



Algebraic geometry

## Remarks on higher-rank ACM bundles on hypersurfaces

## Remarques sur les fibrés ACM de rang supérieur sur les hypersurfaces

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## ABSTRACT

In terms of the number of generators, one of the simplest non-split rank-3 arithmetically Cohen–Macaulay bundles on a smooth hypersurface in  $\mathbb{P}^5$  is 6-generated. We prove that a general hypersurface in  $\mathbb{P}^5$  of degree  $d \geq 3$  does not support such a bundle. We also prove that a smooth positive dimensional hypersurface in projective space of even degree does not support an Ulrich bundle of odd rank and determinant of the form  $\mathcal{O}_X(c)$  for some integer  $c$ . This verifies some cases of conjectures we discuss here.

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## R É S U M É

En termes de nombre de générateurs, le fibré de rang 3 arithmétiquement Cohen–Macaulay, non décomposé, le plus simple sur une hypersurface de  $\mathbb{P}^5$ , est engendré en rang 6. Nous montrons qu'une hypersurface générale dans  $\mathbb{P}^5$ , de degré  $d \geq 3$ , n'admet pas un tel fibré. Nous montrons également qu'une hypersurface lisse de dimension positive dans un espace projectif, de degré pair, n'admet pas de faisceau d'Ulrich de rang impair. Ceci permet de vérifier quelques cas de conjectures, que nous discutons ici.

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## 1. Introduction

We work over a characteristic-zero, algebraically closed field that we denote by  $K$ . Let  $X$  be a smooth hypersurface in  $\mathbb{P}^{n+1}$ . A vector bundle  $E$  on  $X$  is said to be *arithmetically Cohen–Macaulay* if it has no non-zero intermediate cohomology, i.e.

$$H_*^i(X, E) := \bigoplus_{m \in \mathbb{Z}} H^i(X, E(m)) = 0 \text{ for } 0 < i < n.$$

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On a projective space, by a result of Horrocks (see [6]), any ACM bundle is a sum of line bundles. It is easy to construct non-split ACM bundles on smooth hypersurfaces (see, for instance, Proposition 3, [11]). However, these are typically of large rank, and so this raises the question of the existence of low-rank non-split ACM bundles on smooth hypersurfaces. In [3], it has been conjectured that an ACM bundle of rank- $r$  on an  $n$ -dimensional smooth hypersurface is split if  $r < 2^e$ , where  $e := \lfloor \frac{n-1}{2} \rfloor$ . Here, for any real number  $q$ ,  $\lfloor q \rfloor$  is the largest integer  $\leq q$ .

The motivation for the above conjecture comes from the classification of ACM bundles on quadric hypersurfaces (see [8]). In view of this result, the weakest conjecture one may pose is that if there are no non-split ACM bundles of rank- $r \leq 2^m$  for some  $m$  on a smooth hypersurface, then there are none of rank- $r < 2^{m+1}$ . Thus, taking into account the splitting results in [9,10,12], it seems likely that for a general hypersurface of dimension  $n$  and sufficiently high degree, an ACM bundle of rank- $r$  on an  $n$ -dimensional smooth hypersurface is split if  $r < 2^s$ , where  $s := \lfloor \frac{n+1}{2} \rfloor$ . This was alluded to in [12] and made precise in [13]. A more optimistic conjecture can be found in [5]. This note presents some evidence in this direction.

The first higher-rank instance of the above statements, namely that of rank-2 ACM bundles on hypersurfaces in  $\mathbb{P}^4$  is well understood (see, for instance, [1,2,7,9,10,12]). The most general splitting results known so far are:

- in [7,9], it is shown that there are no non-split ACM rank-2 bundles on any smooth hypersurface in  $\mathbb{P}^6$ ;
- in [9,12], it is shown that there are no non-split rank-2 ACM bundles on a general hypersurface of degree  $d \geq 3$  in  $\mathbb{P}^5$ . Results for low-degree hypersurfaces obtained in [4] were used to complete the proof in [12];
- in [10,12], it is shown that there are no non-split rank-2 ACM bundles on a general hypersurface of degree  $d \geq 6$  in  $\mathbb{P}^4$ ;
- in [17], it is shown that there are no rank-3 non-split ACM bundles on any smooth hypersurface in  $\mathbb{P}^6$ ;
- partial results for rank-4 ACM bundles have been obtained in [16].

For a smooth hypersurface  $X \subset \mathbb{P}^{n+1}$  of degree  $d$ , an ACM bundle  $E$  of rank- $r$  on  $X$  comes with a minimal resolution

$$0 \rightarrow F_1 \xrightarrow{\Phi} F_0 \rightarrow E \rightarrow 0, \quad (1)$$

where  $F_0$  and  $F_1$  are sums of line bundles on  $\mathbb{P}^{n+1}$ , and  $\Phi$  is a matrix whose every non-zero entry is a homogeneous polynomial of positive degree. It follows that  $\det \Phi = f^r$ , where  $f$  is the defining polynomial of  $X$ .

We say that  $E$  is *s-generated* if the matrix  $\Phi$  is an  $s \times s$  matrix or equivalently,  $\text{rank}(F_0) = \text{rank}(F_1) = s$ . When  $E$  is indecomposable, we necessarily have  $\text{rank}(F_0) > \text{rank}(E)$  (since  $\text{rank}(F_0) = \text{rank}(E)$  implies that  $E = F_0 \otimes \mathcal{O}_X$  and hence  $E$  is split).  $E$  is *maximally generated* when all the non-zero entries of  $\Phi$  are linear forms. In this case, it follows from  $\det \Phi = f^r$ , that  $s = rd$ . Thus, we see that  $s$  satisfies the inequality  $r + 1 \leq s \leq rd$ . When  $s = rd$ , the matrix  $\Phi$  defines a rank- $r$  ACM bundle  $E'$ :

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(-1)^{rd} \xrightarrow{\Phi} \mathcal{O}_{\mathbb{P}^{n+1}}^{rd} \rightarrow E' \rightarrow 0.$$

Such an ACM bundle is an example of an *Ulrich bundle*.

On restricting the minimal resolution (1) to  $X$ , we get short exact sequences:

$$0 \rightarrow G \rightarrow \bar{F}_0 \rightarrow E \rightarrow 0, \quad \text{and} \quad 0 \rightarrow E(-d) \rightarrow \bar{F}_1 \rightarrow G \rightarrow 0.$$

Here  $\bar{F}_i := F_i \otimes \mathcal{O}_X$  for  $i = 1, 2$ . It follows that the syzygy bundle  $G$  is also ACM, and thus ACM bundles always occur in pairs.

The first result of this note is the following, which proves the base case (with regard to the number of generators) of the conjecture stated above for ACM bundles of rank 3.

**Theorem 1.** *Let  $X \subset \mathbb{P}^5$  be a general hypersurface of degree  $d \geq 3$ . With notation as above, assume that  $E$  is of rank 3 and that  $\wedge^3 G$  is ACM. Then  $E$  splits into a sum of line bundles.*

If  $G$  is a line bundle, then  $G = \mathcal{O}_X(a)$  for some  $a \in \mathbb{Z}$ , by the Lefschetz theorem. In this case, the sequence  $0 \rightarrow G \rightarrow \bar{F}_0 \rightarrow E \rightarrow 0$  splits (such a sequence corresponds to a class in  $\text{Ext}_X^1(E, \mathcal{O}_X(a)) \cong H^1(X, E^\vee(a)) = 0$ ). Since there are no non-split rank-2 bundles on a general hypersurface  $X \subset \mathbb{P}^5$  of degree at least 3 ([9,12]), it follows that the minimal number of generators for a rank-3 ACM bundle  $E$  on such an  $X$  is 6, i.e. the minimal rank of  $G$  is 3. When the rank of  $G$  is 3, then  $\wedge^3 G$  is a line bundle, and hence ACM. Furthermore,  $G$  has rank 4 when  $E$  is 7-generated, in which case we also have  $\wedge^3 G \cong G^\vee(c')$  where  $c' := c_1(G)$ . Thus Theorem 1 has the following corollary.

**Corollary 1.** *There are no indecomposable rank-3 ACM bundles on a general hypersurface in  $\mathbb{P}^5$  of degree  $d \geq 3$  with fewer than 8 generators.*

Our next result is about Ulrich bundles, which are at the other extreme, with regard to the number of generators.

**Theorem 2.** *A smooth positive dimensional hypersurface of even degree does not support an Ulrich bundle of odd rank and determinant equal to  $\mathcal{O}_X(c)$  for some  $c \in \mathbb{Z}$ .*

**Corollary 2.** *An even degree hypersurface of dimension at least 3 or a very general hypersurface of even degree in  $\mathbb{P}^3$  does not support an Ulrich bundle of odd rank.*

1.1. Outline of the proofs

Theorem 2 follows from a Riemann–Roch computation, whereas for Theorem 1, we extend the method in [11], which we describe now. Let  $X \subset \mathbb{P}^5$  be a smooth degree- $d$  hypersurface with defining polynomial  $f$ , and  $E$  be an ACM bundle on  $X$ . Let  $X_k$  denote the  $k$ -th-order thickening of  $X$  given by the vanishing of  $f^{k+1}$ . It is a standard fact that the obstruction for  $E$  to lift to the first-order thickening  $X_1$  is an element  $\eta_E \in H^2(X, \mathcal{E}ndE(-d))$ . To see this, we first note that if  $E$  lifts to a bundle  $\mathcal{E}$  on  $X_1$ , then such an  $\mathcal{E}$  sits in an  $\mathcal{O}_{X_1}$ -sequence

$$0 \rightarrow E(-d) \rightarrow \mathcal{E} \rightarrow E \rightarrow 0,$$

and hence defines an element of the group  $\text{Ext}_{X_1}^1(E, E(-d))$ . A standard Leray spectral sequence argument (see [14], Proposition 2) yields a 4-term sequence

$$0 \rightarrow H^1(X, \mathcal{E}ndE(-d)) \rightarrow \text{Ext}_{X_1}^1(E, E(-d)) \rightarrow H^0(X, \mathcal{E}ndE) \rightarrow H^2(X, \mathcal{E}ndE(-d)).$$

Let  $\eta_E$  denote the image of 1 under the last map. It is then clear that  $\eta_E = 0$  if and only if the element 1 lifts to an element in  $\text{Ext}_{X_1}^1(E, E(-d))$ . This, in turn, as explained in [14], is equivalent to the existence of a bundle  $\mathcal{E}$  on  $X_1$  as above.

To prove that an ACM bundle  $E$  splits, it is enough to show that  $\eta_E = 0$ . To do so, our first step is to show that there is a map (see (5), §2)

$$H_*^0(X, \det(E)) \rightarrow H_*^2(X, \mathcal{E}ndE).$$

Next, we identify the image of this map as the submodule generated by  $\eta_E$ . Finally, we show that this submodule, denoted by  $M$ , is preserved under Serre duality for  $H_*^2(X, \mathcal{E}ndE)$ , and hence is also self-dual. The fact that  $E$  is supported on a general hypersurface implies that  $g \cdot \eta_E = 0 \quad \forall g \in H^0(X, \mathcal{O}_X(d))$  (see Proposition 1) and so  $M = \bigoplus_{-d \leq k < 0} M_k$ . The proof is finished by observing that Serre duality gives an isomorphism  $M_{-d} \cong M_{2d-6}$ , and so  $\eta_E = 0$  when  $d \geq 3$ .

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2. Preliminaries

We refer the interested reader to [14] for more details. Let  $n \geq 4$ , and  $X \subset \mathbb{P}^{n+1}$  be a smooth hypersurface of degree  $d$ , with  $I$  its ideal sheaf, and  $f$  its defining polynomial. Let  $E$  be a rank- $r$  ACM bundle on  $X$ . Since  $\dim X \geq 4$ , we have  $\text{Pic}(X) \cong \mathbb{Z}$ , and using this isomorphism, we let  $c := c_1(E) \in \mathbb{Z}$  so that  $\wedge^r E \cong \mathcal{O}_X(c)$ . Let

$$0 \rightarrow F_1 \xrightarrow{\Phi} F_0 \rightarrow E \rightarrow 0 \tag{2}$$

be a minimal resolution of  $E$  on  $\mathbb{P}^{n+1}$  as in §1. Restricting to  $X$ , we get a 4-term exact sequence

$$0 \rightarrow E(-d) \rightarrow \bar{F}_1 \xrightarrow{\bar{\Phi}} \bar{F}_0 \rightarrow E \rightarrow 0.$$

Let  $G := \text{Image}(\bar{\Phi})$ . Breaking this up into short exact sequences, we get

$$0 \rightarrow G \rightarrow \bar{F}_0 \rightarrow E \rightarrow 0 \text{ and,} \tag{3}$$

$$0 \rightarrow E(-d) \rightarrow \bar{F}_1 \rightarrow G \rightarrow 0. \tag{4}$$

Tensoring (3) and (4) with  $E^\vee$ , we get cohomology long exact sequences:

$$\begin{aligned} H_*^0(X, \mathcal{E}ndE) &\xrightarrow{\partial_1} H_*^1(X, E^\vee \otimes G) \rightarrow H_*^1(X, E^\vee \otimes \bar{F}_0), \text{ and} \\ H_*^1(X, E^\vee \otimes \bar{F}_1) &\rightarrow H_*^1(X, E^\vee \otimes G) \xrightarrow{\partial_2} H_*^2(X, \mathcal{E}ndE(-d)) \rightarrow H_*^2(X, E^\vee \otimes \bar{F}_1). \end{aligned}$$

Since  $F_0$  and  $F_1$  are sums of line bundles, and  $n \geq 4$ , it follows that  $\partial_1$  is a surjection, and  $\partial_2$  is an isomorphism. Composing the coboundary maps, we get

$$\begin{array}{ccccccc} H_*^0(X, \mathcal{E}ndE) & \xrightarrow{\partial_1} & H_*^1(X, E^\vee \otimes G) & \xrightarrow{\partial_2} & H_*^2(X, \mathcal{E}ndE(-d)) & & \\ 1 & \mapsto & \zeta_E & \mapsto & \eta_E & & \end{array} \tag{5}$$

Here  $\zeta_E \in \text{Ext}_X^1(E, G) \cong H^1(X, E^\vee \otimes G)$  is the class of the short exact sequence (3), and  $\eta_E$  is the (obstruction) class which vanishes if and only if  $E$  extends to a vector bundle on  $X_1$  (see §3.3 in [14] for details).

Equivalently, one may start with the dual bundle  $E^\vee$ , and the sequences

$$0 \rightarrow G^\vee(-d) \rightarrow \overline{F}_1^\vee(-d) \rightarrow E^\vee \rightarrow 0, \quad \text{and} \quad 0 \rightarrow E^\vee \rightarrow \overline{F}_0^\vee \rightarrow G^\vee \rightarrow 0$$

to get

$$\begin{array}{ccccc} H_*^0(X, \text{End}E) & \xrightarrow{\partial'_1} & H_*^1(X, E \otimes G^\vee(-d)) & \xrightarrow{\partial'_2} & H_*^2(X, \text{End}E(-d)) \\ 1 & \mapsto & \zeta_{E^\vee} & \mapsto & \eta_{E^\vee} = \eta_E. \end{array} \tag{6}$$

Now assume that  $\text{rank}(E) = 3$ ; then we have an isomorphism  $\wedge^2 E \cong E^\vee \otimes \wedge^3 E$ . Tensoring (3) with  $\wedge^2 E$ , we have a sequence,

$$0 \rightarrow \wedge^2 E \otimes G \rightarrow \wedge^2 E \otimes \overline{F}_0 \rightarrow \wedge^2 E \otimes E \rightarrow 0,$$

which via the inclusion  $\mathcal{O}_X \hookrightarrow \text{End}E$ , yields a pull-back diagram as follows:

$$\begin{array}{ccccccc} 0 & \rightarrow & \wedge^2 E \otimes G & \rightarrow & \mathcal{P}_3 & \rightarrow & \wedge^3 E \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \rightarrow & \wedge^2 E \otimes G & \rightarrow & \wedge^2 E \otimes \overline{F}_0 & \rightarrow & \wedge^2 E \otimes E \rightarrow 0. \end{array} \tag{7}$$

On tensoring this diagram with  $(\wedge^3 E)^{-1}$ , we see that, under the composite map

$$H^0(X, \mathcal{O}_X) \rightarrow H^0(X, \text{End}E) \rightarrow H^1(X, E^\vee \otimes G),$$

the generator  $1 \in H^0(X, \mathcal{O}_X)$  is mapped to the element  $\zeta_E \in H^1(X, E^\vee \otimes G)$ . Hence the Yoneda class of the top-row exact sequence is the element

$$\zeta_E \in \text{Ext}_X^1(\wedge^3 E, G \otimes \wedge^2 E) \cong H^1(X, E^\vee \otimes G).$$

We recall a standard result, which we use quite extensively.

**Lemma 1.** For any short exact sequence of bundles  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  on a variety, we get, on taking exterior powers, sequences  $0 \rightarrow \mathcal{A} \rightarrow \wedge^k B \rightarrow \wedge^k C \rightarrow 0$ , where  $\mathcal{A}$  is a filtered vector bundle with filtration

$$\mathcal{A} = F^0 \supset F^1 \supset \dots \supset F^k = \{0\},$$

and associated graded pieces  $\text{gr}_F^j \mathcal{A} = \wedge^{j+1} A \otimes \wedge^{k-j-1} C$  for  $0 \leq j < k$ .

The surjection  $\overline{F}_0 \rightarrow E$  yields a commutative square

$$\begin{array}{ccc} \wedge^3 \overline{F}_0 & \twoheadrightarrow & \wedge^3 E \\ \downarrow & & \downarrow \\ \wedge^2 \overline{F}_0 \otimes \overline{F}_0 & \twoheadrightarrow & \wedge^2 E \otimes E \end{array}$$

where the bottom row factors via  $\wedge^2 E \otimes \overline{F}_0$ , and so we obtain a commutative diagram:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{K}_3 & \rightarrow & \wedge^3 \overline{F}_0 & \rightarrow & \wedge^3 E \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \wedge^2 E \otimes G & \rightarrow & \wedge^2 E \otimes \overline{F}_0 & \rightarrow & \wedge^2 E \otimes E \rightarrow 0. \end{array} \tag{8}$$

The vertical maps in (8) factor via the top row in (7) by the universal property of pull-backs and so we get the following commutative diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{K}_{3,1} & = & \mathcal{K}_{3,1} & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathcal{K}_3 & \rightarrow & \wedge^3 \overline{F}_0 & \rightarrow & \wedge^3 E \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \rightarrow & \wedge^2 E \otimes G & \rightarrow & \mathcal{P}_3 & \rightarrow & \wedge^3 E \rightarrow 0. \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \tag{9}$$

Here  $\mathcal{K}_3$  comes equipped with a filtration  $F^2 = \wedge^3 G \subset F^1 = \mathcal{K}_{3,1} \subset F^0 = \mathcal{K}_3$  as described in Lemma 1 applied to the exact sequence (3). In particular, we have  $F^1/F^2 = \wedge^2 G \otimes E$ , and so an exact sequence

$$0 \rightarrow \wedge^3 G \rightarrow \mathcal{K}_{3,1} \rightarrow E \otimes \wedge^2 G \rightarrow 0. \tag{10}$$

Let  $\kappa$  denote the Yoneda class of the middle row. Then diagram (9) yields a map

$$\text{Ext}^1(\wedge^3 E, \mathcal{K}_3) \rightarrow \text{Ext}^1(\wedge^3 E, \wedge^2 E \otimes G),$$

under which  $\kappa$  is mapped to  $\zeta_E$ . This map may be identified with the map of cohomology groups in the left vertical sequence

$$H^1(X, \mathcal{K}_3(-c)) \rightarrow H^1(X, \wedge^2 E \otimes G(-c)).$$

Furthermore, doing the same with the dual bundle  $E^\vee$ , we get a commutative diagram analogous to (9):

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{K}'_{3,1} & = & \mathcal{K}'_{3,1} & & \\ & & \downarrow & & \downarrow & & \\ 0 \rightarrow & & \mathcal{K}'_3 & \rightarrow & \wedge^3 \overline{F}_1^\vee & \rightarrow & \wedge^3 E^\vee(3d) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 \rightarrow & & \wedge^2 E^\vee(2d) \otimes G^\vee & \rightarrow & \mathcal{P}'_3 & \rightarrow & \wedge^3 E^\vee(3d) \rightarrow 0, \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} \tag{11}$$

and an exact sequence

$$0 \rightarrow \wedge^3 G^\vee \rightarrow \mathcal{K}'_{3,1} \rightarrow E^\vee(d) \otimes \wedge^2 G^\vee \rightarrow 0. \tag{12}$$

### 3. Proof of Theorem 1

The following result explains what it means for  $E$  to be a vector bundle on a *general* hypersurface.

**Proposition 1.** *Let  $X \subset \mathbb{P}^{n+1}$  be a general hypersurface. For any  $g \in H^0(X, \mathcal{O}_X(d))$ , the image of  $\eta_E$  under the multiplication map*

$$\times g : H^2(X, \mathcal{E}nd E(-d)) \rightarrow H^2(X, \mathcal{E}nd E),$$

*is zero.*

**Proof.** See §3 in [9], or [14], Proposition 2 (i), and Proof of Theorem 2 for details.  $\square$

Tensoring (4) with  $\wedge^2 E$ , we get

$$0 \rightarrow \wedge^2 E \otimes E(-d) \rightarrow \wedge^2 E \otimes \overline{F}_1 \rightarrow \wedge^2 E \otimes G \rightarrow 0.$$

Using the surjection  $\mathcal{K}_3 \twoheadrightarrow \wedge^2 E \otimes G$ , and the isomorphism  $\wedge^2 E \cong E^\vee(c)$ , we get a pull-back diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{K}_{3,1}(-c) & = & \mathcal{K}_{3,1}(-c) & & \\ & & \downarrow & & \downarrow & & \\ 0 \rightarrow & & \mathcal{E}nd E(-d) & \rightarrow & \widetilde{\mathcal{K}}_3 & \rightarrow & \mathcal{K}_3(-c) \rightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 \rightarrow & & \mathcal{E}nd E(-d) & \rightarrow & E^\vee \otimes \overline{F}_1 & \rightarrow & E^\vee \otimes G \rightarrow 0, \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array} \tag{13}$$

Here  $\widetilde{\mathcal{K}}_3$  is defined by the diagram.

In the middle column, we have an isomorphism of graded modules

$$H_*^2(X, \mathcal{K}_{3,1}(-c)) \cong H_*^2(X, \widetilde{\mathcal{K}}_3).$$

Furthermore, we observe from (8) that

- (i) we have a surjection  $H_*^0(X, \mathcal{O}_X) \rightarrow H_*^1(X, \mathcal{K}_3(-c))$ , under which 1 is mapped to  $\kappa$ , and
- (ii)  $H_*^2(X, \mathcal{K}_3) = 0$ .

Hence the cohomology sequence of the middle row yields a (right) exact sequence of graded modules

$$H_*^1(X, \mathcal{K}_3(-c)) \rightarrow H_*^2(X, \mathcal{E}ndE(-d)) \rightarrow H_*^2(X, \mathcal{K}_{3,1}(-c)) \rightarrow 0. \tag{14}$$

A similar analysis will also yield a (right) exact sequence

$$H_*^1(X, \mathcal{K}'_3(c - 3d)) \rightarrow H_*^2(X, \mathcal{E}ndE(-d)) \rightarrow H_*^2(X, \mathcal{K}'_{3,1}(c - 3d)) \rightarrow 0. \tag{15}$$

Since the leftmost maps above factor via  $H_*^1(X, E^\vee \otimes G)$  and  $H_*^1(X, E \otimes G^\vee(-d))$  respectively, we have the following result:

**Lemma 2.** *Let  $X \subset \mathbb{P}^{n+1}$  be as above.*

- (i) *Under the map  $H_*^1(X, \mathcal{K}_3(-c)) \rightarrow H_*^2(X, \mathcal{E}ndE(-d))$ , the generator  $\kappa$  is mapped to  $\eta_E$ .*
- (ii) *Under the map  $H_*^1(X, \mathcal{K}'_3(c - 3d)) \rightarrow H_*^2(X, \mathcal{E}ndE(-d))$ , the generator  $\kappa'$  is mapped to  $\eta_E$ .*

Let  $M$  denote the submodule of  $H_*^2(X, \mathcal{E}ndE(-d))$  generated by  $\eta_E$ . We rewrite sequences (14) and (15) as

$$0 \rightarrow M \rightarrow H_*^2(X, \mathcal{E}ndE(-d)) \rightarrow H_*^2(X, \mathcal{K}_{3,1}(-c)) \rightarrow 0, \text{ and} \tag{16}$$

$$0 \rightarrow M \rightarrow H_*^2(X, \mathcal{E}ndE(-d)) \rightarrow H_*^2(X, \mathcal{K}'_{3,1}(c - 3d)) \rightarrow 0. \tag{17}$$

An immediate consequence of Proposition 1 is the following:

**Corollary 3.** *The module  $M$  is supported in degrees  $k$  where  $-d \leq k < 0$ , i.e.*

$$M = \bigoplus_{-d \leq k < 0} M_k.$$

Now we are ready to prove our first result.

**Proof of Theorem 1.** We shall prove that the obstruction class  $\eta_E$  vanishes in  $H^2(X, \mathcal{E}ndE(-d))$ . From (5), it then follows that  $\zeta_E = 0$ , which means that the sequence (3) splits. This implies that  $E$  splits.

We begin by noting that since  $\wedge^3 G$  is ACM, we have, from (10) and (12),

- $H_*^2(X, \mathcal{K}_{3,1}(-c)) \cong H_*^2(X, E \otimes \wedge^2 G(-c))$ , and
- $H_*^2(X, \mathcal{K}'_{3,1}(c - 3d)) \cong H_*^2(X, E^\vee \otimes \wedge^2 G^\vee(c - 2d))$ .

Using these isomorphisms, sequences (16) and (17) can be rewritten as

$$0 \rightarrow M \rightarrow H_*^2(X, \mathcal{E}ndE(-d)) \rightarrow H_*^2(X, E \otimes \wedge^2 G(-c)) \rightarrow 0, \text{ and} \tag{18}$$

$$0 \rightarrow M \rightarrow H_*^2(X, \mathcal{E}ndE(-d)) \rightarrow H_*^2(X, E^\vee \otimes \wedge^2 G^\vee(c - 2d)) \rightarrow 0. \tag{19}$$

Since the left and the middle terms in (18) and (19) are isomorphic to each other, we have

$$H_*^2(X, E \otimes \wedge^2 G(-c)) \cong H_*^2(X, E^\vee \otimes \wedge^2 G^\vee(c - 2d)).$$

By Serre duality, we also have

$$H^2(X, E \otimes \wedge^2 G(-c)) \cong H^2(X, E^\vee \otimes \wedge^2 G^\vee(c + d - 6)),$$

and hence an isomorphism

$$H^2(X, E^\vee \otimes \wedge^2 G^\vee(c - 2d)) \cong H^2(X, E^\vee \otimes \wedge^2 G^\vee(c + d - 6)). \tag{20}$$

Thus we have the following exact sequences (corresponding to the degree ‘ $-d$ ’ and ‘ $2d - 6$ ’ components in (19)):

$$0 \rightarrow M_{-d} \rightarrow H^2(X, \mathcal{E}ndE(-d)) \rightarrow H^2(X, E^\vee \otimes \wedge^2 G^\vee(c - 2d)) \rightarrow 0, \tag{21}$$

$$0 \rightarrow M_{2d-6} \rightarrow H^2(X, \mathcal{E}ndE(2d - 6)) \rightarrow H^2(X, E^\vee \otimes \wedge^2 G^\vee(c + d - 6)) \rightarrow 0. \tag{22}$$

Since  $H^2(X, \mathcal{E}ndE(-d)) \cong H^2(X, \mathcal{E}ndE(2d - 6))$  by Serre duality, it follows from the above sequences and (20) that  $M_{-d} \cong M_{2d-6}$ . Since  $M_k = 0$  for  $k \geq 0$  by Corollary 3, it follows that  $M_{-d} = 0$  if  $2d - 6 \geq 0$ , or equivalently if  $d \geq 3$ . This means that  $\eta_E = 0$ , and so this finishes the proof.  $\square$

#### 4. Ulrich bundles

As mentioned in the introduction, the proof of the non-existence of Ulrich bundles of odd rank on smooth even-degree hypersurfaces of dimension  $\geq 1$  follows from a Riemann–Roch computation. The following result gives an upper bound for the first Chern class for any ACM bundle.

**Lemma 3.** *Let  $E$  be a rank- $r$  ACM bundle on a smooth, degree- $d$  hypersurface  $X \subset \mathbb{P}^{n+1}$ ,  $n \geq 1$ , with first Chern class  $\mathcal{O}_X(c)$ . Assume that  $E$  is normalized, i.e.  $H^0(X, E(-1)) = 0$ , and  $H^0(X, E) \neq 0$ . Then  $c \leq r(d-1)/2$ .*

**Proof.** Let  $E_H$  denote the restriction of  $E$  to a smooth hyperplane section  $H \subset X$ , so that we have a sequence

$$0 \rightarrow E(-1) \rightarrow E \rightarrow E_H \rightarrow 0.$$

Then  $E_H$  is a normalized rank- $r$  ACM bundle on  $H$ . Since  $\text{Pic}(X) \rightarrow \text{Pic}(H)$  is injective, we have  $c_1(E_H) = \mathcal{O}_X(c)$ . Thus, by induction, we may assume that  $E$  is a normalized vector bundle of rank- $r$  on a smooth, planar curve  $C \subset \mathbb{P}^2$  of degree  $d$ . Since  $E$  is normalized,  $\chi(E(-1)) \leq 0$ . Applying the Riemann–Roch theorem to the bundle  $E(-1)$  gives us

$$\chi(E(-1)) = \text{deg}(E(-1)) + r(1-g) \leq 0,$$

where  $g$  is the genus of  $C$ . Since  $C$  is planar, we have  $g = (d-1)(d-2)/2$ . Using this and the fact that  $\text{deg}(E(-1)) = d(c-r)$ , we get  $c \leq r(d-1)/2$ .  $\square$

**Proof of Theorem 2.** Since  $E$  is Ulrich, it has a resolution of the form

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^{n+1}}(-1)^{rd} \rightarrow \mathcal{O}_{\mathbb{P}^{n+1}}^{rd} \rightarrow E \rightarrow 0.$$

Therefore  $\chi(E(-1)) = 0$ . It follows now that  $c = r(d-1)/2$ . Since  $c \in \mathbb{Z}$ , this is impossible if  $r$  is odd and  $d$  is even.  $\square$

**Remark 1.** The assumption on the degree of the hypersurface or the rank of the bundle in Theorem 2 cannot be weakened. A smooth plane cubic is an elliptic curve; its torsion points of odd order  $r$ , give rise to Ulrich bundles of rank- $r$ , (cf. [15]). Similarly, lines in quadric surfaces in  $\mathbb{P}^3$  correspond to ACM line bundles not isomorphic to  $\mathcal{O}_X(c)$  for any  $c \in \mathbb{Z}$ .

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