C. R. Acad. Sci. Paris, Ser. I 356 (2018) 1156–1166

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Partial differential equations

Note on the fall of an axisymmetric body in a perfect fluid over a horizontal ramp

Note sur la chute d'un solide axisymétrique dans un fluide parfait au-dessus d'un plan horizontal

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ARTICLE INFO

Article history:
Received 27 August 2018
Accepted 18 October 2018
Available online 30 October 2018

Presented by the Editorial Board

ABSTRACT

In this note, we consider the fall of an axisymmetric body in a perfect fluid over a ramp. It was shown in [12] that the possibility of a collision between the body and the ramp is related to the asymptotics of the so-called added mass when the distance between the ramp and the body goes to 0. We propose here a new method to compute this added mass, which provides simultaneously an approximation of an associated fluid velocity field in the gap between the ramp and the body.

RÉSUMÉ

Dans cette note, nous considérons la chute d'un solide axisymétrique dans un fluide parfait au-dessus d'un plan. Il est connu [12] que l'éventualité d'un contact entre le solide et le plan est reliée à l'asymptotique de l'effet de masse ajoutée quand la distance entre le plan et le solide tend vers 0. Nous proposons une nouvelle méthode pour calculer cet effet de masse ajoutée, qui fournit simultanément l'asymptotique d'un champ de vitesses associé entre le solide et le plan.

1. Introduction

The study of fluid–solid problems leads to considerable paradoxical difficulties because of possible contacts between the bodies inside the fluid. Indeed, classical theories for solving partial differential equations of fluid mechanics require the fluid domain to be sufficiently smooth (typically its boundaries should be locally the graph of a Lipschitz function). However, whatever the smoothness of solid boundaries, contact between solid bodies or between one solid body and the container

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https://doi.org/10.1016/j.crma.2018.10.001

boundaries yields geometries in which the fluid domain boundaries are no longer locally Lipschitz. This makes very weak any mathematical theory for fluid–solid problems handling contacts (see [13,3]). Evidences that such theory might be ill posed are given in [14]. Yet, a naive intuition of fluid–solid problems entails that a realistic mathematical theory should handle collisions.

Numerous studies consider the collision problem either experimentally [10] or theoretically [6]. In case the fluid is viscous, first theoretical investigations, based on formal asymptotic expansions for the Stokes problem in thin domains (see [2]), highlight that no collision is to be expected. These formal expansions are justified *via* a sound analysis relying on variational arguments [4,7]. The no-collision paradox is extended to the full Newton–Navier–Stokes system in [5,8]. Fewer references investigate the case of an inviscid fluid. If the fluid is potential, the existence of finite-time contact is obtained in [9,12], coupling a Lagrangian viewpoint with complex analysis methods. Recently, these contact results are extended to non-potential flows in [1] when vorticity in the flow is sufficiently small.

In [4], the formulas of [2] are recovered *via* the introduction of a reduced-functional method. This method turns out to be very robust since it applies to general geometries and boundary conditions. In this note, we apply this method, which is introduced for viscous flows, to study potential flows also. To complement the complex analysis approach of [12], this yields a precise expansion of the flow in the aperture between solid boundaries close to contact. We expect that such a remark is a first step toward a finer understanding of the smallness condition required in [1] to obtain the occurrence of finite-time contacts in the non-potential case.

1.1. Context

Let consider the following simplified configuration taken from [12]. We denote by $\mathcal{S}$ an axisymmetric (homogeneous) rigid body. This body falls vertically in a potential fluid over a fixed horizontal wall $\mathcal{P}$. For simplicity, we neglect gravity, since this has no influence on the contact result. We fix $\mathcal{P} := \{(x, y) \in \mathbb{R}^2,\ \text{such that } y = 0\}$. The solid moves vertically. So, the whole configuration is fixed by the distance h between $\mathcal{S}$ and $\mathcal{P}$. Below we denote by the index h such a configuration:

$$\mathcal{S}_h = h\,\boldsymbol{e}_2 + \mathcal{S} \qquad F_h = \mathbb{R}^2_+ \setminus \bar{\mathcal{S}}_h.$$

We have then that $h = 0$ corresponds to the contact configuration (see Fig. 1).

The unknowns of the fluid–solid interaction problem are then $(\boldsymbol{u}, p)$ the fluid velocity field/pressure and h the distance between the axisymmetric body and the ramp. In case the fluid is assumed inviscid and incompressible, with constant density $\rho_f > 0$, these unknowns are computed by integrating the pde/ode system:

$$\left\{\begin{array}{l} \rho_f\,(\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) + \nabla p = 0 \\ \operatorname{div} \boldsymbol{u} = 0 \end{array}\right. \quad \text{in } F_{h(t)} \tag{1}$$

$$\left\{\begin{array}{l} \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \mathcal{P} \\ \boldsymbol{u}\cdot\boldsymbol{n} = h'(t)\boldsymbol{e}_2\cdot\boldsymbol{n} \text{ on } \partial\mathcal{S}_{h(t)} \end{array}\right. \tag{2}$$

$$mh''(t) = \int_{\partial S_{h(t)}} p\,\boldsymbol{n}\cdot\boldsymbol{e}_2\,\mathrm{d}\sigma. \tag{3}$$

The symbol $\boldsymbol{n}$ stands here for the outer normal to $\partial F_{h(t)}$. More details on the modelling and the resolution of this system can be found in [9,12]. We recall shortly here that, for sufficiently smooth solutions:

- the total kinetic energy:

$$E(t) = \frac{1}{2}m|h'(t)|^2 + \frac{1}{2}\rho_f \int_{F_h} |\boldsymbol{u}(\boldsymbol{x},t)|^2\,\mathrm{d}\boldsymbol{x}$$

  is conserved with time,
- no fluid vorticity is produced by boundaries, so that the vorticity of the flow is simply transported by the velocity field $\boldsymbol{u}$.

As a consequence, if the initial data prescribe a potential velocity, the solution to (1)–(2)–(3) is potential also. In that case, the velocity field $\boldsymbol{u}$ is computed by stating $\boldsymbol{u} = h'(t)\,\boldsymbol{u}_{h(t)}$, where, for $h > 0$, $\boldsymbol{u}_h$ solves:

$$\left\{\begin{array}{ll} \nabla\times\boldsymbol{u}_h = 0 & \text{in } F_h \\ \nabla\cdot\boldsymbol{u}_h = 0 & \text{in } F_h \end{array}\right. \tag{4}$$

with $\nabla\times\boldsymbol{u}_h = \partial_1 u_2 - \partial_2 u_1$ and boundary conditions

$$\left\{\begin{array}{ll} \boldsymbol{u}_h\cdot\boldsymbol{n} = 0 & \text{on } \mathcal{P} \\ \boldsymbol{u}_h\cdot\boldsymbol{n} = \boldsymbol{e}_2\cdot\boldsymbol{n} & \text{on } \partial\mathcal{S}_h. \end{array}\right. \tag{5}$$

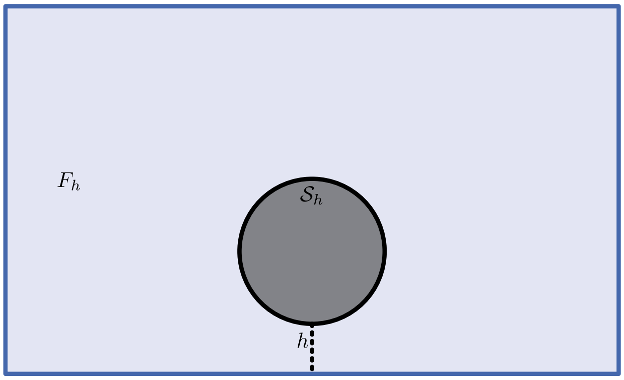

Fig. 1. Example of a configuration with notations.

Such systems yield a unique solution (see below) up to the prescription of the circulation on the internal boundary. We focus herein on circulation-free solutions. So, we prescribe:

$$\int_{\partial \mathcal{S}_h} \boldsymbol{u}_h \cdot \boldsymbol{n}^{\perp} \mathrm{d}\sigma = 0. \tag{6}$$

We introduce here the exponent $\perp$ to denote a rotation of angle $\pi/2$. Though equation (3) requires the computation of the pressure p, we note that the pressure needs not be computed in the potential case. Indeed, fixing that the total kinetic energy E is conserved along trajectories rewrites as an ode for h, which governs the dynamics:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[|h'|^2\left(m_{\mathrm{s}} + \rho_{\mathrm{f}} \int_{F_h} |\boldsymbol{u}_h(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x}\right)\right] = 0.$$

Denoting by $m_{\mathrm{a}}(h)$ the integral appearing in this quantity, we have then (as long as the solution exists and h remains strictly positive):

$$h'(t) = h'(0)\sqrt{\frac{m + \rho_{\mathrm{f}} m_{\mathrm{a}}(h)}{m + \rho_{\mathrm{f}} m_{\mathrm{a}}(h(0))}}. \tag{7}$$

Possible contacts in finite time (if $h'(0) < 0$) depend only on the asymptotics of $m_{\mathrm{a}}(h)$ when $h \to 0$. This quantity $m_{\mathrm{a}}(h)$ measures the "added-mass" effect. Namely, it stands for the minimal fluid kinetic energy that is associated with the velocity field $\boldsymbol{e}_2 \cdot \boldsymbol{n}$ on the body $\mathcal{S}$. It depends on the only geometrical parameter: the distance h between $\mathcal{S}$ and the wall. We refer the reader to [12] for the discussion on the properties that $m_{\mathrm{a}}(h)$ must satisfy in order to allow/forbid contact. Herein, we are interested in a method that yields simultaneously the asymptotic behavior of $m_{\mathrm{a}}(h)$ (when $h \to 0$) and a profile for $\boldsymbol{u}_h$.

1.2. Main result

To state our main results, we make a little more precise the geometry of the body $\mathcal{S}$. We always assume that $\partial\mathcal{S} \in C^2$. We also assume that contact between $\mathcal{S}$ and $\mathcal{P}$ only holds in the origin and is of order "$1+\alpha$." Namely, for $\alpha \geq 1$ given, we define the following assumption.

$$\left\{\begin{array}{ll} \text{There exists } \delta > 0,\ \varepsilon > 0 \text{ and } \kappa > 0 \text{ such that} & \\ \quad x_2 > \varepsilon & \forall (x_1, x_2) \in \partial\mathcal{S} \text{ s.t. } |x_1| > \delta \\ \quad x_2 = \kappa |x|^{1+\alpha} & \forall (x_1, x_2) \in \partial\mathcal{S} \text{ s.t. } |x_1| < \delta,\ |x_2| < \varepsilon \end{array}\right\}. \tag{A_α}$$

Our main result then reads:

Theorem 1. *Let $\alpha \geq 1$ and assume that $\mathcal{S}$ satisfies (A_α). We have the following alternative:*
if $\alpha < 2$ the function $h \mapsto m_{\mathrm{a}}(h)$ remains bounded when $h \to 0$;
if $\alpha > 2$, denoting $\sigma_h(x) = h + \kappa|x|^{1+\alpha}$, we have the following asymptotics when $h \to 0$:

$$m_{\mathrm{a}}(h) = c_{\kappa,\alpha} h^{3/(1+\alpha)-1} + O(1), \quad \textit{with } c_{\kappa,\alpha} = \int_{\mathbb{R}} \frac{t^2 \mathrm{d}t}{1+|t|^{1+\alpha}} \tag{8}$$

$$u_h(x,y) = \nabla^\perp \left[\frac{x_1 x_2}{\sigma_h(x_1)} \right] \mathbf{1}_{|x_1|<\delta \cap x_2 < \varepsilon} + O(1) \quad \text{in } L^2(F_h). \tag{9}$$

Following [12], this theorem entails that collision is bound to occur in finite time whatever the value of $\alpha \geq 1$. The expansion of $m_a(h)$ we state here is already obtained in this reference. Nonetheless, we emphasize that we obtain this same result here with a completely different method, which yields also information on the flow $\boldsymbol{u}_h$. In particular, we are able to compute the diverging part of the flow in the case $\alpha > 2$.

The outline of these notes is as follows. In the next section, we recall the variational viewpoint on system (4)–(5) and explain the method of reduced functionals. In the last section, we perform the explicit computations to extract the asymptotics of $m_a(h)$ and $\boldsymbol{u}_h$ when $h \to 0$.

2. Application of variational methods to (4)–(5)

As explained in the introduction, our proof of **Theorem 1** is based on a fine estimate of the behavior of the added-mass function:

$$\begin{aligned} m_a : (0,\infty) &\longrightarrow \mathbb{R} \\ h &\longmapsto \int_{F_h} |\boldsymbol{u}_h(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \quad (\text{with } \boldsymbol{u}_h \text{ solution to (4)–(5)}). \end{aligned}$$

In this section, at first, we recall the basics on the resolution of (4)–(5). We recall also the variational approach to this problem as presented in [11]. We explain then the method of reduced functionals and how it applies to this problem.

2.1. Resolution of (4)–(5)

In what follows, we assume $\alpha \geq 1$ so that $\partial\mathcal{S}$ is C^2 globally. We denote:

$$L^2_{\mathrm{div}}(F_h) := \{\boldsymbol{v} \in L^2(F_h) \text{ s.t. div } \boldsymbol{v} \in L^2(F_h)\}, \quad L^2_\sigma(F_h) := \{\boldsymbol{v} \in L^2(F_h) \text{ s.t. div } \boldsymbol{v} = 0\}.$$

We recall that $L^2_{\mathrm{div}}(F_h)$ – endowed with the norm obtained by combining the L^2-norm and the L^2-norm of the divergence – is a Hilbert space. The space $L^2_\sigma(F_h)$ is then a Hilbert space when endowed with the L^2-norm. Since ∂F_h splits into the two connected components $\partial\mathcal{S}_h$ and $\partial\mathcal{P}$, which are both at least C^2, we have (see [15, Chapter 1]):

- $C^\infty_c(\overline{F_h})$ is dense in $L^2_{\mathrm{div}}(F_h)$;
- there exists a continuous linear operator $\gamma_{\boldsymbol{n}} : L^2_{\mathrm{div}}(F_h) \to H^{-1/2}(\partial F_h)$ such that, denoting by $\boldsymbol{n}$ the outward normal to ∂F_h, we have $\gamma_{\boldsymbol{n}}(\boldsymbol{v}) = \boldsymbol{v}\cdot\boldsymbol{n}$ for any $\boldsymbol{v} \in C^\infty_c(\overline{F_h})$;
- $E_0(F_h) := \{v \in C^\infty_c(F_h) \text{ s.t. div } \boldsymbol{v} = 0\}$ is dense in $L^2_{\sigma,0}(F_h) := \mathrm{Ker}(\gamma_{\boldsymbol{n}}) \cap L^2_\sigma(F_h)$.

With these notations, we have:

Definition 1. *We call $\boldsymbol{u} \in L^2_\sigma(F_h)$ a weak solution to (4)–(5)–(6) if*

1. *$\gamma_{\boldsymbol{n}}[\boldsymbol{u}] = \boldsymbol{e}_2 \cdot \boldsymbol{n}$ on $\partial\mathcal{S}_h$ and $\gamma_{\boldsymbol{n}}[\boldsymbol{u}] = 0$ on $\partial\mathcal{P}$,*
2. *for any $\boldsymbol{w} \in L^2_{\sigma,0}(F_h)$ there holds:*

$$\int_{F_h} \boldsymbol{u}\cdot\boldsymbol{w} = 0. \tag{10}$$

The weak formulation for (4)–(5) is obtained standardly by remarking that, in the sense of distributions, there holds

$$\langle \nabla \times \boldsymbol{u}, \psi \rangle = -\int_{F_h} \boldsymbol{u}\cdot\nabla^\perp\psi, \quad \forall \psi \in C^\infty_c(F_h).$$

If $\boldsymbol{u}$ solves (4), we obtain then (10) with $\boldsymbol{w} = \nabla^\perp\psi \in L^2_{\sigma,0}(F_h)$. Conversely, for arbitrary $\boldsymbol{w} = (w_1, w_2) \in E_0(F_h)$, setting

$$\psi(x_1,x_2) = \int_0^{x_1} w_2(s,0) - \int_0^{x_2} w_1(x_1,s)\mathrm{d}s,$$

yields $\psi \in C_c^\infty(\mathbb{R}^2_+)$ with ψ constant on $\overline{\mathcal{S}}_h$ and $\boldsymbol{w} = \nabla^\perp \psi$. Denoting ℓ_ψ the value of ψ on $\mathcal{S}_h$, the weak formulation entails (under the assumption that $\boldsymbol{u}$ is sufficiently smooth to make integration by parts meaningful):

$$\int_{F_h} \boldsymbol{u} \cdot \boldsymbol{w} = \int_{F_h} \boldsymbol{u} \cdot \nabla^\perp \psi$$
$$= \ell_\psi \int_{\partial \mathcal{S}_h} \boldsymbol{u} \cdot \boldsymbol{n}^\perp \mathrm{d}\sigma - \int_{F_h} \psi \nabla \times \boldsymbol{u}.$$

Since ℓ_ψ, ψ can be made arbitrary we obtain (4)–(6).

With this formulation, we have the following variant of [11, p. 37] for our specific unbounded domain:

Theorem 2. *Given $h > 0$ there exists a unique weak solution to (4)–(5)–(6). Furthermore, this solution $\boldsymbol{u}_h$ minimizes*

$$\inf \left\{ \int_{F_h} |\boldsymbol{v}|^2 , \quad \boldsymbol{v} \in L^2_\sigma(F_h) \text{ s.t. } \gamma_{\boldsymbol{n}}[\boldsymbol{v}] = \boldsymbol{e}_2 \cdot \boldsymbol{n} \text{ on } \partial \mathcal{S}_h \text{ and } \gamma_{\boldsymbol{n}}[\boldsymbol{v}] = 0 \text{ on } \partial \mathcal{P} \right\}. \tag{11}$$

Proof. The proof of this theorem yields as a standard application of the Lax–Milgram theorem. □

We remark that the above Theorem includes that the quantity $m_{\mathrm{a}}(h)$ that we are interested in is the minimum that is achieved by the solution $\boldsymbol{u}_h$. Hence, we do not need to compute the explicit solution $\boldsymbol{u}_h$, but rather focus on the minimization problem in order to discuss the behavior of the solution to the dynamical equation (7).

2.2. *Computations of bounds for $m_{\mathrm{a}}(h)$ for fixed h*

We now turn to the computations of bounds for the optimization problem (11). To this end, we fix $h > 0$ at first and we adapt the abstract reduced-functional method from [5]. We shall let h tend to 0 in order to justify the relevance of the obtained approximation (yielding Theorem 1) afterwards.

The framework of the reduced-functional method is the following. We assume that we want to minimize the functional m on Y. We assume that we are able to construct a space $\tilde{Y}$ and a functional $\tilde{m}$ on the space $\tilde{Y}$ so that the following holds true.

$$\left\{ \begin{array}{l} \text{There exist two mappings:} \\ \qquad\qquad \Psi : Y \to \tilde{Y} \quad \boldsymbol{V} : \tilde{Y} \to Y \\ \text{such that} \\ \qquad\qquad \tilde{m}(\Psi(\boldsymbol{v})) \le m(\boldsymbol{v}) \quad \forall\, v \in Y. \end{array} \right\} \tag{RF}$$

In this framework, we have the following lemma.

Lemma 1. *Assume that the minimum of $\tilde{m}$ is reached in ψ_0, then there holds:*

$$\inf_{\tilde{Y}} \tilde{m} (= \tilde{m}(\psi_0)) \le \inf_Y m \le m(\boldsymbol{V}[\psi_0]).$$

Proof. The last inequality is straightforward since $\boldsymbol{V}[\psi_0] \in Y$. As for the first one, we remark that, due to (RF), we have:

$$\inf_{\tilde{Y}} \tilde{m} \le \tilde{m}(\boldsymbol{\Psi}[\boldsymbol{v}]) \le m(\boldsymbol{v}) , \quad \forall\, \boldsymbol{v} \in Y.$$

Hence $\tilde{m}(\psi_0)$ is a bound below for $\{m(\boldsymbol{v}),\ \boldsymbol{v} \in Y\}$. This implies the first inequality. □

We emphasize that we do not assume *a priori* that finding a minimum for m on Y is a well-posed problem. This is reminiscent of the fact that the above proposition is relevant if minimizing the functional $\tilde{m}$ on $\tilde{Y}$ is a simpler problem. Also, the above method is relevant only if the “reduced functional” $\tilde{m}$ is sufficiently close to m in the sense that $\tilde{m}(\psi_0) \sim m(\boldsymbol{V}[\psi_0])$. The key-point is to keep in the functional $\tilde{m}$ the relevant/dominating terms of the functional m. This is the motivation of the construction below in the case of the minimization problem (11).

We introduce the space

$$Y_h = \{\boldsymbol{v} \in L^2_\sigma(F_h) \text{ s.t. } \gamma_{\boldsymbol{n}}[\boldsymbol{v}] = \boldsymbol{e}_2 \cdot \boldsymbol{n} \text{ on } \partial \mathcal{S}_h\, ,\ \gamma_{\boldsymbol{n}}[\boldsymbol{v}] = 0 \text{ on } \mathcal{P}\},$$

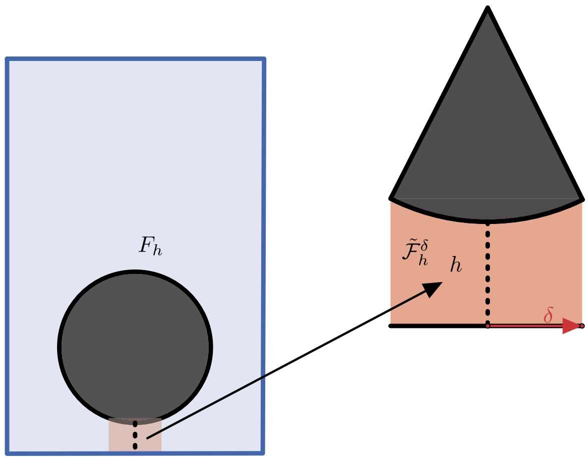

Fig. 2. On the right, a zoom on the gap between $\mathcal{S}_h$ and the ramp with the domain $\tilde{\mathcal{F}}_h^\delta$ in red.

so that our aim is to minimize the functional

$$m(\boldsymbol{v}) = \int_{F_h} |\boldsymbol{v}|^2$$

on Y_h. Following the ideas of [5], we construct the reduced functional as follows. We denote from now on:

$$\tilde{\mathcal{F}}_h^\delta := \{(x_1, x_2) \in \mathbb{R}^2_+ ; x_1 \in]-\delta, \delta[\text{ and } x_2 \in]0, \sigma_h(x)[\}$$

where δ is fixed by assumption (A_α) and we recall that we denote $\sigma_h(x) = h + \kappa|x|^{1+\alpha}$. This domain $\tilde{\mathcal{F}}_h^\delta$ is then the aperture between the solid $\mathcal{S}$ and the ramp $\mathcal{P}$, filled with the fluid, when the solid $\mathcal{S}$ is at distance $h > 0$ of $\mathcal{P}$. (See Fig. 2.)

We then set

$$\tilde{Y}_h = \{\psi \in H^1(\tilde{\mathcal{F}}_h^\delta) \text{ s.t. } \exists C \in \mathbb{R} \text{ with } \psi(x,0) = 0 \quad \psi(x, \sigma_h(x)) = C + x \quad \forall x \in]-\delta, \delta[\},$$

and we consider the energy

$$\tilde{m}(\psi) = \int_{\tilde{\mathcal{F}}_h^\delta} |\partial_2 \psi(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} \quad \forall \psi \in \tilde{Y}_h.$$

The main interest of this construction lies in the following proposition.

Proposition 1. *Given $h > 0$,*

i) there exists a mapping $\Psi : Y_h \to \tilde{Y}_h$ such that

$$\tilde{m}(\Psi[\boldsymbol{v}]) \le m(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in Y_h;$$

ii) there exists a mapping $\boldsymbol{V} : \tilde{Y}_h \to Y_h$ such that:

$$\boldsymbol{V}(\psi) = \nabla^\perp \psi \text{ in } \tilde{\mathcal{F}}_h^\delta \quad \forall \psi \in \tilde{Y}_h.$$

The construction of the mappings Ψ and $\boldsymbol{V}$ is the content of the following subsection. We remark at first that, item *i*) includes that Lemma 1 applies to this construction. Hence, once the above construction is done, the range for $m_a(h)$ stated in **Theorem 1** reduces to obtaining the following lemma.

Lemma 2. *Given $h > 0$, there exists a unique $\psi_0 \in \tilde{Y}_h$ achieving $\tilde{m}(\psi_0) = \min\{\psi,\ \psi \in \tilde{Y}_h\}$. Furthermore, when $h \to 0$, we have the following alternative:*

- *if $\alpha < 2$: $m(\psi_0)$ and $m(\boldsymbol{V}[\psi_0])$ remain bounded;*
- *if $\alpha > 2$:*

$$h^{3/(1+\alpha)} \le \tilde{m}(\psi_0) \le m(\boldsymbol{V}[\psi_0]) \le h^{3/(1+\alpha)} + O(1). \tag{12}$$

2.3. Proof of Proposition 1

We split this subsection into two paragraphs. The first one is devoted to the mapping Ψ, while in the second one, we construct and study the mapping $\boldsymbol{V}$.

Construction of the mapping Ψ. Let $\boldsymbol{v} \in Y_h \cap C^\infty(\overline{F_h})$. The smoothness assumption can be relaxed in a second step by a density argument. We fix then $\Psi[\boldsymbol{v}] = \psi$ by:

$$\psi(x_1, x_2) = \int_0^{x_1} v_2(s, 0)\mathrm{d}s - \int_0^{x_2} v_1(x_1, s)\mathrm{d}s \quad \forall (x_1, x_2) \in \tilde{\mathcal{F}}_h^\delta.$$

This formula yields the unique vector-field vanishing in the origin, so that $\nabla^\perp \psi = \boldsymbol{v}$ (we recall that $\boldsymbol{v}$ is assumed divergence-free). Since $\boldsymbol{v} \in Y_h$, we have then:

- $\psi(x_1, 0) = 0$ (since $v_2 = \boldsymbol{v} \cdot \boldsymbol{n} = 0$ on $\mathcal{P}$)
- $\psi(x_1, x_2) = C + x_1$ on $\partial\mathcal{S}_h$ for some constant C, since, for $x \in (-\delta, \delta)$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\psi(x, \sigma_h(x)) &= \partial_1\psi(x, \sigma_h(x)) + \sigma_h'(x)\partial_2\psi(x, \sigma_h(x)) \\ &= v_2(x, \sigma_h(x)) - \sigma_h'(x)v_1(x, \sigma_h(x)) = (\boldsymbol{v} \cdot n)\sqrt{1 + |\sigma_h'(x)|^2} \\ &= n_2\sqrt{1 + |\sigma_h'(x)|^2} = 1.\end{aligned}$$

This entails that $\psi \in \tilde{Y}_h$. Furthermore, since $v_1 = -\partial_2\psi$ and $\tilde{\mathcal{F}}_h^\delta \subset F_h$, we also have

$$\begin{aligned}\tilde{m}(\psi) &= \int_{\tilde{\mathcal{F}}_h^\delta} |\partial_2\psi|^2 \\ &= \int_{\tilde{\mathcal{F}}_h^\delta} |v_1|^2 \le \int_{F_h} |\boldsymbol{v}|^2 = m(\boldsymbol{v}).\end{aligned}$$

We emphasize that, combining the fact that ψ vanishes on $\mathcal{P}$ with a Hardy inequality, we obtain that the mapping Ψ (constructed on $Y_h \cap C^\infty(\overline{F_h})$) is continuous, $L^2(F_h) \to H^1(\tilde{\mathcal{F}}_h^\delta)$. This enables us to perform a density argument and create an extension $\Psi : Y_h \to \tilde{Y}_h$. The latter inequality comparing $\tilde{m}$ and m extends then to arbitrary $\boldsymbol{v} \in Y_h$ by density. This concludes the proof of item *i)* in Proposition 1.

Construction of the mapping $\boldsymbol{V}$. The construction of the mapping $\boldsymbol{V}$ is a little more technical. Indeed, let us consider $\psi \in \tilde{Y}_h \cap C^\infty(\overline{\tilde{\mathcal{F}}_h^\delta})$ and denote $\boldsymbol{v} = \boldsymbol{V}[\psi]$. Obviously, one wants to fix $\boldsymbol{v} = \nabla^\perp\psi$, but this only defines $\boldsymbol{v}$ on $\tilde{\mathcal{F}}_h^\delta$, so that we need to construct a suitable extension.

To construct the extension mapping, we first define a function that truncates outside $\mathcal{S}$. For this, we note that under assumption (A_α), there exists $\eta > 0$ such that, for arbitrary $(x_1, x_2) \in \mathcal{P}$ such that $|x_1| > \delta/2$ we have $\mathrm{dist}((x_1, x_2), \mathcal{S}) > \eta$. Then, we introduce:

- $\zeta : \mathbb{R} \to [0, 1]$ a cut-off function such that $\zeta = 1$ on $(-1, 1)$ and $\zeta = 0$ on $\mathbb{R} \setminus [-2, 2]$,
- $\chi : \mathbb{R}^2 \to \mathbb{R}$ smooth and satisfying:

$$\mathrm{Supp}\,(\chi) \subset B((0, 0), \eta/6) \qquad \int_{\mathbb{R}^2} \chi = 1,$$

and we set:

$$\begin{aligned}\tilde{T}(x_1, x_2) &= \zeta(3\,\mathrm{dist}((x_1, x_2), \mathcal{S})/\eta) && \forall (x_1, x_2) \in \mathbb{R}^2 \\ T &= \tilde{T} * \chi && \text{with } * \text{ the standard convolution operator.}\end{aligned}$$

By construction, we have $T \in C^\infty(\mathbb{R}^2)$ and $\tilde{T} = 1$ in a $\eta/3$ neighborhood of $\mathcal{S}$ and vanishes outside a $2\eta/3$ neighborhood of $\mathcal{S}$. In particular, for arbitrary $(x_1, x_2) \in \mathcal{S}$, we have $\tilde{T} = 1$ on $B((x_1, x_2), \eta/6)$ and $T(x_1, x_2) = 1$. On the opposite, if $\mathrm{dist}((x_1, x_2), \mathcal{S}) > 5\eta/6$, we have that $\tilde{T} = 0$ on $B((x_1, x_2), \eta/6)$ and $T = 0$.

With this construction at hand, let $\psi \in \tilde{Y}_h \cap C^\infty(\overline{\tilde{\mathcal{F}}_h^\delta})$ for simplicity and C the associated constant such that $\psi(x_1,x_2) = C + x_1$ on $\partial\mathcal{S}_h$. We construct $\boldsymbol{v} := \boldsymbol{V}[\psi] = \nabla^\perp\tilde{\psi}$ with:

$$\tilde{\psi}(x_1,x_2) = \begin{cases} \zeta(2x_1/\delta)\psi(x_1,x_2) + (1-\zeta(2x_1/\delta))(x_1+C)T((x_1,x_2-h)) & \text{if } (x_1,x_2)\in\tilde{\mathcal{F}}_h^\delta, \\ (x_1+C)T((x_1,x_2-h)) & \text{if } (x_1,x_2)\in\mathbb{R}^2_+\setminus\tilde{\mathcal{F}}_h^\delta. \end{cases}$$

Here are some comments about this construction. Since the truncation function ζ appearing on the first line is scaled so that $\tilde{\psi}$ matches smoothly through the lateral boundaries of $\tilde{\mathcal{F}}_h^\delta$, we have that $\tilde{\psi}\in C^\infty(\overline{F_h})$. In particular, $\boldsymbol{v}$ is computed in a standard way and divergence-free. In $\tilde{\mathcal{F}}_h^\delta$, we have,

$$\begin{aligned}\boldsymbol{v}(x_1,x_2) =& \zeta(2x_1/\delta)\nabla^\perp\psi(x_1,x_2) + (1-\zeta(2x_1/\delta))\nabla^\perp((x_1,x_2)\mapsto(x_1+C)T(x_1,x_2-h)) \\ &+ 2/\delta(\psi(x_1,x_2)-(x_1+C)T(x_1,x_2-h))\zeta'(2x_1/\delta)\boldsymbol{e}_2,\end{aligned}$$

and outside, we have:

$$\boldsymbol{v}(x_1,x_2) = T(x_1,x_2-h)\boldsymbol{e}_2 + (x_1+C)\nabla^\perp((x_1,x_2)\mapsto T(x_1,x_2-h)).$$

At this point, we remark that $(x_1,x_2)\in\mathcal{S}_h$ if and only if $(x_1,x_2-h)\in\mathcal{S}$. So $T(x_1,x_2-h)=1$ for $(x_1,x_2)\in\overline{\mathcal{S}}_h$ and $T(x_1,x_2-h)=0$ when $\mathrm{dist}((x_1,x_2),\mathcal{S}_h)\geq\eta$. This entails the required boundary conditions:

1) on $\mathcal{P}$, we have, $x_1\mapsto(1-\zeta(2x_1/\delta))$ and its derivatives vanish if $|x_1|<\delta/2$, while $T(x_1,x_2-h)=0$ and its derivatives vanish when $|x_1|>\delta/2$ (by choice of η w.r.t. δ). Adding that $\psi(x_1,0)=\partial_1\psi(x_1,0)=0$ for $|x_1|<\delta$, we conclude that $\boldsymbol{v}\cdot\boldsymbol{n} = v_2(x_1,0)=0$ globally;
2) on $\partial\mathcal{S}_h$ we have that $T(x_1,x_2-h)=1$ and $\nabla T(x_1,x_2-h)=0$. Hence, $\boldsymbol{v}(x_1,x_2)=\boldsymbol{e}_2$ outside $\tilde{\mathcal{F}}_h^\delta$. Inside $\tilde{\mathcal{F}}_h^\delta$, we have $\psi(x_1,x_2)=(x_1+C)$ so that:
$$\boldsymbol{v}\cdot\boldsymbol{n} = \zeta(2x_1/\delta)\boldsymbol{n}\cdot\nabla^\perp\psi + (1-\zeta(2x_1/\delta))\boldsymbol{n}\cdot\nabla^\perp((x_1+C)T(x_1,x_2-h)).$$
We remark then again that, on $\partial\mathcal{S}_h$, there holds
$$\boldsymbol{n}\cdot\nabla^\perp\psi = \boldsymbol{n}\cdot\nabla^\perp((x_1+C)T(x_1,x_2-h)) = \boldsymbol{e}_2\cdot\boldsymbol{n}$$
(since the operator $\boldsymbol{n}\cdot\nabla^\perp$ corresponds to the tangential derivative of ψ along $\partial\mathcal{S}_h$). This entails $\boldsymbol{v}\cdot\boldsymbol{n}=\boldsymbol{e}_2\cdot\boldsymbol{n}$.

Combining the fact that $\boldsymbol{v}$ is divergence-free and the latter boundary identities, we obtain finally that $\boldsymbol{v}\in Y_h$.

To end up the proof of item *ii)* in Proposition 1, it remains to remark that

$$C = \frac{1}{2\delta}\int_{-\delta}^{\delta}\psi(x_1,\sigma_h(\delta))\,\mathrm{d}x_1$$

so that the mapping $\psi\to C$ is continuous, $H^1(\tilde{Y}_h)\to\mathbb{R}$. The formula for $\boldsymbol{V}[\psi]$ defines then (the restriction to $\tilde{Y}_h\cap C^\infty(\overline{\mathcal{F}_h^\delta})$ of) a continuous linear mapping $\tilde{Y}_h\to Y_h$ when $\tilde{Y}_h$ (resp. Y_h) is endowed with the H^1-topology (resp. L^2 topology).

3. Proof of Theorem 1

We end the paper with the proof of our main result. According to the arguments in the previous section (see in particular Lemma 2), this splits into three steps:

- first to fix h and to compute ψ_0 that minimizes $\tilde{m}$ over $\tilde{Y}_h$,
- second to let $h\to 0$, to compute the asymptotics of the minimum of $\tilde{m}(\psi_0)$ and compute a bound above for $m(\boldsymbol{V}[\psi_0])$,
- to prove the asymptotic formula (9).

These steps correspond to the three subsections below.

3.1. Minimizing $\tilde{m}$

This part of the proof is straightforward. We state the following proposition for completeness.

Proposition 2. *Let $h>0$ and*

$$\psi_0(x_1,x_2) = \frac{x_1x_2}{\sigma_h(x_1)}, \quad \forall (x_1,x_2)\in\tilde{\mathcal{F}}_h^\delta.$$

Then ψ_0 realizes the minimum of $\tilde{m}$ over $\tilde{Y}_h$.

Proof. It is straightforward that $\psi_0 \in \tilde{Y}_h$ satisfies $\partial_{22}\psi_0 = 0$. Then, given $\psi \in \tilde{Y}_h$, we have that $\varphi = \psi - \psi_0 \in H^1(\tilde{\mathcal{F}}_h^\delta)$ satisfies:

$$\begin{aligned} &\varphi(x_1, \sigma_h(x_1)) = C && \text{for } x_1 \in (-\delta, \delta) \text{ and a constant } C \in \mathbb{R} \\ &\varphi(x_1, 0) = 0 && \text{for } x_1 \in (-\delta, \delta)\,. \end{aligned}$$

Integrating by parts and using symmetry arguments yield:

$$\int_{-\delta}^{\delta} \int_0^{\sigma_h(x_1)} \partial_2\psi_0(x_1,x_2)\,\partial_2\varphi(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1 = \int_{-\delta}^{\delta} C\,x_1\,\mathrm{d}x_1 - \int_{-\delta}^{\delta}\int_0^{\sigma_h(x_1)} \partial_{22}\psi_0(x_1,x_2) \quad \varphi(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1 = 0;$$

and consequently:

$$\tilde{m}(\psi) = \int_{-\delta}^{\delta}\int_0^{\sigma_h(x_1)} |\partial_2\psi_0(x_1,x_2)|^2 + \int_{-\delta}^{\delta}\int_0^{\sigma_h(x_1)} |\partial_2\varphi(x_1,x_2)|^2 \geq \tilde{m}(\psi_0)$$

with equality if and only if $\varphi = 0$ *i.e.* $\psi = \psi_0$. □

3.2. Range for m_{a}

We start with finding an equivalent for $\tilde{m}(\psi_0)$.

Proposition 3. *We have two cases:*

- *if* $\alpha < 2$ *the quantity* $\tilde{m}(\psi_0)$ *remains bounded when* $h \to 0$,
- *if* $\alpha > 2$, $\tilde{m}(\psi_0) = c_{\kappa,\alpha} h^{3/(1+\alpha)-1} + O(1)$ *when* $h \to 0$ *with*

$$c_{\kappa,\alpha} = \int_{\mathbb{R}} \frac{t^2\mathrm{d}t}{1+|t|^{1+\alpha}}.$$

Proof. Replacing with the explicit values of ψ_0 we have that $\tilde{m}(\psi_0) = I_h$ with:

$$I_h = \int_{-\delta}^{\delta} \frac{x^2}{\sigma_h(x)}\mathrm{d}x = \int_{-\delta}^{\delta} \frac{x^2}{h+\kappa|x|^{1+\alpha}}\mathrm{d}x.$$

If $\alpha < 2$ we have, for arbitrary h

$$I_h \leq \frac{1}{\kappa}\int_{-\delta}^{\delta} \frac{1}{|x|^{\alpha-1}}\mathrm{d}x$$

and I_h remains bounded when $h \to 0$.

If $\alpha > 2$, we perform the change of variable $x = h^{1/(1+\alpha)}t$. This yields:

$$\begin{aligned} I_h &= h^{3/(1+\alpha)-1} \int_{-\delta/h^{1/(1+\alpha)}}^{\delta/h^{1/(1+\alpha)}} \frac{t^2}{1+|t|^{1+\alpha}}\mathrm{d}t \\ &= h^{3/(1+\alpha)-1}\int_{\mathbb{R}} \frac{t^2}{1+\kappa|t|^{1+\alpha}}\mathrm{d}t + O(1). \end{aligned}$$

This ends the proof. □

We note that, in the case $\alpha = 2$, we can make similar computations and obtain that

$$\tilde{m}(\psi_0) = \frac{2}{3\kappa}|\ln(h)| + O(1).$$

We proceed with the upper bound, *i.e.* computing $m(\boldsymbol{v}_0)$ where $\boldsymbol{v}_0 = \boldsymbol{V}(\psi_0)$. We note that, since the constant C associated with ψ_0 vanishes, the extension of ψ_0 that we construct with $\tilde{\psi}_0$ (see the formula of the previous section) depends on h only through a translation. Hence, the contribution outside $\tilde{\mathcal{F}}_h^\delta$ to the L^2-norm of $\boldsymbol{v}_0$ remains bounded independently of h. Since the singularity in the geometry occurs only in the origin when $h \to 0$, we have also that:

$$\int_{\tilde{\mathcal{F}}_h^\delta} |\boldsymbol{v}_0|^2 = \int_{-\delta/2}^{\delta/2} \int_0^{\sigma_h(x_1)} |\boldsymbol{v}_0|^2(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 + O(1),$$

and, replacing $\boldsymbol{v}_0$ with its values on $\tilde{\mathcal{F}}_h^\delta$, we have:

$$m(\boldsymbol{v}_0) = \tilde{m}(\psi_0) + \int_{-\delta}^{\delta} \int_0^{\sigma_h(x_1)} |\partial_1 \psi_0|^2(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 + O(1).$$

We denote by J_h the second integral on the right-hand side of this last inequality. With the explicit values for ψ_0, we obtain:

$$J_h = \frac{1}{3}\int_{-\delta}^{\delta} \left(\frac{1}{\sigma_h(x_1)} - \frac{x_1 \sigma_h'(x_1)}{\sigma_h(x_1)^2} \right)^2 |\sigma_h(x_1)|^3\,\mathrm{d}x_1.$$

However, there exists an absolute constant K_0 such that $|\sigma_h'(x_1)\,x_1| \leq K_0\,\sigma_h(x_1)$ for $x_1 \in (-\delta, \delta)$. Introducing this inequality in the computation of J_h, we obtain:

$$|J_h| \leq \int_{-\delta}^{\delta} \sigma_h(x_1)\,\mathrm{d}x_1 \leq K_1,$$

with K_1 independent of h. This ends the computations of the bounds for $m_{\mathrm{a}}(h)$.

3.3. Asymptotic expansion

To complete the proof of Theorem 1, we obtain the asymptotics (9). For this, we re-introduce $(\boldsymbol{u}_h)_{h>0}$, the family containing the unique weak solutions to (4)–(5)–(6), and we recall the notations $\boldsymbol{v}_0 = \boldsymbol{V}[\psi_0]$ that we introduced above. We recall that, for fixed $h > 0$, since $\boldsymbol{v}_0 \in Y_h$ and $\boldsymbol{u}_h$ is a weak solution to (4)–(5)–(6), we have, by definition, that:

$$\int_{F_h} \boldsymbol{u}_h \cdot (\boldsymbol{u}_h - \boldsymbol{v}_0) = 0 \quad \textit{i.e.} \quad \int_{F_h} |\boldsymbol{u}_h|^2 = \int_{F_h} \boldsymbol{u}_h \cdot \boldsymbol{v}_0.$$

Consequently, expanding the square and applying this last identity, we obtain:

$$\int_{F_h} |\boldsymbol{u}_h - \boldsymbol{v}_0|^2 = \int_{F_h} |\boldsymbol{v}_0|^2 - \int_{F_h} |\boldsymbol{u}_h|^2.$$

However, by construction of the reduced functional method, we have:

$$\tilde{m}(\psi_0) \leq \int_{F_h} |\boldsymbol{u}_h|^2 \leq \int_{F_h} |\boldsymbol{v}_0|^2 = m(\boldsymbol{v}_0).$$

Plugging these bounds in the previous identity yields the expected formula by recalling that $m(\boldsymbol{v}_0) = \tilde{m}(\psi_0) + O(1)$ when $h \to 0$. This ends the proof.

Acknowledgements

The first author is partially supported by ANR project ANR-15-CE40-0010. The second author is partially supported by NLAGA project, http://nlaga-simons.ucad.sn.

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