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Algebraic geometry

A note on 1-cycles on the moduli space of rank-2 bundles over a curve

Une note sur les 1-cycles des espaces de module des fibrés vectoriels de rang 2 sur une courbe

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ABSTRACT

Over a smooth complex projective curve of genus ≥ 3, we study 1-cycles on the moduli space of rank-2 stable vector bundles with fixed determinant of degree 1. We show the first Chow group of the moduli space is isomorphic to the zeroth Chow group of the curve.

RÉSUMÉ

Nous étudions les 1-cycles des espaces de module de fibrés vectoriels de rang 2 et déterminant de degré 1 fixé, sur une courbe complexe, projective, lisse, de genre ≥ 3. Nous montrons que le groupe de Chow d'indice 1 des espaces de module est isomorphe au groupe de Chow d'indice 0 de la courbe.

The moduli space of vector bundles over a smooth projective curve has been intensively studied for decades. But there are still basic questions to be answered. For example, the Chow groups of the moduli space are not known in general, although there are conjectures [3] in the rank-2 case. The Picard group of the moduli space, singular or not, is isomorphic to $\mathbb{Z}$ [7,4]. Our study, which is modest, lies on the other end. We will study the Chow group of 1-cycles on the moduli space of rank-2 vector bundles with a fixed determinant of degree 1. In this note, we will improve Choe–Hwang's result [2] using [8].

Let X be a smooth complex projective curve of genus $g \geq 3$. Let x be a point on X. We denote by M_x the moduli space of rank-2 stable vector bundles on X with determinant $\mathcal{O}_X(x)$. The isomorphism class of M_x is independent of the choice of x. The variety M_x is a nonsingular Fano variety of dimension $3g-3$. According to [1], the singular cohomology of M_x with integral coefficients is torsion free.

We want to prove the following.

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Theorem. *There is an isomorphism* $\mathrm{CH}_1(M_x) \cong \mathrm{CH}_0(X)$.

This is a slight extension of Choe–Hwang's result, which is over rational coefficients.

Let J be the Jacobian of X and $\mathscr{P} \to J \times X$ be the Poincaré bundle. For $L \in J$, let

$$D_L = \{[E] \in M_x \mid \exists L \hookrightarrow E\} \cong \mathbb{P}\,\mathrm{Ext}^1(L^{-1}(x), L).$$

These form a family over J:

$$\pi : \mathcal{D} = \mathbb{P} R^1 \pi_{J*} \mathcal{H}\mathrm{om}(\mathscr{P}^{-1} \otimes \pi_X^* \mathcal{O}(x), \mathscr{P}) \to J.$$

Let $\rho \colon \mathcal{G} \to J$ be the Grassmannian of lines on D_L and $\mathcal{U} \to \mathcal{G} \times_J \mathcal{D}$ be the corresponding universal family. There is a natural embedding $\mathcal{U} \hookrightarrow \mathcal{G} \times M_x$ which makes $\mathcal{U}$ into the universal family of lines on M_x [5]. Then, using $\mathcal{U}$ as the correspondence,

$$\begin{array}{ccccc} & & \mathcal{U} & & \\ & \swarrow & & \searrow & \\ J \longleftarrow & \mathcal{G} & & & M_x \end{array}, \qquad (1)$$

we can define a homomorphism:

$$\mathrm{CH}_0(\mathcal{G}) \to \mathrm{CH}_1(M_x).$$

On the other hand, $\mathrm{CH}_0(\mathcal{G}) \cong \mathrm{CH}_0(J)$, since $\mathcal{G}$ is a Grassmannian bundle over J. We thus have a homomorphism:

$$\chi : \mathrm{CH}_0(J) \to \mathrm{CH}_1(M_x).$$

Proposition 1. *The homomorphism* χ *is surjective.*

Let us first recall some basic facts and notions. Because the cohomology of M_x is torsion free, $H_2(M_x;\mathbb{Z})$ is torsion free and $H^2(M_x;\mathbb{Z}) \cong \mathbb{Z}$ [1, Proposition 9.13]. Therefore, $H_2(M_x;\mathbb{Z}) \cong \mathbb{Z}$. Because we have plenty of lines on M_x, the cycle map

$$\mathrm{CH}_1(M_x) \to H_2(M_x;\mathbb{Z})$$

is surjective.

Next, let us recall the construction of Hecke curves and their degenerations. Given a point $y \in X$, let M_{y-x} be the moduli space of semi-stable vector bundles of rank 2 whose determinant is $\mathcal{O}_X(y-x)$ and let $M^s_{y-x} \subset M_{y-x}$ be the open subset consisting of stable bundles. For any stable bundle $[E] \in M^s_{y-x}$ and any point $\zeta \in \mathbb{P}(E_y^\vee)$, we define $\varepsilon(E,\zeta)$ by the exact sequence:

$$0 \to \varepsilon(E,\zeta) \to E \xrightarrow{\zeta} \mathcal{O}_y \to 0.$$

Then $[\varepsilon(E,\zeta)]$ belongs to M_{-x} and its dual $[\varepsilon(E,\zeta)^*]$ belongs to M_x. As ζ varies on $\mathbb{P}(E_y^\vee)$, the family of vector bundles $\varepsilon(E,\zeta)^*$ defines a smooth rational curve on M_x. Thus, each point of M^s_{y-x} gives rise to a rational curve on M_x, defining a morphism:

$$\varphi_y \colon M^s_{y-x} \to \mathrm{CH}_1(M_x).$$

Let $\mathcal{H}_y$ be the closure of the image of φ_y. A rational curve on M_x is called a *Hecke curve* if it arises this way from a point of M^s_{y-x} for some $y \in X$.

Note that a general point of $\mathcal{H}_y$ represents a free curve of degree two (with respect to the ample generator of the Picard group of M_x) and, at some special point, Hecke curves degenerate into two lines.

Proof of Proposition 1. Let $\mathcal{H}_y$ be as above. Firstly, we claim that $\mathrm{CH}_1(M_x)$ is generated by irreducible reduced 1-cycles that are irreducible components of members of $\mathcal{H}_y$. There are elements in $\mathcal{H}_y$, which are unions of two lines. We denote one of these two lines by l. Let $[l]$ denote the homology class represented by l. Then $H_2(M_x;\mathbb{Z}) \cong \mathbb{Z}.[l]$. By the following exact sequence,

$$0 \to \mathrm{CH}_1(M_x)_{\mathrm{hom}} \to \mathrm{CH}_1(M_x) \to H_2(M_x;\mathbb{Z}) \to 0,$$

we know that, for each 1-cycle $D \in \mathrm{CH}_1(M_x)$, there exists an integer m such that $D - ml$ belongs to $\mathrm{CH}_1(M_x)_{\mathrm{hom}}$.

The following assertions are well known:

(1) the first Griffiths group $\mathrm{Griff}_1(X)$ ($\triangleq \mathrm{CH}_1(X)_{\mathrm{hom}}/\mathrm{CH}_1(x)_{\mathrm{alg}}$) is a birational invariant, see [9];
(2) the moduli space M_x is rational, see [6].

Thus, $\mathrm{Griff}_1(M_x)$ is trivial. It follows that $D-ml$ is algebraically equivalent to zero.

Since $\mathrm{CH}_1(x)_{\mathrm{alg}}$ is a divisible group, for any positive integer N, there exists a 1-cycle E_N such that $D-ml$ is rationally equivalent to NE_N. For the dominating family $\mathcal{H}_y$ of rational curves, by Proposition 3.1 of [8], there exists an integer N_0 such that for any 1-cycle D', N_0D' is rationally equivalent to a sum of rational curves, which are irreducible components of members of $\mathcal{H}_y$. So, there exists an E_{N_0} such that $D-ml$ is rationally equivalent to $N_0E_{N_0}$ and $N_0E_{N_0}$ is a sum of rational curves, which are irreducible components of members of $\mathcal{H}_y$. This completes the proof of our claim.

Since members of $\mathcal{H}_y$ always degenerate into lines, the morphism $\chi : \mathrm{CH}_0(J) \to \mathrm{CH}_1(M_x)$ is surjective. $\square$

Still using the diagram (1), we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{CH}_0(J)_{\mathrm{hom}} & \longrightarrow & \mathrm{CH}_0(J) & \longrightarrow & H_0(J;\mathbb{Z}) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle\chi} & & \big\downarrow & & \\ 0 & \longrightarrow & \mathrm{CH}_1(M_x)_{\mathrm{hom}} & \longrightarrow & \mathrm{CH}_1(M_x) & \longrightarrow & H_2(M_x;\mathbb{Z}) & \longrightarrow & 0. \end{array}$$

We can fill in the dashed arrow, which is also surjective by a simple diagram trace.

Let $I=\mathrm{CH}_0(J)_{\mathrm{hom}}$. Let I^{*2} be the Pontryagin square of I, namely, the subgroup generated by cycles of the form $\{a+b\}-\{a\}-\{b\}+\{o\}$, where we denote the identity element in J by o. There is a canonical isomorphism $J \cong I/I^{*2}$. The following is crucial in [2] and for us as well.

Proposition 2. *The homomorphism χ annihilates the subgroup I^{*2}.*

Therefore, the homomorphism χ thus induces a surjective homomorphism

$$\bar{\chi} : \mathrm{CH}_0(X)_{\mathrm{hom}} = J \twoheadrightarrow \mathrm{CH}_1(M_x)_{\mathrm{hom}}.$$

Furthermore, it was shown in [2, Proposition 10] that $\bar{\chi}$ is injective. Thus, $\bar{\chi}$ is an isomorphism. By identifying $H_0(X;\mathbb{Z})$ and $H_2(M_x;\mathbb{Z})$, we obtain an isomorphism

$$\mathrm{CH}_0(X) \overset{\cong}{\to} \mathrm{CH}_1(M_x).$$

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