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Numerical analysis

A variant of Nitsche's method

Une variante de la méthode de Nitsche

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ABSTRACT

We present a method that aims to reconcile Nitsche's method with the traditional finite element method ('weak' versus 'strong implementation' of essential boundary conditions). We retain the original idea of a variational formulation based on an extended energy, but replace the original boundary terms by domain terms involving weak derivatives. The solution of the proposed method coincides, for the Poisson problem, with the one of the traditional method, which in particular shows monotonicity under the standard angle condition for the Courant element. For more general second-order problems, it allows for the weighting of boundary terms inherent to Nitsche's method. This is of particular interest for singularly perturbed problems.

RÉSUMÉ

Nous présentons une méthode qui vise à concilier la méthode de Nitsche avec la méthode traditionnelle (implémentation « faible » versus « forte » de conditions aux limites essentielles). L'idée originelle d'une formulation faible basée sur une énergie étendue est préservée, mais les termes de bord sont remplacés par des termes sur le domaine utilisant des dérivées faibles. La solution de la méthode proposée coïncide, pour le problàme de Poisson, avec celle de la méthode traditionnelle ; cela montre, en particulier, la monotonie sous la condition d'angle maximal pour l'élément de Courant. Pour des problèmes plus généraux, notre modification permet une pondération des termes de bord comme la méthode de Nitsche. Cela est particulièrement intéressant pour les perturbations singulières.

1. Introduction

Nitsche's method [13] overcame the difficulties inherent to the penalty method [3], without the introduction of an additional variable as in the mortar method [2,6]. The interest of the method is its flexibility. This makes it a cornerstone for

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subsequent developments such as the discontinuous Galerkin method [1], domain decomposition with non-matching meshes [5], cut-FEM for interface problems [11], composite grids [12], fluid–structure interaction [8], and contact in elasticity [9].

We consider the Poisson problem on a bounded polygonal domain $\Omega \subset \mathbb{R}^d$, $d \ge 1$, with boundary $\Gamma = \partial\Omega$, data $f \in L^2(\Omega)$ and $u^{\mathrm{D}} \in H^{\frac{1}{2}}(\Gamma)$, together with its standard variational formulation:

$$-\Delta u = f \quad \text{in } \Omega, \quad u = u^{\mathrm{D}} \quad \text{on } \Gamma, \quad u \in \mathrm{Pr}(u^{\mathrm{D}}) + V_0 : \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \quad \forall v \in V_0 := H_0^1(\Omega). \tag{1}$$

We denote by $\gamma^{\mathrm{tr}} : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$ the trace operator and by $\mathrm{Pr} : H^{\frac{1}{2}}(\Gamma) \to H^1(\Omega)$ a right-inverse of γ^{tr}, $\gamma^{\mathrm{tr}} \circ \mathrm{Pr} = \mathrm{id}_{H^{\frac{1}{2}}(\Gamma)}$.

The Poisson problem (1) is the stationarity condition for minimization of the convex quadratic energy $J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega f u$ over the constrained space $V_{u^{\mathrm{D}}} := \left\{u \in H^1(\Omega) : \gamma^{\mathrm{tr}}(u) = u^{\mathrm{D}}\right\}$.

The augmented Lagrange functional with parameter $\gamma \ge 0$ for this optimization problem reads

$$\mathcal{L}(u, \lambda) = J(u) + \langle u - u^{\mathrm{D}}, \lambda \rangle_\Gamma + \frac{\gamma}{2} |||u - u^{\mathrm{D}}|||_\Gamma^2, \qquad \mathcal{L} : V \times Q \to \mathbb{R}, \quad Q = H^{-\frac{1}{2}}(\Gamma) \tag{2}$$

where $\langle \cdot, \cdot \rangle_\Gamma$ and $|||\cdot|||_\Gamma$ denote the duality pairing and the $H^{\frac{1}{2}}(\Gamma)$-norm, respectively. Then the stationarity of $\mathcal{L}$ over the unconstrained spaces $H^1(\Omega)$ and $H^{\frac{1}{2}}(\Gamma)$ is equivalent to the stationarity of J over the constrained space $V_{u^{\mathrm{D}}}$ and yields the additional equation ($\mathcal{L}'_u = 0$ for $\gamma = 0$):

$$\langle v, \lambda \rangle_\Gamma = \int_\Omega f v - \int_\Omega \nabla u \cdot \nabla v, \quad \text{providing the boundary flux } \lambda = -\frac{\partial u}{\partial n} = T u, \quad T : V \to Q. \tag{3}$$

This shows that, in contrast to general saddle-point problems, there is a *local* relation between the variables, which is then used to define Nitsche's modified energy by eliminating λ in (2):

$$J_{\mathrm{N}}(u) = J(u) - \langle u - u^{\mathrm{D}}, \frac{\partial u}{\partial n} \rangle_\Gamma + \frac{\gamma}{2} \|h^{-\frac{1}{2}}(u - u^{\mathrm{D}})\|_\Gamma^2, \tag{4}$$

where we have, in addition, replaced the computationally cumbersome norm $|||\cdot|||_\Gamma$ by the meshsize-weighted $L^2(\Gamma)$-norm, supposed to be equivalent on the discrete spaces. The stationarity of J_{N} then gives the well-known variational formulation of Nitsche's method: find $u \in V_h$ such that for all $v \in V_h$

$$\int_\Omega \nabla_h u \cdot \nabla_h v - \int_\Gamma \frac{\partial u}{\partial n} v - \int_\Gamma u \frac{\partial v}{\partial n} + \int_\Gamma \frac{\gamma}{h} u v = \int_\Omega f v - \int_\Gamma u^{\mathrm{D}} \frac{\partial v}{\partial n} + \int_\Gamma \frac{\gamma}{h} u^{\mathrm{D}} v, \tag{5}$$

with a finite element approximation $V_h \approx H^1(\Omega)$. We have used here the subscript h in ∇_h to indicate the piecewise-defined gradient operator, such that the method can be applied to nonconforming spaces; accordingly, with a slight abuse of notation, $\frac{\partial u}{\partial n}$ means $\nabla_h u \cdot n$.

In the next section, we present our modification of Nitsche's method for the Poisson problem. Then we discuss the relation to the mortar method. In section 4, we consider the case of the Crouzeix–Raviart element. As an example of generalization, we treat the singularly perturbed linear diffusion–reaction and convection–diffusion equation in section 5. Finally, some conclusions are drawn in section 6.

2. A modification of Nitsche's method

We start with equation (3) on the continuous level. Introducing a mollifier $\phi_\varepsilon \in H^1(\Omega)$ such that $\gamma^{\mathrm{tr}}(\phi_\varepsilon) = 1$ and $\mathrm{supp}(\phi_\varepsilon) \subset \Gamma_\varepsilon := \left\{x \in \bar{\Omega} : \mathrm{dist}(x, \Gamma) \le \varepsilon\right\}$, we write

$$v = (1 - \phi_\varepsilon) v + \phi_\varepsilon v =: v_\varepsilon^{\mathrm{int}} + v_\varepsilon^{\mathrm{bdr}}$$

and it follows that

$$\langle v, \lambda \rangle_\Gamma = \int_\Omega f v_\varepsilon^{\mathrm{bdr}} - \int_\Omega \nabla u \cdot \nabla v_\varepsilon^{\mathrm{bdr}}. \tag{6}$$

Letting $\varepsilon \to 0$, we recover the flux relation $\lambda = -\frac{\partial u}{\partial n}$, but on the discrete level we might as well stick to $\varepsilon \approx h$. Accordingly, we introduce a splitting of the finite element space in its interior and boundary parts. Let either V_h be the conforming space P_h^1 or the nonconforming space CR_h^1 and $\{\phi_i : 1 \le i \le N\}$ be the canonical basis (either associated with the nodes or the sides). Then we have:

$$V_h = V_h^{\text{int}} \oplus V_h^{\text{bdr}}, \quad V_h^{\text{int}} := \text{Vect}(\phi_i : \int_\Gamma \phi_i = 0), \quad V_h^{\text{bdr}} = \text{Vect}(\phi_i : \int_\Gamma \phi_i \neq 0). \tag{7}$$

An element v of V_h can the be written as (we omit the subscript h for readability) $v = v^{\text{int}} + v^{\text{bdr}}$.

Let $u_h^{\text{D}} \in V_h^{\text{bdr}}$ be an approximation of the boundary data. We define the modified Nitsche energy by

$$J_{\text{M}}(u) := J(u) + \int_\Omega f(u^{\text{bdr}} - u_h^{\text{D}}) - \int_\Omega \nabla_h u \cdot \nabla_h (u^{\text{bdr}} - u_h^{\text{D}}) + \int_\Omega \left|\nabla_h (u^{\text{bdr}} - u_h^{\text{D}})\right|^2, \tag{8}$$

which can be rewritten as

$$J_{\text{M}}(u) = \frac{1}{2}\int_\Omega \nabla_h u \cdot \nabla_h u - \int_\Omega f(u^{\text{int}} + u_h^{\text{D}}) - \int_\Omega \nabla_h (u^{\text{int}} + u_h^{\text{D}}) \cdot \nabla_h (u^{\text{bdr}} - u_h^{\text{D}}).$$

The stationarity condition then gives: find $u \in V_h$ such that for all $v \in V_h$

$$\int_\Omega \nabla_h u^{\text{int}} \cdot \nabla_h v^{\text{int}} + \int_\Omega \nabla_h u^{\text{bdr}} \cdot \nabla_h v^{\text{bdr}} = \int_\Omega f v^{\text{int}} - \int_\Omega \nabla_h u_h^{\text{D}} \cdot \nabla_h (v^{\text{int}} - v^{\text{bdr}}). \tag{9}$$

Now we write the equations corresponding to each part

$$\begin{cases} \int_\Omega \nabla_h u^{\text{int}} \cdot \nabla_h v^{\text{int}} = \int_\Omega f v^{\text{int}} - \int_\Omega \nabla_h u_h^{\text{D}} \cdot \nabla_h v^{\text{int}}, \\ \int_\Omega \nabla_h u^{\text{bdr}} \cdot \nabla_h v^{\text{bdr}} = \int_\Omega \nabla_h u_h^{\text{D}} \cdot \nabla_h v^{\text{bdr}}. \end{cases} \tag{10}$$

The problem is regular, since the matrix corresponding to $\int_\Omega \nabla_h u^{\text{bdr}} \cdot \nabla_h v^{\text{bdr}}$ is regular (by the Poincaré inequality, and is even spectrally equivalent to the scaled boundary mass matrix). It follows that u^{int} equals the interior part of the traditional method, and also $u^{\text{bdr}} = u_h^{\text{D}}$.

Taking v as the characteristic test function in (9), we have:

$$\int_\Omega f = \int_\Omega f v^{\text{bdr}} + \int_\Omega f v^{\text{int}} = \int_\Omega f v^{\text{bdr}} - \int_\Omega \nabla_h u \cdot \nabla_h v^{\text{bdr}}, \tag{11}$$

such that the right-hand side is the approximation to the total boundary flux as for example considered in [4,10].

Next we describe some variants. Skipping the right-hand side term in the modified energy (8), we obtain an expression similar to the Nitsche energy (with $\gamma = 2$)

$$\tilde{J}_{\text{M}}(u) := J(u) - \int_\Omega \nabla_h u \cdot \nabla_h (u^{\text{bdr}} - u_h^{\text{D}}) + \int_\Omega \left|\nabla_h (u^{\text{bdr}} - u_h^{\text{D}})\right|^2. \tag{12}$$

Since $\tilde{J}_{\text{M}}(u) = J(u) - \int_\Omega \nabla_h (u^{\text{int}} + u_h^{\text{D}}) \cdot \nabla_h (u^{\text{bdr}} - u_h^{\text{D}})$, this leads to the equations

$$\int_\Omega \nabla_h u \cdot \nabla_h v - \int_\Omega \nabla_h (u^{\text{int}} + u_h^{\text{D}}) \cdot \nabla_h v^{\text{bdr}} - \int_\Omega \nabla_h (u^{\text{bdr}} - u_h^{\text{D}}) \cdot \nabla_h v^{\text{int}} = \int_\Omega f v. \tag{13}$$

From this we can define a ‘non-symmetric’ variant:

$$\int_\Omega \nabla_h u \cdot \nabla_h v - \int_\Omega \nabla_h (u^{\text{int}} + u_h^{\text{D}}) \cdot \nabla_h v^{\text{bdr}} = \int_\Omega f v. \tag{14}$$

Letting $\sigma \in \{0, 1\}$, we obtain for (13) ($\sigma = 1$) and (14) ($\sigma = 0$) the system

$$\begin{cases} \int_\Omega \nabla_h u^{\text{int}} \cdot \nabla_h v^{\text{int}} = \int_\Omega f v^{\text{int}} - \int_\Omega \nabla_h \left((1-\sigma) u^{\text{bdr}} + \sigma u_h^{\text{D}}\right) \cdot \nabla_h v^{\text{int}}, \\ \int_\Omega \nabla_h u^{\text{bdr}} \cdot \nabla_h v^{\text{bdr}} = \int_\Omega \nabla_h u_h^{\text{D}} \cdot \nabla_h v^{\text{bdr}} + \int_\Omega f v^{\text{bdr}}. \end{cases} \tag{15}$$

The main difference with (10) is the appearance of the last term in the boundary equations. Therefore, the method does not reproduce the interpolated Dirichlet data and is nonconforming in any case. We briefly sketch the error analysis for the nonsymmetric formulation (the case of the symmetric formulation is very similar).

First we remark that

$$|||v_h|||^2 := \|\nabla_h v_h^{\rm int}\|^2 + \|\nabla_h v_h^{\rm bdr}\|^2$$

defines a norm on the discrete space V_h and the method is coercive with respect to this norm by the Cauchy–Schwarz inequality, since for any $v \in V_h$:

$$\int_\Omega \nabla_h v \cdot \nabla_h v - \int_\Omega \nabla_h v^{\rm int} \cdot \nabla_h v^{\rm bdr} = \|\nabla_h v^{\rm int}\|^2 + \|\nabla_h v^{\rm bdr}\|^2 + \int_\Omega \nabla_h v^{\rm bdr} \cdot \nabla_h v^{\rm int} \geq \frac{1}{2}|||v|||^2.$$

Let $I_h u$ be an interpolation of the exact solution $u \in H^2(\Omega)$, which coincides with $u_h^{\rm D}$ on the boundary, and u_h the solution to (14). Then with $v := u_h - I_h u$

$$\begin{aligned}
\frac{1}{2}|||u_h - I_h u|||^2 &\leq \int_\Omega \nabla_h (u_h - I_h u) \cdot \nabla_h v - \int_\Omega \nabla_h (u_h - I_h u)^{\rm int} \cdot \nabla_h v^{\rm bdr} \\
&= \int_\Omega f v + \int_\Omega \nabla_h u_h^{\rm D} \cdot \nabla_h v^{\rm bdr} - \int_\Omega \nabla_h I_h u \cdot \nabla_h v + \int_\Omega \nabla_h (I_h u)^{\rm int} \cdot \nabla_h v^{\rm bdr} \\
&= \int_\Omega \nabla_h (u - I_h u) \cdot \nabla_h v - \int_{\partial\Omega} \frac{\partial u}{\partial n} v + \int_\Omega \nabla_h (I_h u) \cdot \nabla_h v^{\rm bdr}
\end{aligned}$$

The first term is bounded with Cauchy–Schwarz and, for the second one, we have, with partial integration:

$$\begin{aligned}
\int_{\partial\Omega} \frac{\partial u}{\partial n} v - \int_\Omega \nabla_h (I_h u) \cdot \nabla_h v^{\rm bdr} &= \int_{\partial\Omega} \frac{\partial (u - I_h u)}{\partial n} v - \sum_{S \in \mathcal{S}_h} \int_S [\frac{\partial I_h u}{\partial n}] v^{\rm bdr} \\
&= \int_{\partial\Omega} \frac{\partial (u - I_h u)}{\partial n} v + \sum_{S \in \mathcal{S}_h} \int_S [\frac{\partial (u - I_h u)}{\partial n}] v^{\rm bdr}.
\end{aligned}$$

With the trace inequality and the properties of the interpolation operator, we conclude that the method is first-order accurate in the energy-norm.

3. Relation to the mortar method

We have seen that the modified Nitsche method is very closely related to the traditional method. But it is also very close to the mortar method with the special choice

$$\Lambda_h = \gamma(V_h) := \left\{\gamma(v_h) :\ v_h \in V_h\right\} \quad (= \gamma^{\rm tr}(V_h^{\rm bdr}). \tag{16}$$

Let us consider the stationarity of (2) with $\gamma = 0$, which corresponds to the original mortar method. Using our splitting $V_h = V_h^{\rm bdr} + V_h^{\rm int}$, we find the three-field mortar equations:

$$\begin{cases}
\int_\Omega \nabla_h u^{\rm int} \cdot \nabla_h v^{\rm int} + \int_\Omega \nabla_h u^{\rm bdr} \cdot \nabla_h v^{\rm int} = \int_\Omega f v^{\rm int} \quad \forall v^{\rm int} \in V_h^{\rm int} \\
\int_\Omega \nabla_h u \cdot \nabla_h v^{\rm bdr} + \int_\Gamma \lambda v^{\rm bdr} = \int_\Omega f v^{\rm bdr} \quad \forall v^{\rm bdr} \in V_h^{\rm bdr} \\
\int_\Gamma \mu u^{\rm bdr} = \int_\Gamma \mu u^{\rm D} \quad \forall \mu \in \Lambda_h
\end{cases} \tag{17}$$

Now we see that with the choice (16) we can solve the saddle-point problem by the following sequential steps:

(i) $u^{\rm bdr} = P_{\Lambda_h}(u^{\rm D})$, the $L^2(\Gamma)$-projection on Λ_h,

(ii) $\int_\Omega \nabla_h u^{\rm int} \cdot \nabla_h v^{\rm int} = \int_\Omega f v^{\rm int} - \int_\Omega \nabla_h u^{\rm bdr} \cdot \nabla_h v^{\rm int} \quad \forall v^{\rm int} \in V^{\rm int}$,

(iii) $\int_\Gamma \lambda v^{\mathrm{bdr}} = \int_\Omega f v^{\mathrm{bdr}} - \int_\Omega \nabla_h u \cdot \nabla_h v^{\mathrm{bdr}} \quad \forall v^{\mathrm{bdr}} \in V_h^{\mathrm{bdr}}$.

If we use $u_h^{\mathrm{D}} = P_{\Lambda_h}(u^{\mathrm{D}})$ in the method based on J_M, we end up with the identical solution. The third step computes in addition an approximation to the normal flux, which corresponds to (11).

4. The case of the Crouzeix–Raviart element

Our space definition (7) implies that the boundary space is made up from the basis functions corresponding to the boundary sides. (The neighboring basis functions have mean zero on the boundary sides and are considered as interior.) The properties of the Crouzeix–Raviart space lead to the relation

$$\int_\Omega \nabla_h u \cdot \nabla_h v = \int_\Gamma \frac{\partial u}{\partial n} v \qquad \forall u \in V_h, v \in V_h^{\mathrm{bdr}}. \tag{18}$$

This follows from integration by parts and the facts that the gradients are piecewise constant and the means of $v \in V_h^{\mathrm{bdr}}$ vanish on all interior sides. In addition, we have, for the boundary mesh function h defined by $h_{|S} = \frac{|K|}{|S|}$ for each boundary side S of the mesh included in the simplex K,

$$\int_\Omega \nabla_h u \cdot \nabla_h v = \int_\Gamma \frac{1}{h} v u \qquad \forall u, v \in V_h^{\mathrm{bdr}}. \tag{19}$$

Now we can rewrite the modified Nitsche energy (12) with (18) and (19) as

$$\tilde{J}_{\mathrm{M}}(u) := J(u) - \int_\Gamma \frac{\partial u}{\partial n}(u^{\mathrm{bdr}} - u_h^{\mathrm{D}}) + \int_\Gamma \frac{1}{h}(u^{\mathrm{bdr}} - u_h^{\mathrm{D}})^2$$

which is equivalent to Nitsche's method (5) with $\gamma = 2$. With the relation (18), it is easy to see that $\gamma = 0$ also leads to a stable discretization. In the latter case, the second equation of (15) reads

$$-\int_\Omega \nabla_h u^{\mathrm{bdr}} \cdot \nabla_h v^{\mathrm{bdr}} = -\int_\Omega \nabla_h u_h^{\mathrm{D}} \cdot \nabla_h v^{\mathrm{bdr}} + \int_\Omega f v^{\mathrm{bdr}}, \tag{20}$$

which has opposite sign, such that the whole discrete system is indefinite. We recover a result of [7]: Nitsche's method for the Crouzeix–Raviart element does not need stabilization. We notice, however, that, due to the presence of the last integral in (20), the Nitsche method without stabilization does not yield monotonicity.

5. Applications

5.1. Linear reaction–diffusion equation

We consider the singularly-perturbed problem

$$u - \varepsilon \Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

We use the non-symmetric method (14) in the form

$$\int_\Omega uv + \varepsilon \int_\Omega \nabla_h u \cdot \nabla_h v - \varepsilon \int_\Omega \nabla_h u^{\mathrm{int}} \cdot \nabla_h v^{\mathrm{bdr}} = \int_\Omega f v + \varepsilon \int_\Omega \nabla_h u_h^{\mathrm{D}} \cdot \nabla_h v^{\mathrm{bdr}}. \tag{21}$$

As for Nitsche's method, the limit $\varepsilon = 0$ is well defined and yields the $L^2(\Omega)$-projection. In the following test, we employ mass lumping. We compare the proposed method with the traditional method and Nitsche's method with $\gamma = 2$ at hand of the one-dimensional equation on $\Omega =]-1, 1[$. The analytical solution is given by $u(x) = 1 - \cosh(x\varepsilon^{-\frac{1}{2}}) / \cosh(\varepsilon^{-\frac{1}{2}})$. We take $\varepsilon = 10^{-4}$. The error with respect to h is shown in log–log scale on the left of Fig. 1. In the middle, the discrete solutions on a coarse mesh and, on the right, the dependence of the $L^1(\Omega)$ errors on ε are shown.

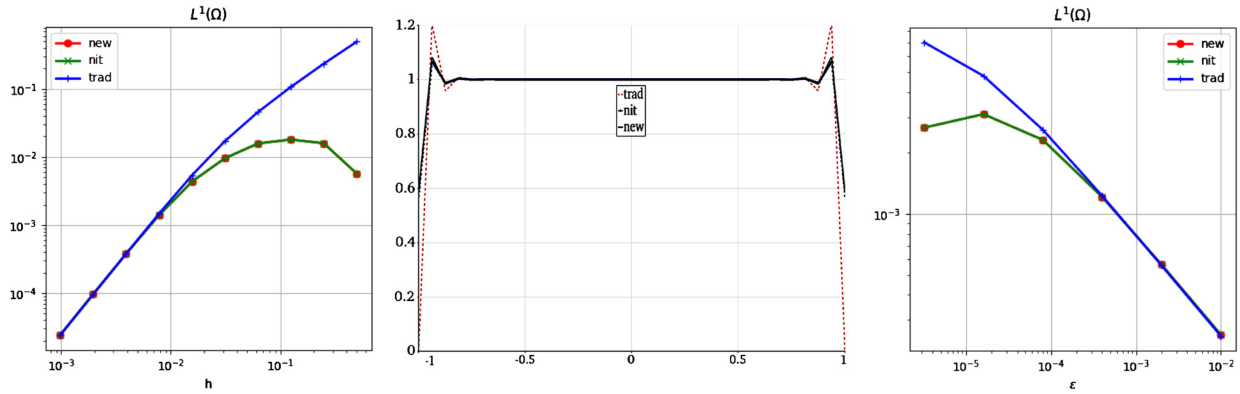

Fig. 1. Comparison of $L^1(\Omega)$ for $\varepsilon = 10^{-4}$ (left), solutions for $h = 1/16$ (middle), and $L^1(\Omega)$ for $h = 0.01$ (right).

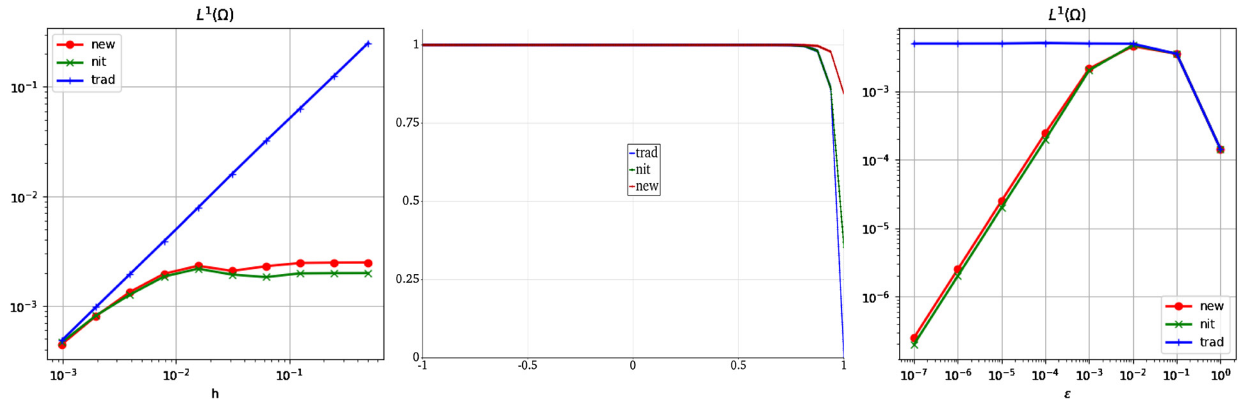

Fig. 2. Comparison of $L^1(\Omega)$ for $\varepsilon = 10^{-3}$ (left), solutions for $h = 1/16$ (middle), and $L^1(\Omega)$ for $h = 0.01$ (right).

5.2. *Linear convection–diffusion equation*

We consider the singularly-perturbed problem

$$\beta \cdot \nabla u - \varepsilon \Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

We use the following bilinear and linear forms

$$a_{\mathrm{d}}(u,v) = \int_\Omega \nabla_h u \cdot \nabla_h v - \int_\Omega \nabla_h u^{\mathrm{int}} \cdot \nabla_h v^{\mathrm{bdr}} - \int_\Omega \nabla_h u^{\mathrm{bdr}} \cdot \nabla_h v^{\mathrm{int}}, \qquad l_{\mathrm{d}}(v) = \int_\Omega \nabla_h u_h^{\mathrm{D}} \cdot \nabla_h v^{\mathrm{bdr}},$$

$$a_{\mathrm{c}}(u,v) = \int_\Omega \beta \cdot \nabla u\,(v + \delta\beta \cdot \nabla v) + \int_{\partial\Omega} |\beta_n^-| u^{\mathrm{bdr}} v^{\mathrm{bdr}}, \quad l_{\mathrm{c}}(v) = \int_\Omega f \delta\beta \cdot \nabla v + \int_{\partial\Omega} |\beta_n^-| u_h^{\mathrm{D}} v^{\mathrm{bdr}}$$

Then the SUPG-stabilized discrete system is: Find $u \in V_h$ such that for all $v \in V_h$

$$a_{\mathrm{c}}(u,v) + \varepsilon a_{\mathrm{d}}(u,v) = \int_\Omega f v + l_{\mathrm{c}}(v) + \varepsilon\, l_{\mathrm{d}}(v). \tag{22}$$

As a numerical test, we consider the outflow-layer problem on $\Omega =]-1,+1[$ with $\beta = 1$ and exact solution $u(x) = \frac{1-\exp((x-1)/\varepsilon)}{1-\exp(-2/\varepsilon)}$. For $\varepsilon = 10^{-3}$, the $L^1(\Omega)$ error with respect to h is shown in log–log scale on the left of Fig. 2. In the middle, the discrete solutions on a coarse mesh and, on the right, the dependence of the $L^1(\Omega)$ errors on ε are shown.

6. Conclusions

We have presented variants of Nitsche's method. The first coincides for the Poisson problem with the traditional method. However, for more general elliptic equations, the method is non-conforming in the $H_0^1(\Omega)$ setting. This allows us to treat singularly perturbed problems up to the limit. We have illustrated this at hand of the linear reaction–diffusion equation.

The extension to domain decomposition with non-matching meshes and interface problems in the spirit of [11] is the subject of further work.

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