C. R. Acad. Sci. Paris, Ser. I 357 (2019) 74–83

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Differential geometry

Hypoelliptic Laplacian and twisted trace formula

Laplacien hypoelliptique et formule des traces tordue

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ARTICLE INFO

Article history:
Received 18 July 2018
Accepted 16 November 2018
Available online 22 November 2018

Presented by the Editorial Board

ABSTRACT

In this Note, we give an explicit geometric formula for twisted orbital integrals using the method of the hypoelliptic Laplacian developed by Bismut. We apply this formula to evaluate the leading term in the asymptotic expansion of the equivariant Ray–Singer analytic torsion on compact locally symmetric spaces.

RÉSUMÉ

On donne une formule géométrique explicite pour des intégrales orbitales tordues en utilisant la méthode du laplacien hypoelliptique développée par Bismut. On utilise cette formule explicite pour évaluer le terme dominant dans l'asymptotique de la torsion équivariante de Ray–Singer sur un espace localement symétrique compact.

0. Introduction

Let G be a connected real reductive group, and let K be a maximal compact subgroup of G. Let $X = G/K$ be the associated symmetric space. In [4], the author constructed a family of hypoelliptic Laplacians $\mathcal{L}_b^X|_{b>0}$, which converges in the proper sense to a Bochner-like Laplacian $\mathcal{L}^X$ on X as $b \to 0$. Using a geometric description of the orbital integrals associated with semisimple elements in G, Bismut obtained an explicit formula for the semisimple orbital integrals for the heat kernel and the wave kernel of $\mathcal{L}^X$ [4, Theorems 6.1.1, 6.3.2]. In this Note, we introduce a twist $\sigma \in \mathrm{Aut}(G)$. We consider the σ-twisted orbital integrals associated with $\gamma \in G$. We show that if $\gamma\sigma$ is semisimple, we can adapt the method in [4] to get an explicit geometric formula for the σ-twisted orbital integrals associated with the heat kernel of $\mathcal{L}^X$.

As an application, following ideas in [7, Section 8], we use our explicit formula to compute the leading term in the asymptotic expansion of the equivariant Ray–Singer analytic torsions associated with a family of flat vector bundles $F_d|_{d\in\mathbb{N}}$ on compact locally symmetric spaces Z. We show that the leading term can be evaluated in terms of a finite set of locally computable invariants on the associated fixed point set in Z.

The proofs of the results contained in this Note are developed in [13].

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https://doi.org/10.1016/j.crma.2018.11.010

1. Hypoelliptic Laplacian

Let G be a connected real reductive group with Lie algebra $\mathfrak{g}$, and let $\theta \in \mathrm{Aut}(G)$ be a Cartan involution. Let K be the fixed point set of θ in G. Then K is a maximal compact subgroup of G, and let $\mathfrak{k}$ be its Lie algebra. Let $\mathfrak{p} \subset \mathfrak{g}$ be the eigenspace of θ associated with the eigenvalue -1. The Cartan decomposition of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}. \tag{1}$$

Put $m = \dim \mathfrak{p}$, $n = \dim \mathfrak{k}$.

Let B be a G and θ-invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$, which is positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. It induces a symmetric bilinear form B^* on $\mathfrak{g}^*$, which extends to a symmetric bilinear form on $\Lambda^\cdot(\mathfrak{g}^*)$.

Let $\sigma \in \mathrm{Aut}(G)$. We assume that σ commutes with θ and preserves B. In particular, σ preserves K. Let Σ^σ be the closed subgroup of $\mathrm{Aut}(G)$ generated by σ. Then Σ^σ is compact. Let $G^\sigma = G \rtimes \Sigma^\sigma$ be the semidirect product of G and Σ^σ, so that if $g, g' \in G$, $\tau, \tau' \in \Sigma^\sigma$, the group multiplication is given by

$$(g, \tau)(g', \tau') = (g\tau(g'), \tau\tau'). \tag{2}$$

Put

$$K^\sigma = K \rtimes \Sigma^\sigma. \tag{3}$$

In the sequel, we write $g\tau \in G^\sigma$ instead of (g, τ).

We now recall the construction of the hypoelliptic Laplacian in [4, Chapter 2]. Let $U\mathfrak{g}$ be the enveloping algebra of $\mathfrak{g}$, let $C^{\mathfrak{g}} \in U\mathfrak{g}$ be the Casimir operator associated with B. If $e_1, \cdots, e_{m+n}$ is a basis of $\mathfrak{g}$, and if $e_1^*, \cdots, e_{m+n}^*$ is the dual basis of $\mathfrak{g}$ with respect to B, then

$$C^{\mathfrak{g}} = -\sum_{i=1}^{m+n} e_i^* e_i. \tag{4}$$

Let $c(\mathfrak{g})$, $\widehat{c}(\mathfrak{g})$ be the Clifford algebras associated with $(\mathfrak{g}, B)$, $(\mathfrak{g}, -B)$. These algebras act on $\Lambda^\cdot(\mathfrak{g}^*)$. Let $\kappa^{\mathfrak{g}} \in \Lambda^3(\mathfrak{g}^*)$ be such that if $a, b, c \in \mathfrak{g}$,

$$\kappa^{\mathfrak{g}}(a, b, c) = B([a, b], c). \tag{5}$$

Let $\widehat{c}(-\kappa^{\mathfrak{g}}) \in \widehat{c}(\mathfrak{g})$ be the canonical element associated with $-\kappa^{\mathfrak{g}}$ so that

$$\widehat{c}(-\kappa^{\mathfrak{g}}) = -\frac{1}{6}\sum_{i,j,k} \kappa^{\mathfrak{g}}(e_i^*, e_j^*, e_k^*)\widehat{c}(e_i)\widehat{c}(e_j)\widehat{c}(e_k). \tag{6}$$

Let $\widehat{D}^{\mathfrak{g}} \in \widehat{c}(\mathfrak{g}) \otimes U\mathfrak{g}$ be the Dirac operator of Kostant [12], which is given by

$$\widehat{D}^{\mathfrak{g}} = \sum_{i=1}^{m+n} \widehat{c}(e_i^*)e_i + \frac{1}{2}\widehat{c}(-\kappa^{\mathfrak{g}}). \tag{7}$$

We have

$$\widehat{D}^{\mathfrak{g},2} = -C^{\mathfrak{g}} - \frac{1}{4}B^*(\kappa^{\mathfrak{g}}, \kappa^{\mathfrak{g}}). \tag{8}$$

We assume that $e_1, \cdots, e_m$ is an orthonormal basis of $(\mathfrak{p}, B|_{\mathfrak{p}})$, and that $e_{m+1}, \cdots, e_{m+n}$ is an orthonormal basis of $(\mathfrak{k}, -B|_{\mathfrak{k}})$. Let $\nabla_{e_i}|_{1 \le i \le m+n}$ denote the corresponding differential operators on $\mathfrak{g}$.

Let Y be the generic point of $\mathfrak{g}$, so that $Y = Y^{\mathfrak{p}} + Y^{\mathfrak{k}}$ with $Y^{\mathfrak{p}} \in \mathfrak{p}$, $Y^{\mathfrak{k}} \in \mathfrak{k}$. Put

$$\begin{aligned} \mathcal{D}^{\mathfrak{p}} &= \sum_{j=1}^{m} c(e_j^*)\nabla_{e_j}, \quad \mathcal{E}^{\mathfrak{p}} = \widehat{c}(Y^{\mathfrak{p}}), \\ \mathcal{D}^{\mathfrak{k}} &= \sum_{j=m+1}^{m+n} c(e_j^*)\nabla_{e_j}, \quad \mathcal{E}^{\mathfrak{k}} = \widehat{c}(Y^{\mathfrak{k}}). \end{aligned} \tag{9}$$

Let $\Delta^{\mathfrak{g}}$ be the standard Laplace operator on $\mathfrak{g}$, and let $N^{\Lambda^\cdot(\mathfrak{g}^*)}$ be the number operator on $\Lambda^\cdot(\mathfrak{g}^*)$. We have the identity of operators acting on $C^\infty(\mathfrak{g}, \Lambda^\cdot(\mathfrak{g}^*) \otimes_{\mathbb{R}} \mathbb{C})$,

$$\frac{1}{2}\big(\mathcal{D}^{\mathfrak{p}}+\mathcal{E}^{\mathfrak{p}}-\mathrm{i}\mathcal{D}^{\mathfrak{k}}+\mathrm{i}\mathcal{E}^{\mathfrak{k}}\big)^2=\frac{1}{2}\big(-\Delta^{\mathfrak{g}}+|Y|^2-(m+n)\big)+N^{\Lambda^{\cdot}(\mathfrak{g}^*)}. \tag{10}$$

The kernel of the operator in (10) is concentrated in degree 0. It is spanned by the function $\exp(-|Y|^2/2)$.

We regard $\widehat{D}^{\mathfrak{g}}$ as a first-order differential operator acting on $C^\infty(G,\Lambda^{\cdot}(\mathfrak{g}^*))$. Then it also acts on $C^\infty(G\times\mathfrak{g},\Lambda^{\cdot}(\mathfrak{g}^*)\otimes_{\mathbb{R}}\mathbb{C})$. As in [4, Definition 2.9.1], for $b>0$, let $\mathfrak{D}_b$ be the first-order differential operator acting on $C^\infty(G\times\mathfrak{g},\Lambda^{\cdot}(\mathfrak{g}^*)\otimes_{\mathbb{R}}\mathbb{C})$ given by

$$\mathfrak{D}_b=\widehat{D}^{\mathfrak{g}}+\mathrm{i}c([Y^{\mathfrak{k}},Y^{\mathfrak{p}}])+\frac{1}{b}(\mathcal{D}^{\mathfrak{p}}+\mathcal{E}^{\mathfrak{p}}-\mathrm{i}\mathcal{D}^{\mathfrak{k}}+\mathrm{i}\mathcal{E}^{\mathfrak{k}}). \tag{11}$$

The group K acts on $\Lambda^{\cdot}(\mathfrak{g}^*)$ by the adjoint action $\mathrm{Ad}(\cdot)$. The right multiplication of K on G induces an action of K on $C^\infty(G\times\mathfrak{g},\Lambda^{\cdot}(\mathfrak{g}^*)\otimes_{\mathbb{R}}\mathbb{C})$ such that if $k\in K$, $s\in C^\infty(G\times\mathfrak{g},\Lambda^{\cdot}(\mathfrak{g}^*)\otimes_{\mathbb{R}}\mathbb{C})$, then

$$(k.s)(g,Y)=\mathrm{Ad}(k)s(gk,\mathrm{Ad}(k^{-1})Y). \tag{12}$$

The operators $\widehat{D}^{\mathfrak{g}}$, $\mathfrak{D}_b$ commute with K.

Let $X=G/K$ be the symmetric space associated with G. Let $p:G\to X$ denote the canonical projection, which defines a K-principal bundle. The splitting in (1) produces a connection on $p:G\to X$.

If $\rho^E:K\to\mathrm{Aut}(E)$ is a finite dimensional unitary representation of K, then $F=G\times_K E$ is a Hermitian vector bundle on X. The adjoint action of K on $\mathfrak{p}$, $\mathfrak{k}$ define the real vector bundles TX, N. The connection form induces canonical Hermitian or Euclidean connections ∇^F, ∇^{TX}, ∇^N on F, TX, N. Also $B|_{\mathfrak{p}}$ induces a Riemannian metric g^{TX} on X, so that ∇^{TX} is just the corresponding Levi-Civita connection, which is of nonpositive sectional curvature.

Let $\widehat{\mathcal{X}}$ be the total space of $\widehat{\pi}:TX\oplus N=G\times_K\mathfrak{g}\to X$, so that $\widehat{\mathcal{X}}\simeq X\times\mathfrak{g}$. We can define the action of K on $C^\infty(G\times\mathfrak{g},\Lambda^{\cdot}(\mathfrak{g}^*)\otimes E)$ as in (12). Let $C^\infty_K(G\times\mathfrak{g},\Lambda^{\cdot}(\mathfrak{g}^*)\otimes E)$ be the subspace of K-invariant elements. Then we have the identification

$$C^\infty_K(G\times\mathfrak{g},\Lambda^{\cdot}(\mathfrak{g}^*)\otimes E)=C^\infty(\widehat{\mathcal{X}},\widehat{\pi}^*(\Lambda^{\cdot}(T^*X\oplus N^*)\otimes F)). \tag{13}$$

Similarly, we can identify $C^\infty(X,F)$ with $C^\infty_K(G,E)$.

Since $\widehat{D}^{\mathfrak{g}}$, $\mathfrak{D}_b$ commutes with K, using (13), these operators descend to differential operators $\widehat{D}^{\mathfrak{g},X}$, $\mathfrak{D}^X_b$ acting on $C^\infty(\widehat{\mathcal{X}},\widehat{\pi}^*(\Lambda^{\cdot}(T^*X\oplus N^*)\otimes F))$. Also $C^{\mathfrak{g}}$ descends to a generalized Laplacian acting on $C^\infty(X,F)$.

Definition 1.1 *(Bismut [4]).* For $b>0$, let $\mathcal{L}^X_b$ be the operator acting on $C^\infty(\widehat{\mathcal{X}},\widehat{\pi}^*(\Lambda^{\cdot}(T^*X\oplus N^*)\otimes F))$ given by

$$\mathcal{L}^X_b=-\frac{1}{2}\widehat{D}^{\mathfrak{g},X,2}+\frac{1}{2}\mathfrak{D}^{X,2}_b. \tag{14}$$

Let $\mathcal{L}^X$ be the operator acting on $C^\infty(X,F)$

$$\mathcal{L}^X=\frac{1}{2}C^{\mathfrak{g}}+\frac{1}{8}B^*(\kappa^{\mathfrak{g}},\kappa^{\mathfrak{g}}). \tag{15}$$

An explicit formula for $\mathcal{L}^X_b$ is given in [4, equation (2.13.5)]. By a result of Hörmander [10], $\mathcal{L}^X_b$ is hypoelliptic. This operator is called the hypoelliptic Laplacian. By [4, Proposition 2.15.1], we have

$$[\mathfrak{D}^X_b,\mathcal{L}^X_b]=0. \tag{16}$$

If $x=pg$, $g\in G$, σ acts isometrically on X by the map $x\mapsto\sigma(x)=p\sigma(g)$. Then G^σ acts on X isometrically, and $X=G^\sigma/K^\sigma$.

The adjoint action of K on $\mathfrak{p}$, $\mathfrak{k}$ extend to K^σ. In the sequel, we assume that (E,ρ^E) extends to a representation of K^σ, which we still denote by ρ^E. Then $F=G^\sigma\times_{K^\sigma}E$, and we have similar identifications for TX, N. The action of G^σ on X lifts to these vector bundles, so that G^σ acts on $C^\infty(X,F)$ and on $C^\infty(\widehat{\mathcal{X}},\widehat{\pi}^*(\Lambda^{\cdot}(T^*X\oplus N^*)\otimes F))$. If $\gamma\in G$, $\tau\in\Sigma^\sigma$, $s\in C^\infty_K(G\times\mathfrak{g},\Lambda^{\cdot}(\mathfrak{g}^*)\otimes E)$, then

$$(\gamma\tau)s(g,Y)=\rho^{\Lambda^{\cdot}(\mathfrak{g}^*)\otimes E}(\tau)s(\tau^{-1}(\gamma^{-1}g),\tau^{-1}Y). \tag{17}$$

Proposition 1.2. *The operators $\mathfrak{D}^X_b$, $\mathcal{L}^X_b$, $\mathcal{L}^X$ commute with the action of G^σ.*

Proof. It is enough to prove that the corresponding operators on $G\times\mathfrak{g}$ or G commute with the action of σ. This follows from the fact that σ preserves the splitting in (1) and the bilinear form B. □

Let $A\in\mathrm{Aut}(E)$ be self-adjoint. Assume that A commutes with K^σ. Then A defines a self-adjoint parallel section of $\mathrm{End}(F)$ on X, which also lifts to $\widehat{\mathcal{X}}$. The section A commutes with the action of G^σ. Put

$$\mathcal{L}^X_A=\mathcal{L}^X+A\,,\ \ \mathcal{L}^X_{A,b}=\mathcal{L}^X_b+A. \tag{18}$$

2. Twisted orbital integrals

If $h, \gamma \in G$, the σ-twisted conjugation of γ by h is given by

$$C_\sigma(h)\gamma = h\gamma\sigma(h^{-1}). \tag{19}$$

Then $C_\sigma(\cdot)$ defines an action of G on itself. Let $[\gamma]_\sigma$ be the orbit of γ under this action, and let $Z_\sigma(\gamma) \subset G$ be the corresponding stabilizer of γ. Then

$$[\gamma]_\sigma \simeq Z_\sigma(\gamma)\backslash G. \tag{20}$$

Then $g \in Z_\sigma(\gamma)$ if and only if $\gamma\sigma g = g\gamma\sigma \in G^\sigma$. Set $K_\sigma(\gamma) = Z_\sigma(\gamma) \cap K$. Let $\mathfrak{z}_\sigma(\gamma)$, $\mathfrak{k}_\sigma(\gamma)$ be the Lie algebras of $Z_\sigma(\gamma)$, $K_\sigma(\gamma)$. Then

$$\mathfrak{z}_\sigma(\gamma) = \{f \in \mathfrak{g} : \mathrm{Ad}(\gamma)\sigma(f) = f\}. \tag{21}$$

Let $d(\cdot,\cdot)$ be the Riemannian distance in X. If $\gamma \in G$, the displacement function on X associated with $\gamma\sigma \in G^\sigma$ is given by

$$d_{\gamma\sigma}(x) = d(x, \gamma\sigma(x)). \tag{22}$$

By [8, Chapter 1, Example 1.6.6], $d_{\gamma\sigma}$ is a convex function on X.

By definition, $\gamma\sigma$ is semisimple if $d_{\gamma\sigma}$ reaches its infimum in X. In this case, let $X(\gamma\sigma)$ be the minimizing set of $d_{\gamma\sigma}$, i.e., the set of critical points of $d_{\gamma\sigma}$, which is a convex subset of X. If $\gamma\sigma$ has fixed points in X, $\gamma\sigma$ is said to be elliptic.

If $\gamma\sigma$ is semisimple, after conjugation by an element in G, we can assume that

$$\gamma = e^a k^{-1},\ a \in \mathfrak{p},\ k \in K,\ \mathrm{Ad}(k^{-1})\sigma a = a, \tag{23}$$

and the decomposition in (23) is unique. In this case, $\mathfrak{z}_\sigma(\gamma)$ splits as

$$\mathfrak{z}_\sigma(\gamma) = \mathfrak{p}_\sigma(\gamma) \oplus \mathfrak{k}_\sigma(\gamma), \tag{24}$$

where $\mathfrak{p}_\sigma(\gamma)$, $\mathfrak{k}_\sigma(\gamma)$ are the intersections of $\mathfrak{z}_\sigma(\gamma)$ with $\mathfrak{p}$, $\mathfrak{k}$. Let $\mathfrak{z}^\perp_\sigma(\gamma) = \mathfrak{p}^\perp_\sigma(\gamma) \oplus \mathfrak{k}^\perp_\sigma(\gamma)$ be the orthogonal space of $\mathfrak{z}_\sigma(\gamma)$ in $\mathfrak{g}$.

Then $X(\gamma\sigma)$ is the symmetric space associated with $(Z_\sigma(\gamma), K_\sigma(\gamma))$. Let $Z^0_\sigma(\gamma)$, $K^0_\sigma(\gamma)$ be the identity component of $Z_\sigma(\gamma)$, $K_\sigma(\gamma)$. Then we also have

$$X(\gamma\sigma) = Z^0_\sigma(\gamma)/K^0_\sigma(\gamma). \tag{25}$$

Let $N_{X(\gamma\sigma)/X}$ be the normal bundle to $X(\gamma\sigma)$. Then $N_{X(\gamma\sigma)/X}$ is just the vector bundle associated with $\mathfrak{p}^\perp_\sigma(\gamma)$.

We identify the total space $\mathcal{N}_{X(\gamma\sigma)/X}$ of $N_{X(\gamma\sigma)/X}$ with X using normal geodesic coordinates. We can adapt the arguments of [4, Theorem 3.4.1] to show that if $(x_0, f) \in \mathcal{N}_{X(\gamma\sigma)/X}$ is such that $|f| \geq 1$,

$$d_{\gamma\sigma}(x_0, f) \geq |a| + C|f|. \tag{26}$$

Let $\mathrm{d}x$, $\mathrm{d}y$ be the Riemannian volumes on X, $X(\gamma\sigma)$, and let $\mathrm{d}f$ be the volume element on $\mathfrak{p}^\perp_\sigma(\gamma)$. Then there exists a smooth positive function $r(f)$ on $\mathfrak{p}^\perp_\sigma(\gamma)$ such that

$$dx = r(f)\,\mathrm{d}y\,\mathrm{d}f. \tag{27}$$

By [4, eq. (3.4.36)], there exists $C > 0$, $C' > 0$ such that

$$r(f) \leq C\exp(C'|f|). \tag{28}$$

For $t > 0$, let $p^X_t(x, x')$ be the smooth kernel for the heat operator $\exp(-t\mathcal{L}^X_A)$. By [4, Proposition 4.4.2], for $t > 0$, there exist $C > 0$, $C' > 0$ such that

$$|p^X_t(x, x')| \leq C\exp(-C'd^2(x, x')). \tag{29}$$

Let $[\gamma\sigma]$ be the conjugacy class of $\gamma\sigma$ in G^σ. We extend the formula for the semisimple orbital integrals in [4, Definition 4.2.2] to our case.

Definition 2.1 *(Elliptic twisted orbital integrals).* Put

$$\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_A)] = \int_{\mathfrak{p}^\perp_\sigma(\gamma)} \mathrm{Tr}^F[\gamma\sigma p^X_t(e^f p1, \gamma\sigma e^f p1)]\, r(f)\,\mathrm{d}f. \tag{30}$$

The existence of the integral in (30) follows from the estimates in (22), (28) and (29). Using (20), we can interpret (30) as an integral on the twisted adjoint orbit $[\gamma]_\sigma$ in G.

By [4, Section 11.8], for $t > 0$, there is a smooth kernel $q^X_{b,t}((x,Y),(x',Y'))$ associated with the heat operator $\exp(-t\mathcal{L}^X_{A,b})$. Let $\mathbf{P}$ be the projection from $\Lambda^\cdot(T^*X \oplus N^*) \otimes F$ onto $\Lambda^0(T^*X \oplus N^*) \otimes F$.

We recall some results established in [4, Theorem 4.5.2 & Chapter 14]. Given $0 < \epsilon \le M$, there exist $C, C' > 0$ such that for $0 < b \le M, \epsilon \le t \le M, (x,Y),(x',Y') \in \widehat{\mathcal{X}}$,

$$\left|q^X_{b,t}((x,Y),(x',Y'))\right| \le C \exp\left(-C'(d^2(x,x') + |Y|^2 + |Y'|^2)\right). \tag{31}$$

Moreover, as $b \to 0$, we have the convergence in any C^k-norm on any compact subset,

$$q^X_{b,t}((x,Y),(x',Y')) \to \mathbf{P} p^X_t(x,x')\pi^{-(m+n)/2}\exp(-\frac{1}{2}(|Y|^2 + |Y'|^2))\mathbf{P}. \tag{32}$$

As in [4, Definition 4.3.3], we define the hypoelliptic twisted orbital integrals.

Definition 2.2. We put

$$\begin{aligned}&\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_{A,b}))] = \\ &\int_{\mathfrak{p}^\perp_\sigma(\gamma)} \left[\int_{TX\oplus N} \mathrm{Tr_s}^{\Lambda^\cdot(T^*X\oplus N^*)\otimes F}[\gamma\sigma q^X_{b,t}((e^f p1, Y), \gamma\sigma(e^f p1, Y))]dY\right] r(f)\, df.\end{aligned} \tag{33}$$

The existence of the integral follows from the estimates in (22), (28), and (31).

3. An explicit geometric formula for twisted orbital integrals

We will establish a fundamental identity, which extends [4, Theorem 4.6.1].

Theorem 3.1. *For $t > 0$, $b > 0$,*

$$\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_{A,b})] = \mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_A)]. \tag{34}$$

Proof. First, we use the fact that the orbital integral as in (33) can be interpreted as a generalized supertrace on a suitable algebra of G^σ-invariant kernel operators. By (16), Proposition 1.2, we get

$$\frac{\partial}{\partial b}\mathrm{Tr_s}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X_{A,b})] = 0. \tag{35}$$

Then we use the convergence in (32) to get (34). □

Put

$$\mathfrak{z}_0 = \ker \mathrm{ad}(a). \tag{36}$$

Then $\mathfrak{z}_\sigma(\gamma) \subset \mathfrak{z}_0$. Let $\mathfrak{z}^\perp_{\sigma,0}(\gamma)$ be the orthogonal space of $\mathfrak{z}_\sigma(\gamma)$ in $\mathfrak{z}_0$. We have the splitting

$$\mathfrak{z}^\perp_{\sigma,0}(\gamma) = \mathfrak{p}^\perp_{\sigma,0}(\gamma) \oplus \mathfrak{k}^\perp_{\sigma,0}(\gamma). \tag{37}$$

In [4, Chapter 5], the author defined an analytic function J_γ on $\mathfrak{k}(\gamma)$, which plays an important role in the formula for semisimple orbital integrals [4, Theorem 6.1.1]. We now extend the definition of this function.

Definition 3.2. Let $J_{\gamma\sigma}(Y^\mathfrak{k}_0)$ be the analytic function of $Y^\mathfrak{k}_0 \in \mathfrak{k}_\sigma(\gamma)$ given by

$$\begin{aligned}J_{\gamma\sigma}(Y^\mathfrak{k}_0) = &\frac{1}{|\det(1-\mathrm{Ad}(\gamma\sigma))|_{\mathfrak{z}^\perp_0}|^{1/2}} \frac{\widehat{A}(\mathrm{i}\,\mathrm{ad}(Y^\mathfrak{k}_0)|_{\mathfrak{p}_\sigma(\gamma)})}{\widehat{A}(\mathrm{i}\,\mathrm{ad}(Y^\mathfrak{k}_0)|_{\mathfrak{k}_\sigma(\gamma)})} \\ &\left[\frac{1}{\det(1-\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{z}^\perp_{\sigma,0}(\gamma)}} \frac{\det(1-\exp(-\mathrm{i}\,\mathrm{ad}(Y^\mathfrak{k}_0))\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{k}^\perp_{\sigma,0}(\gamma)}}{\det(1-\exp(-\mathrm{i}\,\mathrm{ad}(Y^\mathfrak{k}_0))\mathrm{Ad}(k^{-1}\sigma))|_{\mathfrak{p}^\perp_{\sigma,0}(\gamma)}}\right]^{1/2}.\end{aligned} \tag{38}$$

Put

$$p = \dim \mathfrak{p}_\sigma(\gamma),\ q = \dim \mathfrak{k}_\sigma(\gamma). \tag{39}$$

We have the following main result of this Note, which extends [4, Theorem 6.1.1].

Theorem 3.3. *For any $t > 0$, the following identity holds:*

$$\begin{aligned}&\mathrm{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}_A^X)] = \frac{\exp(-|a|^2/2t)}{(2\pi t)^{p/2}} \times \\ &\int_{\mathfrak{k}_\sigma(\gamma)} J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\mathrm{Tr}^E[\rho^E(k^{-1}\sigma)\exp(-\mathrm{i}\rho^E(Y_0^{\mathfrak{k}}) - tA)]\,\mathrm{e}^{-|Y_0^{\mathfrak{k}}|^2/2t}\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.\end{aligned} \tag{40}$$

Proof. We adapt the proof of [4, Theorem 6.1.1] to our case. The main idea is to make $b \to +\infty$ in (34). Then the evaluation of the hypoelliptic orbital integral concentrates near $X(\gamma\sigma)$. After rescaling near $X(\gamma\sigma)$, as $b \to +\infty$, $q_{b,t}^X((x,Y), \gamma\sigma(x,Y))$ converges to the heat kernel of a model operator involving the geometry of normal bundle $N_{X(\gamma\sigma)/X}$. The function $J_{\gamma\sigma}$ appears when we compute the rescaled twisted orbital integral of this model operator. □

Let $P_\sigma^\perp(\gamma) \subset X$ be the image of $\mathfrak{p}_\sigma^\perp(\gamma)$ by the map $f \to p e^f$. Put

$$\Delta_X^{\gamma\sigma} = \{(x, \gamma\sigma(x)) : x \in P_\sigma^\perp(\gamma)\}. \tag{41}$$

Then $\Delta_X^{\gamma\sigma}$ is a submanifold of $X \times X$. Let $(a, \mathfrak{k}_\sigma(\gamma))$ denote the affine subspace of $\mathfrak{z}_\sigma(\gamma) = \mathfrak{p}_\sigma(\gamma) \oplus \mathfrak{k}_\sigma(\gamma)$. Set

$$H_\sigma^\gamma = \{0\} \times (a, \mathfrak{k}_\sigma(\gamma)) \subset \mathfrak{z}_\sigma(\gamma) \times \mathfrak{z}_\sigma(\gamma). \tag{42}$$

By proceeding as in [4, Section 6.3] and using Theorem 3.3, we get an extension of [4, Theorem 6.3.2] for the twisted orbital integrals for the wave operator.

Theorem 3.4. *We have the identity of even distributions on $\mathbb{R}$ supported on $\{s \in \mathbb{R} : |s| \geq \sqrt{2}|a|\}$ with singular support included in $\pm\sqrt{2}|a|$,*

$$\int_{\Delta_X^{\gamma\sigma}} \mathrm{Tr}^F[\gamma\sigma\cos(s\sqrt{\mathcal{L}^X + A})] = \int_{H_\sigma^\gamma} \mathrm{Tr}^E[\cos(s\sqrt{-\Delta^{\mathfrak{z}_\sigma(\gamma)}/2 + A})\,J_{\gamma\sigma}(Y_0^{\mathfrak{k}})\rho^E(k^{-1}\sigma)\exp(-\mathrm{i}\rho^E(Y_0^{\mathfrak{k}}))]. \tag{43}$$

4. An application: asymptotics of equivariant Ray–Singer torsion

Müller [14] initiated the study of asymptotic Ray–Singer analytic torsion for symmetric powers of a given flat vector bundle on hyperbolic manifolds. Bismut–Ma–Zhang [7] and Müller–Pfaff [15] studied the case of a sequence of flat vector bundles associated with multiples of a given highest weight defining a representation of the compact form of G. Here, we will be concerned with the asymptotics of the equivariant Ray–Singer analytic torsion on compact locally symmetric spaces. A similar problem has already been considered by Ksenia Fedosova [9] using methods of harmonic analysis on the reductive group G. Following ideas in [7, Section 8], we will exploit instead the explicit formula in Theorem 3.3.

Assume that G has compact center. Let U be the compact form of G with Lie algebra $\mathfrak{u} = \mathfrak{k} \oplus \mathrm{i}\mathfrak{p}$. We assume that the action of σ on $\mathfrak{u}$ lifts to U. Put $U^\sigma = U \rtimes \Sigma^\sigma$.

If $\rho^E : U^\sigma \to \mathrm{Aut}(E)$ is a finite dimensional complex representation of U^σ, there is a Hermitian metric h^E on E that is such that ρ^E is unitary. Now, we will use the unitary trick. By [11, Proposition 5.7], this representation ρ^E extends uniquely to a representation of G^σ.

The map $(g, v) \in G \times_K E \to \rho^E(g)v \in E$ gives an identification

$$F = G \times_K E = X \times E. \tag{44}$$

Then F is equipped with a canonical flat connection $\nabla^{F,f}$, which is invariant by the action of G^σ. Let h^F be the Hermitian metric on F induced by h^E.

Let $(\Omega^{\cdot}(X,F), d^{X,F})$ be the de Rham complex associated with $(F, \nabla^{F,f})$, and let $d^{X,F,*}$ be the formal adjoint of $d^{X,F}$ with respect to g^{TX}, h^F. Put

$$\mathbf{D}^{X,F} = d^{X,F} + d^{X,F,*}. \tag{45}$$

Let $C^{\mathfrak{g},E}$ be the action of $C^{\mathfrak{g}}$ on E. By [7, Proposition 8.4], we have

$$\frac{\mathbf{D}^{X,F,2}}{2} = \mathcal{L}^X - \frac{1}{2}C^{\mathfrak{g},E} - \frac{1}{8}B^*(\kappa^{\mathfrak{g}}, \kappa^{\mathfrak{g}}). \tag{46}$$

Let $\Gamma \subset G$ be a discrete, cocompact, torsion-free subgroup of G. Put

$$Z = \Gamma\backslash X. \tag{47}$$

Then Z is a compact smooth manifold.

Lemma 4.1. *If $\sigma(\Gamma)=\Gamma$, if $\gamma\in\Gamma$, $\gamma\sigma$ is semisimple and $\Gamma\cap Z_\sigma(\gamma)$ is a cocompact discrete subgroup of $Z_\sigma(\gamma)$.*

Proof. The first part follows from the fact that Γ is cocompact, so that the $d_{\gamma\sigma}$ always reaches its infimum in X. The second part is a consequence of [18, Lemmas 1, 2]. □

We now assume that $\sigma(\Gamma)=\Gamma$. Then Σ^σ acts on Z isometrically.

The vector bundle F descends to a flat vector bundle on Z, which we still denote F. Also TX descends to the tangent bundle TZ of Z. The action of Σ^σ on Z lifts to an action on F preserving $\nabla^{F,f}$. Then $\mathcal{L}^X$, $d^{X,F}$, $\mathbf{D}^{X,F}$ descend to the corresponding operators $\mathcal{L}^Z$, $d^{Z,F}$, $\mathbf{D}^{Z,F}$.

Let $\mathcal{T}_\sigma(F)$ be the equivariant Ray–Singer analytic torsion associated with the action of σ ([5,6,16,17]). Let $P^\perp$ be the orthogonal projection onto $(\ker \mathbf{D}^{Z,F})^\perp$. Then $\mathcal{T}_\sigma(F)$ is the derivative at 0 of the Mellin transform of $-\mathrm{Tr_s}\big[\big(N^{\Lambda^\cdot(T^*Z)}-\frac{m}{2}\big)\sigma\exp(-t\mathbf{D}^{Z,F,2}/2)P^\perp\big]$.

If $\gamma\sigma$ is given by (23), put $\delta_\sigma(\gamma)=\mathrm{rk}_{\mathbb{C}}(Z_\sigma(\gamma))-\mathrm{rk}_{\mathbb{C}}(K_\sigma(\gamma))\in\mathbb{N}$. Then $\delta_\sigma(\gamma)$ only depends on the class $[\gamma]_\sigma$.

Proposition 4.2. *If one of the following three assumptions is verified:*

(i) m is even and σ preserves the orientation of $\mathfrak{p}$;
(ii) m is odd and σ does not preserve the orientation of $\mathfrak{p}$;
(iii) for $\gamma\in\Gamma$, $\delta_\sigma(\gamma)\neq 1$,

then

$$\mathcal{T}_\sigma(F)=0. \tag{48}$$

Proof. Let $[\Gamma]_\sigma$ be the set of σ-twisted conjugacy classes in Γ. If $\gamma\in\Gamma$, let $[\gamma]'_\sigma\in[\Gamma]_\sigma$ be the corresponding class. We have Selberg's twisted trace formula,

$$\begin{aligned}&\mathrm{Tr_s}\Big[\Big(N^{\Lambda^\cdot(T^*Z)}-\frac{m}{2}\Big)\sigma\exp(-t\mathbf{D}^{Z,F,2}/2)\Big]\\&=\sum_{[\gamma]'_\sigma\in[\Gamma]_\sigma}\mathrm{Vol}(\Gamma\cap Z(\gamma\sigma)\backslash X(\gamma\sigma))\mathrm{Tr_s}^{[\gamma\sigma]}\Big[\Big(N^{\Lambda^\cdot(T^*X)}-\frac{m}{2}\Big)\exp(-t\mathbf{D}^{X,F,2}/2)\Big].\end{aligned}\tag{49}$$

By (46), the orbital integrals in the right-hand side of (49) can be computed using Theorem 3.3. Then one can verify that they vanish identically if any of the three above assumptions is verified. □

We now construct a family of flat vector bundles $F_d|_{d\in\mathbb{N}}$ on Z.

Let $\mathfrak{h}$ be a maximal Cartan subalgebra of $\mathfrak{u}$ preserved by σ with a Weyl chamber $\mathfrak{c}$ fixed by σ. This determines a root system R and a positive root system R_+ preserved by σ. Let P_{++} be the set of dominant weights equipped with the action of σ.

Here we will consider a special case of an irreducible unitary representation (E,ρ^E) of U^σ, which is also irreducible as a representation of U. Let $\lambda\in P_{++}$ be the highest weight of E. Then $\sigma\cdot\lambda=\lambda$.

Let $U(\lambda)$ be the centralizer of λ in U. Then

$$U^\sigma(\lambda)=U(\lambda)\times\Sigma^\sigma. \tag{50}$$

Let $M_\lambda=U/U(\lambda)$ be the associated flag manifold. We also have

$$M_\lambda=U^\sigma/U^\sigma(\lambda). \tag{51}$$

Let L_λ be the canonical complex line bundle on M_λ. Then by [19, Lemmas 6.2.9 and 6.2.13], M_λ is a complex manifold, L_λ is a positive holomorphic line bundle, and U^σ acts holomorphically and isometrically on $L_\lambda\to M_\lambda$. Then for $d\in\mathbb{N}$, U^σ acts on $H^{(0,0)}(M_\lambda,L_\lambda^{\otimes d})$.

Definition 4.3. For $d\in\mathbb{N}$, let (E_d,ρ^{E_d}) be the unitary representation of U^σ given by $H^{(0,0)}(M_\lambda,L_\lambda^{\otimes d})$.

The character χ_d of (E_d,ρ^{E_d}) can be evaluated by the Lefschetz fixed point theorem of Atiyah–Bott [1,2] or the fixed point theorem of Berline–Vergne [3, Theorem 3.23].

If $u\in U$, let $M_\lambda^{u\sigma}$ be the fixed point set of $u\sigma$ acting on M_λ, and let $M_\lambda^{u\sigma,\max}$ be the part of maximal dimension $n(u\sigma)$. Then

$$M_\lambda^{u\sigma,\max} = \cup_{j\in I(u)} M_\lambda^{u\sigma,j}, \tag{52}$$

where $I(u)$ describes the set of connected components of $M_\lambda^{u\sigma,\max}$.

Let $\mu : M_\lambda \to \mathfrak{u}^*$ be the moment map associated with the action of U on $L_\lambda \to M_\lambda$. Let $c_1(L_\lambda)$ be the first Chern class of L_λ.

The component of degree 0 of the equivariant Todd form $\mathrm{Td}^{u\sigma}(TM_\lambda|_{M_\lambda^{u\sigma,j}}, g^{TM_\lambda|_{M_\lambda^{u\sigma,j}}})$ is a locally constant function on $M_\lambda^{u\sigma}$, and if $j \in I(u)$, we denote its value on $M_\lambda^{u\sigma,j}$ by c_j. The action of $u\sigma$ on $L_\lambda|_{M_\lambda^{u\sigma}}$ is represented by a locally constant function valued in $\mathbb{S}^1$, and if $j \in I(u)$, we denote its value on $M_\lambda^{u\sigma,j}$ by $h_j \in \mathbb{S}^1$.

If $j \in I(u)$, if $y \in \mathfrak{u}$ is such that $\mathrm{Ad}(u)\sigma(y) = y$, put

$$R_{u\sigma}^j(y) = c_j \int_{M_\lambda^{u\sigma,j}} \exp(2\pi i\langle \mu, y\rangle + c_1(L_\lambda)). \tag{53}$$

By the fixed point theorem of Berline–Vergne [3, Theorem 3.23], we get that as $d \to +\infty$,

$$\chi_d(u\sigma e^{y/d}) = d^{n(u\sigma)} \sum_{j\in I(u)} h_j^d R_{u\sigma}^j(y) + \mathcal{O}(d^{n(u\sigma)-1}). \tag{54}$$

The representations (E_d, ρ^{E_d}) extend to representations of G^σ. This way, we get a family of flat Hermitian vector bundles $F_d|_{d\in\mathbb{N}}$ on X and on Z equipped with the action of Σ^σ. Let $\mathbf{D}^{X,F_d}$, $\mathbf{D}^{Z,F_d}$ denote the corresponding Hodge–de Rham operators.

We will consider K as a subgroup of U. If $k \in K$, the function $R_{k^{-1}\sigma}^j(y)$ extends to an analytic function in $y \in \mathfrak{z}_\sigma(k^{-1})$. If $j \in I(u)$, by [7, Subsection 2.5], associated with the pair $(Z_\sigma^0(k^{-1}), X(k^{-1}\sigma))$ and the function $R_{k^{-1}\sigma}^j$, one can associate locally computable invariant differential forms e_t^j, d_t^j on $X(k^{-1}\sigma)$ twisted by the orientation bundle $o(TX(k^{-1}\sigma))$.

Let $[\cdot]^{\max} \in \mathbb{R}$ be the component of top degree with respect to the given volume on $X(k^{-1}\sigma)$. Because of the group invariance, here $[e_t^j]^{\max}$, $[d_t^j]^{\max}$ are constant functions on $X(k^{-1}\sigma)$.

Theorem 4.4. *Suppose that $\gamma = k^{-1} \in K$ and that $\delta_\sigma(\gamma) = 1$. For $t > 0$, as $d \to +\infty$,*

$$\begin{aligned}
&d^{-n(\gamma\sigma)-1}\mathrm{Tr_s}^{[\gamma\sigma]}\Big[\Big(N^{\Lambda^\cdot(T^*X)} - \frac{m}{2}\Big)\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\Big] \\
&= 2\sum_{j\in I(\gamma)} h_j^d[e_{t/2}^j]^{\max} + \mathcal{O}(d^{-1}), \\
&d^{-n(\gamma\sigma)-1}\mathrm{Tr_s}^{[\gamma\sigma]}\Big[\Big(N^{\Lambda^\cdot(T^*X)} - \frac{m}{2}\Big)\Big(1 - \frac{t\mathbf{D}^{X,F_d,2}}{d^2}\Big)\exp(-t\mathbf{D}^{X,F_d,2}/2d^2)\Big] \\
&= 2\sum_{j\in I(\gamma)} h_j^d[d_{t/2}^j]^{\max} + \mathcal{O}(d^{-1}).
\end{aligned} \tag{55}$$

Proof. Using (46), we apply Theorem 3.3 to the twisted orbital integral in the left-hand side of the first identity in (55). Then we adapt the proof of [7, Theorem 8.14] to prove the first identity, where we use (54) to evaluate the character of (E_d, ρ^{E_d}) that appears in the right-hand side of (40). The second identity follows from [7, Theorem 2.10], which says that

$$\Big(1 + 2t\frac{\partial}{\partial t}\Big)[e_t^j]^{\max} = [d_t^j]^{\max}. \quad \square \tag{56}$$

The pair (M_λ, μ) is said to be nondegenerate if $\mu(M_\lambda) \cap \mathfrak{k}^* = \emptyset$. In [7, Proposition 8.12], the authors gave an explicit condition for the nondegeneracy of (M_λ, μ) using the Weyl group of U.

In the sequel, we assume that (M_λ, μ) is nondegenerate. Then by [7, Theorem 4.4 and Remark 4.5], there exist $c > 0$, $C > 0$ such that, for $d \in \mathbb{N}$,

$$\mathbf{D}^{X,F_d,2} \geq cd^2 - C. \tag{57}$$

Moreover, if $j \in I(k^{-1})$, set

$$W^j = -\int_0^{+\infty} d_t^j \frac{dt}{t}. \tag{58}$$

By [7, Section 2.6], W^j is a locally computable smooth form on $X(k^{-1}\sigma)$. Such forms are called W-invariants associated with $(Z_\sigma(k^{-1}), R^j_{k-1})$. Here $[W^j]^{\max}$ is a constant function on $X(k^{-1}\sigma)$.

If $[\gamma]'_\sigma \in [\Gamma]_\sigma$, if $\gamma\sigma$ is elliptic, we say that $[\gamma]'_\sigma$ is an elliptic class. Let E_σ be the set of the elliptic classes in $[\Gamma]_\sigma$ with $\delta_\sigma(\gamma) = 1$, which is a finite set. If $[\gamma]'_\sigma \in E_\sigma$, then γ is C_σ-conjugate to some $k^{-1} \in K$. Then we associate the class $[\gamma]'_\sigma$ with the family $(n(k^{-1}\sigma), I(k^{-1}), h_j|_{j\in I(k^{-1})}, W^j|_{j\in I(k^{-1})})$.

Let $m(\sigma)$ be the maximum of the $n(k^{-1}\sigma)$ associated with $[\gamma]'_\sigma \in E_\sigma$, and let E'_σ be the subset of $[\gamma]'_\sigma \in E_\sigma$ such that $n(k^{-1}\sigma)$ equals $m(\sigma)$.

Theorem 4.5. *If (M_λ, μ) is nondegenerate, as $d \to +\infty$,*

$$d^{-m(\sigma)-1}\mathcal{T}_\sigma(F_d) = \sum_{[\gamma]'_\sigma \in E'_\sigma} \mathrm{Vol}(\Gamma \cap Z_\sigma(\gamma)\backslash X(\gamma\sigma)) \Big[\sum_{j\in I(k^{-1})} h^d_j [W^j]^{\max} \Big] + o(1). \tag{59}$$

In particular, if $E_\sigma = \emptyset$, as $d \to +\infty$,

$$\mathcal{T}_\sigma(F_d) = \mathcal{O}(\mathrm{e}^{-cd}). \tag{60}$$

Proof. Using the lower bound in (57) and small time estimates for the heat kernel of $\mathbf{D}^{Z,F_d,2}/2$ as in [7, Subsection 7.3], one can show that the contribution of the nonelliptic classes $[\gamma]'_\sigma$ in $[\Gamma]_\sigma$ is exponentially small. By the third condition in Proposition 4.2, only the elliptic classes in E_σ contribute to the evaluation of $\mathcal{T}_\sigma(F_d)$; this proves (60). Then (59) follows from Theorem 4.4 and (58). □

Let Z^σ be the fixed point set of σ in Z. Then one can show that

$$Z^\sigma = \bigcup_{[\gamma]'_\sigma \text{ elliptic}} \Gamma \cap Z_\sigma(\gamma)\backslash X(\gamma\sigma). \tag{61}$$

Then the differential forms W^j are a finite set of locally computable invariants associated with Z^σ. The result of Theorem 4.5 can be regarded as a fixed point theorem for the asymptotics of the equivariant Ray–Singer torsions.

Remark 1. The presence of oscillating terms h^d_j in (59) and the exponential decay in (60) are compatible with the results of Ksenia Fedosova [9] on compact hyperbolic orbifolds.

Acknowledgements

I would like to express my great appreciation to Prof. Jean-Michel Bismut for encouraging me to work on this subject, and for many useful discussions. I also thank the referees, who provided many comments that improved the article.

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