\frac{11}{10} \le x \le \frac{12}{10}$. Then we have

$$|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)(x)| \ge (2\varepsilon)^{-\frac{1}{p_1}} (2\varepsilon)^{-\frac{1}{p_2}} \int\limits_{|y_1|<\varepsilon} \int\limits_{|y_2|<\varepsilon} \frac{\Omega\big(\frac{(x-y_1, x-y_2)}{|(x-y_1, x-y_2)|}\big)}{|(x-y_1, x-y_2)|^2} \, dy_1 \, dy_2. \tag{3}$$

Let $P_{\varepsilon,x}$ be all projections of points of the form $(x - y_1, x - y_2)$ onto the circle $\mathbf{S}^1$, where (y_1, y_2) is an arbitrary point in $(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$. As the point $(x - y_1, x - y_2)$ lies near the positive diagonal (that forms 45° with the positive horizontal axis), this projection will only intersect circular caps I_n^+ and will never intersect caps I_n^-. In this case, every term in the sum that defines Ω and appears in (3) is positive. We obtain

$$|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)(x)| \ge c\varepsilon^{-\frac{1}{p_1}} \varepsilon^{-\frac{1}{p_2}} \varepsilon \sum_{\substack{n \ge 10 \\ I_n^+ \subseteq P_{\varepsilon,x}}} \ell_n h_n$$

as $|(x - y_1, x - y_2)|^2 \sim 1$ and if $I_n^+ \subseteq P_{\varepsilon,x}$, then the set of those (y_1, y_2) satisfying $|y_1| < \varepsilon$, $|y_2| < \varepsilon$ and $(x - y_1, x - y_2)/|(x - y_1, x - y_2)| \in I_n^+$ has measure comparable to $\varepsilon \ell_n$, since x is so close to 1. As $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, we obtain, for $\frac{11}{10} \le x \le \frac{12}{10}$, that

$$|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)(x)| \gtrsim \varepsilon^{-\frac{1}{p}+1} \sum_{\substack{n: \\ 2^{-n} < c\varepsilon}} 2^{n\delta - n} \gtrsim \varepsilon^{2-\frac{1}{p}-\delta},$$

which yields that $\|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)\|_{L^{p,\infty}(\mathbb{R})} \gtrsim \varepsilon^{2-\frac{1}{p}-\delta}$, and

$$\|T_{K_\Omega}\|_{L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{p,\infty}(\mathbb{R})} \ge \frac{\|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)\|_{L^{p,\infty}(\mathbb{R})}}{\|f_\varepsilon\|_{L^{p_1}(\mathbb{R})} \|g_\varepsilon\|_{L^{p_2}(\mathbb{R})}} \gtrsim \varepsilon^{2-\frac{1}{p}-\delta}.$$

Choosing δ sufficiently close to $1/q$, we conclude that, if $2 - \frac{1}{p} - \frac{1}{q} < 0$, then

$$\|T_{K_\Omega}\|_{L^{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})\to L^{p,\infty}(\mathbb{R})}=\infty.$$

To complete the proof of the main theorem, we need to know that K_Ω satisfies Hörmander's condition (1). For this, we prove the following lemma in which points in $\mathbb{R}^2$ will be denoted by capital letters.

Lemma 4. *Let $r>1$ and $\Omega_t=t^{-\frac{1}{r}}\chi_{I_t}$, where I_t is a circular arc of small length $t>0$ on the circle $\mathbb{S}^1$. Then there is a constant $C_r<\infty$ such that*

$$\sup_{t>0}\sup_{Y\neq 0}\int_{|X|\geq 2|Y|}\left|K_{\Omega_t}(X-Y)-K_{\Omega_t}(X)\right|\mathrm{d}X\leq C_r.$$

As the proof of Lemma 4 is contained in that of Lemma 5 proved later, we do not include it here. Since $\delta<\frac{1}{q}\leq 1$, we can choose r such that $\delta<\frac{1}{r}<1$, then Lemma 4 gives that

$$\begin{aligned}\|K_\Omega\|_H&\leq\sum_{n=10}^{\infty}h_n\ell_n^{\frac{1}{r}}\Big(\Big\|\frac{1}{\ell_n^{\frac{1}{r}}}\chi_{I_n^+}\Big\|_H+\Big\|\frac{1}{\ell_n^{\frac{1}{r}}}\chi_{I_n^-}\Big\|_H\Big)\\&\leq C\sum_{n=10}^{\infty}h_n\ell_n^{\frac{1}{r}}=C\sum_{n=10}^{\infty}2^{n\delta-n\frac{1}{r}}\end{aligned}$$

and this sum is convergent. This concludes the proof of Theorem 1 when $d=1$.

3. Proof of Theorem 1 when $d\geq 2$

We now extend the proof to higher dimensions. Fix a point

$$a=(\tfrac{1}{\sqrt{2d}},\dots,\tfrac{1}{\sqrt{2d}})\in\mathbb{S}^{2d-1}$$

and for $n=10,11,12,\dots$ define spherical annuli

$$A_n^+=\mathbb{S}^{2d-1}\cap\Big(B(a,2^{-n})\setminus B(a,2^{-n-1})\Big).$$

Let A_n^- be the reflection about the origin of A_n^+. We observe that the measure ν_n of both A_n^+ and A_n^- is approximately $2^{-n(2d-1)}$. Consider the function

$$\Omega=\sum_{n=10}^{\infty}h_n\big(\chi_{A_n^+}-\chi_{A_n^-}\big)$$

where $h_n=2^{n\delta}$ for some $\delta<\frac{2d-1}{q}$. Note that

$$\|\Omega\|_{L^q(\mathbb{S}^{2d-1})}\leq c\Big(\sum_{n=10}^{\infty}h_n^q\nu_n\Big)^{\frac{1}{q}}\leq c\Big(\sum_{n=10}^{\infty}2^{n(\delta q-(2d-1))}\Big)^{\frac{1}{q}}<\infty$$

and that Ω is an odd function on $\mathbb{S}^{2d-1}$.

For $0<\varepsilon<\frac{1}{100d}$, define $f_\varepsilon=(2\varepsilon)^{-\frac{d}{p_1}}\chi_{[-\varepsilon,\varepsilon]^d}$, $g_\varepsilon=(2\varepsilon)^{-\frac{d}{p_2}}\chi_{[-\varepsilon,\varepsilon]^d}$; these functions satisfy $\|f_\varepsilon\|_{L^{p_1}}=\|g_\varepsilon\|_{L^{p_2}}=1$.

Let us fix an interval on the diagonal line in $\mathbb{R}^d$ defined by

$$I_d=\Big\{x\in\mathbb{R}^d: x_1=x_2=\cdots=x_d\in[\tfrac{1}{\sqrt{d}}+\tfrac{1}{100d},\tfrac{1}{\sqrt{d}}+\tfrac{2}{100d}]\Big\}.\tag{4}$$

Then, for $x\in I_d$, we have

$$|T_{K_\Omega}(f_\varepsilon,g_\varepsilon)(x)|\geq\quad(2\varepsilon)^{-\frac{d}{p_1}}(2\varepsilon)^{-\frac{d}{p_2}}\int_{[-\varepsilon,\varepsilon]^d}\int_{[-\varepsilon,\varepsilon]^d}\frac{\Omega\big(\frac{(x-y_1,x-y_2)}{|(x-y_1,x-y_2)|}\big)}{|(x-y_1,x-y_2)|^2}\,\mathrm{d}y_1\,\mathrm{d}y_2.\tag{5}$$

Let $P_{\varepsilon,x}$ be the set of all projections onto the sphere $\mathbb{S}^{2d-1}$ of points of the form $(x-y_1,x-y_2)$, where (y_1,y_2) is an arbitrary point in $[-\varepsilon,\varepsilon]^{2d}$. As the point $(x-y_1,x-y_2)$ lies near the positive diagonal, this projection will only intersect spherical annuli A_n^+ and will never intersect annuli A_n^-. In this case, every term in the sum that defines Ω and appears in (5) is positive. We obtain

$$|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)(x)| \geq c\varepsilon^{-\frac{d}{p_1}}\varepsilon^{-\frac{d}{p_2}}\varepsilon \sum_{\substack{n\geq 10 \\ A_n^+\subseteq P_{\varepsilon,x}}} v_n h_n$$

as $|(x-y_1, x-y_2)|^2 \sim 1$ and if $A_n^+ \subseteq P_{\varepsilon,x}$, then the set of those (y_1, y_2) satisfying $(y_1, y_2) \in [-\varepsilon, \varepsilon]^{2d}$ and $(x-y_1, x-y_2)/|(x-y_1, x-y_2)| \in A_n^+$ has measure comparable to εv_n, since x is so close to the unit sphere. Since $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, we obtain

$$|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)(x)| \gtrsim \varepsilon^{-\frac{d}{p}+1} \sum_{\substack{n: \\ 2^{-n}<c_d\varepsilon}} 2^{n\delta-n(2d-1)} \gtrsim \varepsilon^{(2-\frac{1}{p})d-\delta},$$

whenever $x \in I_d$. In particular, in the last summation the term with $2^{-n_\varepsilon} \sim \frac{c_d}{10}\varepsilon$ would contribute essentially the same lower bound $\varepsilon^{(2-\frac{1}{p})d-\delta}$.

We now fix a point $x_0 \in I_d$. For any x such that $|x-x_0| \leq c'_d\varepsilon$ with c'_d a small positive constant, we define $P_{\varepsilon,x}$ as the projection of $(x,x)+[-\varepsilon,\varepsilon]^{2d}$ onto $\mathbb{S}^{2d-1}$. Recalling that P_{ε,x_0} contains $A_{n_\varepsilon}^+$ and that the distance between $A_{n_\varepsilon}^+$ and $\mathbb{S}^{2d-1} \setminus P_{\varepsilon,x_0}$ is greater than $\frac{c_d}{2}\varepsilon$, we obtain that $A_{n_\varepsilon}^+ \subset P_{\varepsilon,x}$ if c'_d is small enough, since the distance between the boundary of P_{ε,x_0} and the boundary of $P_{\varepsilon,x}$ is bounded by $c'_d\varepsilon$. In summary, for any point $x \in N_\varepsilon$, the $c'_d\varepsilon$-neighborhood of I_d with volume about ε^{d-1}, we have

$$|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)(x)| \gtrsim \varepsilon^{-\frac{d}{p}+1} 2^{n_\varepsilon(\delta-2d+1)} \sim \varepsilon^{(2-\frac{1}{p})d-\delta}. \tag{6}$$

This yields

$$\|T_{K_\Omega}\|_{L^{p_1}(\mathbb{R}^d)\times L^{p_2}(\mathbb{R}^d)\to L^{p,\infty}(\mathbb{R}^d)} \geq \frac{\|T_{K_\Omega}(f_\varepsilon, g_\varepsilon)\|_{L^{p,\infty}(\mathbb{R}^d)}}{\|f_\varepsilon\|_{L^{p_1}(\mathbb{R}^d)}\|g_\varepsilon\|_{L^{p_2}(\mathbb{R}^d)}} \gtrsim \varepsilon^{\frac{d-1}{p}+(2-\frac{1}{p})d-\delta}.$$

Choosing δ sufficiently close to $\frac{2d-1}{q}$, we conclude that, if

$$2d - \frac{1}{p} - \frac{2d-1}{q} < 0,$$

then

$$\|T_{K_\Omega}\|_{L^{p_1}(\mathbb{R}^d)\times L^{p_2}(\mathbb{R}^d)\to L^{p,\infty}(\mathbb{R}^d)} = \infty.$$

We have the following d-dimensional extension of Lemma 4.

Lemma 5. *Let $r > \frac{1}{2d-1}$ and $\Omega_t = t^{-\frac{1}{r}}\chi_{A_t}$, where A_t is a spherical cap of small radius t on the sphere $\mathbb{S}^{2d-1}$. Then there is a constant C that depends on d and r such that*

$$\sup_{t>0}\sup_{Y\neq 0} \int_{|X|\geq 2|Y|} \left|K_{\Omega_t}(X-Y) - K_{\Omega_t}(X)\right| \mathrm{d}X \leq C. \tag{7}$$

We note that each spherical annulus A_n^+, A_n^- can be written as $B_n^+ \setminus C_n^+$ or $B_n^- \setminus C_n^-$, where B_n^+, C_n^+ and B_n^-, C_n^- are spherical caps of radius approximately 2^{-n} centered at a and $-a$, respectively. Therefore, assuming Lemma 5, we obtain

$$\begin{aligned}\|K_\Omega\|_H &\leq \sum_{n=10}^{\infty} h_n 2^{-\frac{n}{r}} \left\| 2^{\frac{n}{r}} \left(\chi_{B_n^+} - \chi_{C_n^+} - \chi_{B_n^-} + \chi_{C_n^-}\right)\right\|_H \\ &\leq C\sum_{n=10}^{\infty} h_n 2^{-\frac{n}{r}} = C\sum_{n=10}^{\infty} 2^{n\delta-\frac{n}{r}}\end{aligned}$$

and this sum is convergent if we choose $\delta < \frac{1}{r} < 2d-1$, which is possible since $\delta < \frac{2d-1}{q} \leq 2d-1$.

This finishes the proof of Theorem 1 for $d \geq 2$ assuming Lemma 5, which is proved in the next section.

4. Proof of Lemma 5

Let $X \in \mathbb{R}^{2d}$ and $X' = X/|X|$. It suffices to prove that

$$\int_{|X|\geq 2|Y|} \left|\Omega_t((X-Y)') - \Omega_t(X')\right| \frac{\mathrm{d}X}{|X-Y|^{2d}} \leq C < \infty$$

as the part

$$\int_{|X|\geq 2|Y|} \left| \frac{\Omega_t(X')}{|X-Y|^{2d}} - \frac{\Omega_t(X')}{|X|^{2d}} \right| \mathrm{d}X$$

is trivially bounded by $\|\Omega_t\|_{L^1(\mathbb{S}^{2d-1})} \leq C$ since $r > \frac{1}{2d-1}$.

But $|X-Y| \sim |X|$, and so we look at

$$\int_{2|Y|}^{\infty} \int_{\mathbb{S}^{2d-1}} \left|\Omega_t((s\theta - Y)') - \Omega_t(\theta)\right| \mathrm{d}\theta \frac{\mathrm{d}s}{s}. \tag{8}$$

The interior integral vanishes if both terms $\chi_{A_t}((s\theta - Y)')$ and $\chi_{A_t}(\theta)$ are 1 or 0. Thus we may consider the case when one term is one and the other is zero. In this case, we estimate the expression on the left in (7) by

$$t^{-\frac{1}{r}} \int_{2|Y|}^{\infty} |\{\theta \in A_t, \left(\theta - \tfrac{Y}{s}\right)' \notin A_t\}| \frac{\mathrm{d}s}{s} + t^{-\frac{1}{r}} \int_{2|Y|}^{\infty} |\{\theta \notin A_t, \left(\theta - \tfrac{Y}{s}\right)' \in A_t\}| \frac{\mathrm{d}s}{s}.$$

Both A_t and the set of all $\theta \in \mathbb{S}^{2d-1}$ for which $\left(\theta - \frac{Y}{s}\right)' \in A_t$ have spherical measure at most ct^{2d-1}, where to show the latter we use the fact that $|\frac{Y}{s}| \leq \frac{1}{2}$. Let us now assume that $\frac{|Y|}{s} \leq \frac{t}{100} \ll 1$. In the first integral, the set has spherical measure at most $c\frac{|Y|}{s}t^{2d-2}$, because it is comparable to $|A_t' \setminus A_t|$ with A_t' an appropriate rotation of A_t with displacement $\sim \frac{|Y|}{s}$. Similarly, the set in the second integral has spherical measure at most $c\frac{|Y|}{s}t^{2d-2}$ as well. We therefore obtain the estimate for (8)

$$ct^{-\frac{1}{r}} \left[\int_{2|Y|}^{\frac{100|Y|}{t}} t^{2d-1} \frac{\mathrm{d}s}{s} + \int_{\frac{100|Y|}{t}}^{\infty} \frac{|Y|}{s} t^{2d-2} \frac{\mathrm{d}s}{s} \right] \leq ct^{-\frac{1}{r}} [t^{2d-1} \log(t^{-1})] \leq C,$$

where $C < \infty$, since $2d - 1 - \frac{1}{r} > 0$ and $t \leq 1$. This proves (7).

5. The multilinear case

The argument needed to prove a multilinear version of Theorem 1 is similar to the one performed above. We sketch it below for completeness.

Let Ω be an integrable function on the sphere $\mathbb{S}^{md-1}$ with vanishing integral. We define

$$K_\Omega(x_1, \ldots, x_m) = \Omega((x_1, \ldots, x_m)/|(x_1, \ldots, x_m)|)|(x_1, \ldots, x_m)|^{-md}$$

for $(x_1, \ldots, x_m) \in \mathbb{R}^{md}$. The m-linear rough singular integral operator T_{K_Ω} is then defined by

$$T_{K_\Omega}(f_1, \ldots, f_m)(x) = \text{p.v.} \int_{\mathbb{R}^{md}} K_\Omega(x - y_1, \ldots, x - y_m) f_1(y_1) \cdots f_m(y_m) \, \mathrm{d}y_1 \cdots \mathrm{d}y_m.$$

Let $1 \leq q < \infty$. We choose $a = (\frac{1}{\sqrt{md}}, \ldots, \frac{1}{\sqrt{md}}) \in \mathbb{S}^{md-1}$, and define $\Omega = \sum_n h_n(\chi_{A_n^+} - \chi_{A_n^-})$ with $h_n = 2^{n\delta}$ and $\delta < (md-1)/q$. Here, A_n^+ is a spherical annulus centered at point a whose radius is 2^{-n} and measure $\sim 2^{-(md-1)n}$, and A_n^- is its reflection with respect to the origin. We can easily check that $\Omega \in L^q(\mathbb{S}^{md-1})$.

Let $1 \leq p_1, \ldots, p_m \leq \infty$ and $p > 0$ be such that $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$. We take $f_j = (2\varepsilon)^{-d/p_j} \chi_{[-\varepsilon,\varepsilon]^d}$; then $\|f_j\|_{L^{p_j}(\mathbb{R}^d)} = 1$ for $j = 1, \ldots, m$. Let I_d be as in (4) and let N_ε be a $c_d'\varepsilon$-neighborhood of I_d, then we can verify that

$$T_{K_\Omega}(f_1, \ldots, f_m)(x) \geq c\varepsilon^{-\frac{d}{p}} \varepsilon \sum_{n:\, 2^{-n} \leq \varepsilon} |A_n^+| h_n \sim c\varepsilon^{-\frac{d}{p} + md - \delta}$$

for all $x \in N_\varepsilon$. Therefore,

$$\|T_{K_\Omega}\|_{L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_m}(\mathbb{R}^d) \to L^{p,\infty}(\mathbb{R}^d)} \gtrsim \varepsilon^{md - \frac{1}{p} - \delta},$$

which tends to ∞ as $\varepsilon \to 0$ when $md < \frac{1}{p} + \frac{md-1}{q}$ if we choose δ close to $\frac{md-1}{q}$. It is straightforward to verify Lemma 5 in the multilinear setting under the condition $r > \frac{1}{md-1}$. In summary, we have showed the following.

Proposition 6. *For any $1 \le q < \infty$, there is an odd function Ω in $L^q(\mathbb{S}^{md-1})$ such that the associated kernel K_Ω satisfies Hörmander's condition (1) but the Calderón–Zygmund operator T_{K_Ω} does not map $L^{p_1}(\mathbb{R}^d) \times \cdots \times L^{p_m}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ whenever $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$, $1 \le p_1, \ldots, p_m \le \infty$, and $\frac{1}{p} + \frac{md-1}{q} > md$. In particular, this operator is not of weak type $(1, \ldots, 1, \frac{1}{m})$ when $1 \le q < \frac{md-1}{m(d-1)}$.*

Remark 2. It is known from [1] that the m-linear operator T_{K_Ω} is bounded from $L^2(\mathbb{R}^d) \times \cdots \times L^2(\mathbb{R}^d)$ to $L^{2/m}(\mathbb{R}^d)$ whenever $\Omega \in L^q(\mathbb{S}^{md-1})$ with $q > \frac{2m}{m+1}$. Thus, in the multilinear case, boundedness on the product of L^2 spaces and Hörmander's condition are not sufficient to yield the weak-type $(1, 1, \ldots, 1, 1/m)$ endpoint when $d \le 2$.

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