C. R. Acad. Sci. Paris, Ser. I 356 (2018) 1167–1178

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Partial differential equations

Scalar conservation laws: Initial and boundary value problems revisited and saturated solutions ☆

Lois de conservation scalaires : retour sur les problèmes aux limites et solutions saturées

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ARTICLE INFO

Article history:
Received 25 September 2018
Accepted 27 September 2018
Available online 6 November 2018

Presented by Pierre-Louis Lions

ABSTRACT

We revisit the classical theory of multidimensional scalar conservation laws. We reformulate the notion of the classical Kruzkov entropy solutions and study some new properties as well as the well-posedness of the initial value problem with inhomogeneous fluxes and general initial data. We also consider Dirichlet boundary value problems. We put forward a new and transparent definition for solutions and give a simple proof for their well-posedness in domains with smooth boundaries. Finally, we introduce the notion of saturated solutions and show that it is well-posed.

RÉSUMÉ

Nous revenons sur la théorie classique des lois de conservation scalaires multidimensionnelles. Nous introduisons une notion nouvelle de sous- et de sur-solutions, qui est équivalente à la notion classique de solutions entropiques à la Kruzkov. Nous utilisons ensuite cette notion pour établir des propriétés nouvelles de ces solutions. Nous proposons également une formulation nouvelle et claire des solutions pour les problèmes aux limites et nous donnons une preuve simple du caractère bien posé de ces problèmes. Enfin, nous introduisons la notion de solutions saturées et montrons que de tels problèmes sont bien posés.

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☆ Partially supported by the National Science Foundation grants DMS-1600129 and the Office of Naval Research grant N000141712095.
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https://doi.org/10.1016/j.crma.2018.09.010

Version française abrégée

Nous considérons dans cette Note les lois de conservation scalaires multidimensionnelles du type $u_t + \mathrm{div}_x \mathbf{f}(u) = 0$ dans $U \times (0,T), u(x,0) = u_0$ dans U, où $T > 0$ est fixé, $d \geq 1$, $\mathbf{f} = (f_1, \dots, f_d)$, où les fonctions f_i (de $\mathbb{R}$ dans $\mathbb{R}$) sont supposées être de classe C^1 (pour simplifier la présentation) et la condition initiale u_0 est supposée appartenir à $(L^1 \cap L^\infty)(U)$ (à nouveau pour simplifier la présentation). L'inconnue $u(x,t)$ prend ses valeurs dans $\mathbb{R}$ et sera supposée appartenir à $C([0,T]; L^1(U)) \cap L^\infty(U \times (0,T))$. Quelques informations bibliographiques sont données dans la version en anglais qui suit.

Nous introduisons la notion de sous-solution (respectivement sur-solution) entropique en imposant que, pour tout $k \in \mathbb{R}, u \vee k = \max(u,k)$ (respectivement $u \wedge k = \min(u,k)$) vérifie au sens des distributions

$$(u \vee k)_t + \mathrm{div}_x \mathbf{f}(u \vee k) \leq 0 \ \text{(resp. } (u \wedge k)_t + \mathrm{div}_x \mathbf{f}(u \wedge k) \geq 0) \ \text{ dans } \mathbb{R}^d \times (0,T).$$

Enfin, u est une solution entropique si elle est à la fois sous-solution et sur-solution entropique.

On vérifie alors aisément que cette notion est équivalente à la notion de solution entropique à la Kruzkov. Et on observe que la méthode de doublement de variables de Kruzkov permet d'établir le résultat suivant.

Théorème 0.1. *Si $u, v \in C([0,T]; L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d \times (0,T))$ sont deux sous-solutions entropiques, alors $u \vee v$ l'est également.*

Grâce à la conservation de $\int_{\mathbb{R}^d} u(x,t)\,\mathrm{d}x$, ce théorème implique à l'évidence tous les résultats classiques de la théorie de Kruzkov (unicité, principe de comparaison, contraction dans L^1).

Dans le cas où U est borneé, on considère le problème aux limites

$$u_t + \mathrm{div}_x \mathbf{f}(u) = 0 \ \text{ dans } U \times (0,T), \quad u(\cdot,0) = u_0 \ \text{ dans } U, \quad u = 0 \ \text{ dans } \partial U \times (0,T),$$

et on introduit la notion suivante de sous- et sur-solution entropique : pour $u \in C([0,T]; L^1(U)) \cap L^\infty(U \times (0,T))$, on dit que u est sous-solution (resp., sur-solution) entropique si on a, pour tout $k \in \mathbb{R}$,

$$(u \vee k)_t + \mathrm{div}_x \mathbf{f}(u \vee k) \leq 0 \ \text{ dans } U \times (0,T),$$
$$\text{(resp.,} \quad (u \wedge k)_t + \mathrm{div}_x \mathbf{f}(u \wedge k) \geq 0 \ \text{ dans } U \times (0,T),)$$

et pour tout $k \geq 0$ (resp. $k \leq 0$)

$$(u \vee k)_t + \mathrm{div}_x \mathbf{f}(u \vee k) - f_1(k)\delta_0(x_1) \leq 0 \ \text{ dans } \overline{U} \times (0,T),$$
$$\text{(resp. } (u \wedge k)_t + \mathrm{div}_x \mathbf{f}(u \wedge k) - f_1(k)\delta_0(x_1) \geq 0 \ \text{ dans } \overline{U} \times (0,T))$$

où l'inégalité sur $\overline{U}$ signifie que, pour tout $k \geq 0$ et $\varphi \in C^1(\mathbb{R}^d), \varphi \geq 0$, à support compact dans $\overline{U}$, on a

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\overline{U}} (u \vee k)\varphi \,\mathrm{d}x - \int_{\overline{U}} \mathbf{f}(u \vee k) \cdot \mathrm{D}\varphi \,\mathrm{d}x + \int_{\partial U} f_1(k)\varphi(0,x)\,\mathrm{d}x \leq 0$$

(resp., pour tout $k \leq 0$ et $\varphi \in C^1(\mathbb{R}^d), \varphi \geq 0$, à support compact dans $\overline{U}$, on a

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\overline{U}} (u \wedge k)\varphi \,\mathrm{d}x - \int_{\overline{U}} \mathbf{f}(u \wedge k) \cdot \mathrm{D}\varphi \,\mathrm{d}x + \int_{\partial U} f_1(k)\varphi(0,x)\,\mathrm{d}x \geq 0.)$$

Enfin, u est une solution entropique si u est à la fois sous- et sur-solution entropique.

On démontre alors le Théroème 2.

Théorème 0.2. *i) Si u (resp., v) $\in C([0,T]; L^1(U) \cap L^\infty(U \times (0,T))$ est sous-solution (resp. sur-solution) entropique, alors on a, pour tout $t \in [0,T]$*

$$\int_U (u(x,t) - v(x,t))_+ \,\mathrm{d}x \leq \int_U (u(x,0) - v(x,0))_+ \,\mathrm{d}x$$

ii) Si $u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d_-)$, il existe une unique solution entropique $u \in C([0,T]; L^1(\overline{U}) \cap L^\infty(\overline{U} \times (0,T))$ vérifiant $u(.,0) = u_0$ p.p. sur U et $u = 0$ sur $\partial U \times (0,T)$.

Signalons que la démonstration de l'existence prouve la convergence de l'approximation par viscosité évanescente. Pour simplicite nous prenons $U = \mathbb{R}^d_-$.

Enfin, nous introduisons une notion nouvelle, que nous appelons « solution saturée » dans le cas où $U = \mathbb{R}^d_-$. Par exemple : $u \in L^\infty(0,T; L^1_{x_1}(L^\infty_{x'})) \cap C([0,T]; L^1_{x_1}(L^1_{\text{loc},x'})) \cap L^\infty(\mathbb{R}^d_- \times (0,T))$ est solution saturée si elle est sous-solution entropique sur $\mathbb{R}^d_-$ et sur-solution entropique sur $\overline{\mathbb{R}^d_-}$ pour tout $k \in \mathbb{R}$, i.e.

$$(u \vee k)_t + \mathbf{f}(u \vee k) + f_1(k)\delta_0(x_1) \geq 0 \ \text{ sur } \overline{\mathbb{R}^d_-} \times (0,T)$$

pour tout $k \in R$. Et on démontre le théorème suivant.

Théorème 0.3. *Si $u_0 \in L^1_{x_1}(L^\infty_{x'}) \cap L^\infty_x$ et si le flux f_1 vérifie la condition suivante*

$$\exists\, \zeta_n \underset{n}{\rightarrow} +\infty, \ \ \forall \zeta \geq \zeta_n, \ \ f_1(\zeta_n) \geq f_1(\zeta),$$

alors il existe une unique solution saturée. De plus, $u \geq v$ p.p. dans $\mathbb{R}^d_- \times (0,T)$ pour toute sous-solution entropique $u \in L^\infty(0,T; L^1_{x_1}(L^\infty_{x'})) \cap C([0,T]; L^1_{x_1}(L^1_{\text{loc},x'})) \cap L^\infty(\mathbb{R}^d_- \times (0,T))$ sur $\mathbb{R}^d_- \times (0,T)$ vérifiant $v(x,0) \leq u_0(x)$ p.p. dans $\mathbb{R}^d_-$.

1. Introduction

We introduce in section 2 a simple formulation of sub- and super-solutions for multi-dimensional scalar conservation laws. This new formulation, which is equivalent to the classical Kruzkov-entropy solutions, is technically very flexible and allows us to study some new properties of the solutions. We also study the existence and uniqueness of solutions for general inhomogeneous fluxes and initial data in $L^1 + L^\infty$. In section 3, we consider initial boundary value problems and put forward a new definition, which is transparent, avoids the need to use special entropies and traces on the boundary, and yields a rather easy proof of the comparison principle. Finally, in section 4 we introduce the new notion of saturated solutions, which are the analogues of state-constraint solutions for Hamilton–Jacobi equations, and prove that they are well posed and maximal.

Due to page limitations, we only state with sketches of proofs some of the results without any attempt to optimize assumptions. Details can be found in a forthcoming paper of the authors [6] as well as the lectures of the first author in the fall of 2017 at the "Collège de France". Some of the results were announced in a lecture of the first author at the University of Chicago in January 2017.

2. The initial value problem

We introduce a simple formulation for solutions to the initial value problem for scalar conservation laws, which is, of course, equivalent to the classical Kruzkov entropy solution.

For a fixed $T > 0$, we consider the equation

$$u_t + \text{div}_x \mathbf{f}(u,x) = 0 \ \text{ in } \mathbb{R}^d \times (0,T), \tag{1}$$

with

$$\mathbf{f} = (f_1, \ldots, f_d) \in (W^{1,1}_{\text{loc},x}(C_u))^d; \tag{2}$$

we say that $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \in (W^{1,1}_{\text{loc},x}(C_u))^d$ if, for all $R > 0$, $\sup\limits_{|u| \leq R, \ 1 \leq i,j \leq d} |\dfrac{\partial f_i}{\partial x_j}| \in L^1_{\text{loc}}(\mathbb{R}^d)$.

The new formulation. In what follows, $a \wedge b = \min(a,b)$, $a \vee b = \max(a,b)$ and $\text{sign}_+ z = 1$ if $z \geq 0$ and 0 otherwise. Moreover, $\mathbf{1}$ denotes the characteristic function of the set A. Finally, we write $\mathcal{D}'(U)$ to signify that an expression holds in the sense of distributions in U.

Definition 2.1. A function $u \in L^\infty_{\text{loc}}(\mathbb{R}^d \times (0,T))$ is a local sub- (resp. super-) solution to (1) if, for every $k \in \mathbb{R}$,

$$(u \vee k)_t + \text{div}_x \mathbf{f}(u \vee k, x) - \mathbf{1}_{\{u<k\}} \text{div}_x \mathbf{f}(k,x) \leq 0 \ \text{ in } \mathcal{D}'(\mathbb{R}^d \times (0,T)), \tag{3}$$

$$(\text{resp.} \quad (u \wedge k)_t + \text{div}_x \mathbf{f}(u \wedge k, x) - \mathbf{1}_{\{u>k\}} \text{div}_x \mathbf{f}(k,x) \geq 0 \ \text{ in } \mathcal{D}'(\mathbb{R}^d \times (0,T)). \tag{4}$$

We remark that it easy to check that (3) (resp. (4)) also holds if $\mathbf{1}_{\{u<k\}}$ (resp. $\mathbf{1}_{\{u>k\}}$) is replaced by $\mathbf{1}_{\{u\leq k\}}$ (resp. $\mathbf{1}_{\{u\geq k\}}$). Moreover, it is straightforward to see that Definition 2.1 is in fact equivalent to the classical Kruzkov ([5]) entropy sub- and super-solution.

The following result, which is an immediate consequence of Definition 2.1, is nevertheless surprisingly new in the theory of scalar conservation laws.

Proposition 2.2. *The maximum (resp. minimum) of two local sub- (resp. super-) solutions to* (1) *in* $L^\infty_{\mathrm{loc}}(\mathbb{R}^d \times (0,T))$ *is a local sub- (resp. super-) solution.*

Proof. We only sketch the proof of the sub-solution property, since the other claim follows similarly. The argument is based on the classical "doubling the variables" technique introduced in the theory of scalar conservation laws in [5]. For simplicity, we only discuss the case of homogeneous fluxes.

Fix $k \in \mathbb{R}$ and, for $(y,s),(x,t) \in \mathbb{R}^d \times (0,T)$, use the definition of the local sub-solution for u with constant $v(y,s) \vee k$ and for v with constant $u(x,t) \vee k$. It follows that, a.e. in (y,s) and (x,t),

$$(u \vee (v(y,s) \vee k))_t + \mathrm{div}_x \mathbf{f}(u \vee (v(y,s) \vee k)) \le 0 \ \text{ in } \ \mathcal{D}'(\mathbb{R}^d \times (0,T)),$$

and

$$(v \vee (u(x,t) \vee k))_s + \mathrm{div}_y \mathbf{f}(v \vee (u(x,t) \vee k)) \le 0 \ \text{ in } \ \mathcal{D}'(\mathbb{R}^d \times (0,T)).$$

Employing $\rho_\eta(t-s)\phi_\theta(x-y)\psi(\frac{x+y}{2}, \frac{t+s}{2})$ as the test functions above, where ρ_η and ϕ_θ are approximations of the identity, after integrating in x, y, t, s and passing to the limits $\eta, \theta \to 0$, we find

$$\int_0^T \int_{\mathbb{R}^d} [(u \vee v \vee k)_+ \psi_t - \mathbf{f}(u \vee v \vee k)) \cdot \mathrm{D}_x \psi], \mathrm{d}x\,\mathrm{d}t \le 0,$$

and, hence, the claim. □

We state next without proof an immediate consequence of Proposition 2.2.

Corollary 2.3. *If* $u, v \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d \times (0,T))$ *are local sub- (resp. super-) solutions to* (1)*, then* $u \vee v$ *(resp.* $u \wedge v$*) is a local sub- (resp. super-) solution to* (1) *in the sense of distributions in* $\mathbb{R}^d \times (0,T)$*.*

A regularization. We discuss a regularization of entropy sub- and super-solutions that resembles the so-called sup- and inf-convolutions of viscosity solutions.

Notice that, if $u \in C_t(L^1_x) \cap L^\infty_{x,t}$ is an entropy sub- (resp. super-) solution to the homogeneous problem

$$u_t + \mathrm{div}_x \mathbf{f}(u) = 0 \ \text{ in } \ \mathbb{R}^d \times (0,T), \tag{5}$$

then, for each $z \in R^d$, $u(\cdot + z, \cdot)$ is a sub- (resp. super-) solution to (5).

For $\epsilon > 0$, let

$$u^\epsilon(x,t) := \mathrm{esssup}_{|z|\le\epsilon}\, u(x+z,t) \quad \text{and} \quad u_\epsilon(x,t) := \mathrm{essinf}_{|z|\le\epsilon}\, u(x+z,t). \tag{6}$$

It is not clear if and in what sense u^ε and u_ϵ converge to u in the generality we are discussing here. They are, however, more regular than u and preserve the sub- and super-solution properties.

The following holds.

Proposition 2.4. *Let* $u \in L^\infty_{x,t}$ *be a sub- (resp. super-) solution to* (5)*. Then* u^ϵ *(resp.* u_ϵ*)* $\in L^\infty_{x,t}$ *is a sub- (resp. super-) solution to* (5)*. Moreover, for all* $t \in (0,T)$*,* $u^\epsilon(\cdot,t), u_\epsilon(\cdot,t) \in BV_x$*.*

Proof. We sketch the sub-solution property and refer to [6] for the full details. Also, since ϵ is fixed, we write $\overline{u}$ in place of u^ϵ.

Since $\overline{u}(x,t) \vee k = \mathrm{esssup}_{|z|\le\epsilon}(u(x+z,t) \vee k)$ and, for each $z \in \mathbb{R}^d$, $u(\cdot + z, \cdot) \vee k$ is a sub-solution to (5), the claim follows if we show that $\overline{u}$ is a distributional sub-solution to (5). Fix $t_0 \in [0,T)$ and consider the initial value problem

$$v_t + \mathrm{div}_x \mathbf{f}(v) = 0 \text{ in } \mathbb{R}^d \times (t_0,T) \quad \text{and} \quad v(\cdot,t_0) = \overline{u}(\cdot,t_0). \tag{7}$$

Recall that, for each $z \in \mathbb{R}^d$, $\overline{u}(\cdot,t_0) \ge u(\cdot + z, t_0)$ a.e. and $u(\cdot + z, \cdot)$ is a sub-solution to (5). It then follows from the comparison principle that $v \ge u(\cdot + z, \cdot)$ in $\mathbb{R}^d \times (t_0,T)$, and, hence,

$$v \ge \overline{u} \text{ in } \mathbb{R}^d \times (t_0,T).$$

For the inequality above to be true, it is necessary to use a more involved property of the esssup, which is discussed in [6].

Note that we do not know if $\overline{u}(\cdot,t_0) \in L^1(\mathbb{R}^d)$. Instead, we have $\overline{u}(\cdot,t_0) \in L^\infty(\mathbb{R}^d)$, and it is, hence, necessary to have comparison in L^∞, as it is shown in the next section.

Next fix a non-negative $\phi \in C_c^\infty(\mathbb{R}^d \times (0,T))$. Since v is an entropy solution, for $h > 0$, we find that

$$\int_{\mathbb{R}^d} \frac{v(x,t_0+h) - v(x,t_0)}{h} \phi(x,t_0)\,\mathrm{d}x = \frac{1}{h}\int_{t_0}^{t_0+h}\int_{\mathbb{R}^d} \mathbf{f}(v(x,s))\cdot \mathrm{D}\phi(x,s)\,\mathrm{d}x\,\mathrm{d}s,$$

and, since $v \in C_t(L^1_{x,\mathrm{loc}})$,

$$\lim_{h\to 0}\frac{1}{h}\int_{t_0}^{t_0+h}\int_{\mathbb{R}^d} \mathbf{f}(v(x,s))\cdot \mathrm{D}\phi(x,s)\,\mathrm{d}x\,\mathrm{d}s = \int_{\mathbb{R}}^{d} \mathbf{f}(v(x,t_0))\cdot \mathrm{D}\phi(x,t_0)\,\mathrm{d}x.$$

It follows that

$$\int_{\mathbb{R}^d} \frac{\overline{u}(x,t_0+h) - \overline{u}(x,t_0)}{h} \phi(x,t_0)\,\mathrm{d}x \le \frac{1}{h}\int_{t_0}^{t_0+h}\int_{\mathbb{R}^d} \mathbf{f}(v(x,s))\cdot \mathrm{D}\phi(x,s)\,\mathrm{d}x\,\mathrm{d}s,$$

and thus, for each $t_0 \in (0,T)$,

$$\limsup_{h\to 0^+}\int_{\mathbb{R}^d} \frac{\overline{u}(x,t_0+h) - \overline{u}(x,t_0)}{h} \phi(x,t_0)\,\mathrm{d}x \le \int_{\mathbb{R}^d} \mathbf{f}(\overline{u}(x,t_0))\cdot \mathrm{D}\phi(x,t_0)\,\mathrm{d}x,$$

and, hence, the claim. □

Remark 2.5. A statement similar to Proposition 2.2 and the sub- (resp. super-) solution property of Proposition 2.4, but under stronger assumptions, appeared later in Silvestre [11].

The well-posedness. The most general comparison result we know for (1) is for $u_0 \in (L^1 + L^\infty)(\mathbb{R}^d)$ and, hence, solutions in $C_t((L^1+L^\infty)_x)$. To deal with such generality, it is necessary to make assumptions on the modulus of continuity of $\mathbf{f}$ in u. For example, for comparison with solutions in $C_t((L^1+c))$, the condition is about the modulus of continuity of $\mathbf{f}$ at c. The details appear in [6].

Here we discuss solutions in $C([0,T];L^1(\mathbb{R}^d)) \cap L^\infty_{x,t}$. To formulate the conditions on $\mathbf{f}$, we need to introduce some additional notation. For $\delta > 0$ and $C > 0$, let

$$M_2^\delta := \begin{cases} \sup_{|z|\le\delta} \dfrac{|f(z,x)-f(0,x)|}{|z|^\theta} & \text{if } d \ge 2 \text{ with } \theta =: \dfrac{d-1}{d}, \\ \sup_{|z|\le\delta} |f(z,x)-f(0,x)| & \text{if } d = 1, \end{cases} \qquad \text{and } M_{1,C} := \sup_{|z|\le C} |f(z,x) - f(0,z)|,$$

and recall that, for any $p \in [1,\infty)$,

$$L^p_{\mathrm{unif}} = \{ f \in L^p_{\mathrm{loc}} : \sup_{n\in\mathbb{Z}^d} \|f\||_{L^p(n+B_1)} < \infty \}.$$

We assume that

$$\mathrm{div}_x \mathbf{f}(0,x) \in L^1(\mathbb{R}^d), \tag{8}$$

and, for all for $\delta > 0$ and $C > 0$, if $d \ge 2$,

$$M_2^\delta \in L^d_{\mathrm{unif}} \text{ and } M_{1,C} \in L^d_{\mathrm{unif}}, \tag{9}$$

or, if $d = 1$,

$$\begin{cases} M_{1,C} \in L^1_{\mathrm{unif}} \text{ and } (M_{1,C}(n+\cdot))_{n\in\mathbb{Z}} \text{ is uniformly integrable in } [-1,+1], \\ M_2^\delta \in L^1_{\mathrm{unif}} \text{ and } M_2^\delta \to 0 \text{ in } L^1_{\mathrm{unif}} \text{ as } \delta \to 0. \end{cases} \tag{10}$$

We remark that "$M_{1,C} \in L^1_{\mathrm{unif}}$ and $(M_{1,C}(n+\cdot))_{n\in\mathbb{Z}}$ is uniformly integrable in $[-1,+1]$" is equivalent to

$$M_{1,C} \in \mathrm{closure}_{L^1_{\mathrm{unif}}}(L^p_{\mathrm{unif}}) \text{ for some } p > 1.$$

Finally, we also assume, for some $c > 0$,

$$\mathrm{div}_x \mathbf{f}(z,x)\mathrm{sign}(z) \ge -c(1+|z|) \text{ for all } x \in \mathbb{R}^d \text{ and } z \in \mathbb{R}. \tag{11}$$

We remark that (8), (9) or (10), and (11) are considerably weaker than

$$D^2_{x,z}\mathbf{f} \in L^\infty([-M, M] \times \mathbb{R}^d) \text{ and } D_x \mathrm{div}_x \mathbf{f} \in L^1_x(C([-M, M]) \text{ for each } M > 0 \tag{12}$$

which were usually assumed in the existing literature.

The new comparison and existence result is stated next.

Theorem 2.6. *Assume (2). If $u_1, u_2 \in C_t(L^1_x) \cap L^\infty_{x,t}$ are solutions to (1), then, for all $t, s \geq 0$ such that $t \geq s$,*

$$\|u_1(\cdot, t) - u_2(\cdot, t)\|_1 \leq \|u_1(\cdot, s) - u_2(\cdot, s)\|_1. \tag{13}$$

Assume, in addition, (8) and (11). For each $u_0 \in (L^1 \cap L^\infty)(\mathbb{R}^d)$, (1) has a solution $u \in C_t(L^1_x) \cap L^\infty_{x,t}$.

The L^1-contraction follows from the assumed continuity in time and the fact that Corollary 2.3 yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int (u \vee v)(x, t) \, \mathrm{d}x \leq 0 \text{ and } -\frac{\mathrm{d}}{\mathrm{d}t} \int (u \wedge v)(x, t) \, \mathrm{d}x \leq 0,$$

and, hence,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int |u - v|(x, t) \, \mathrm{d}x \leq 0.$$

For the existence, we use the approximate viscous initial value problem

$$u_{\epsilon,t} + \mathrm{div}_x \mathbf{f}(u_\epsilon, x) = \epsilon \Delta u_\epsilon \text{ in } \mathbb{R}^d \times (0, T) \quad \text{and} \quad u_\epsilon(\cdot, 0) = u_0, \tag{14}$$

and employ (8) and (11) to obtain independent of $\epsilon \in (0, 1)$ bounds on $\|u_\epsilon(\cdot, t)\|_1$ and $\|u^\epsilon(\cdot, t)\|_\infty$.

Then, instead of establishing a bound on $\|Du_\epsilon(\cdot, t)\|_1$ that requires a hypothesis like (12), we use a new strategy that does not rely on such a strong condition on the flux, but, instead, is based on a new stability result, which we state next.

Theorem 2.7. *Let $(u_n)_{n\in\mathbb{N}} \subset C_t(L^1_x) \cap L^1_{x,t}$ be a family of solutions to (1) with fluxes $\mathbf{f}_n$ satisfying (8), (9) if $d \geq 2$ or (10) if $d = 1$, and (11) with bounds uniform in n and initial data $(u_{0,n})_{n\in\mathbb{N}}$ uniformly bounded in $L^1 \cap L^\infty$ and such that $u_{0,n} \to u_0$ in $L^1 \cap L^\infty$. Then there exists a solution $u \in C_t(L^1_x) \cap L^1_{x,t}$ to (1) with initial datum u_0 such that, as $n \to \infty$, $u_n \to u$ in $C_t(L^1_x) \cap L^1_{x,t}$.*

The proof of Theorem 2.7 is based on showing that there exist moduli $\omega_\delta(n, m)$ and $\alpha(\delta)$ such that, as $n, m \to \infty$ and $\delta \to 0$, $\omega_\delta(n, m) \to 0$ and $\alpha(\delta) \to 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} ((u_n \vee u_m)(x, t) - u_m(y, t) \wedge u_n(y, t))_+ \rho_\delta(x - y) \, \mathrm{d}x \, \mathrm{d}y \leq \omega_\delta(n, m) + \alpha(\delta), \tag{15}$$

where ρ_δ is a standard mollifier. We refer to [6] for the details.

3. The initial boundary value problem

For $T > 0$ fixed we consider the initial boundary value problem

$$u_t + \mathrm{div}\mathbf{f}(u, x) = 0 \text{ in } U \times (0, T), \qquad u = h \text{ on } \Sigma, \qquad u(\cdot, 0) = u_0 \text{ on } U, \tag{16}$$

where $U \subset \mathbb{R}^d$ is an open set with smooth, for example C^1-boundary ∂U and external normal vector ν, $\Sigma := \partial U \times [0, T]$, and $h : \Sigma \to \mathbb{R}$.

We continue with a by no means comprehensive review of the existing literature for solutions to (16). Here we try to mention some of the most important references. We recognize, however, that we may have missed some.

The study of boundary value problems for scalar conservation laws with homogeneous fluxes goes back to the work of Bardos, le Roux and Nedelec [3], who considered when $d = 1$ solutions in $L^\infty_t(\mathrm{BV}_x)$, which in view of the BV-regularity have traces; see also Dubois and Le Floch [4]. Then Otto [7,8] introduced the notion of boundary entropies and used weak traces, whose existence is a simple consequence of the Kruzkov definition, to study solutions with boundary data in $h \in L^\infty(\Sigma)$ without assuming the existence of (strong) traces. Later Porretta and Vovelle [10] and Ammar, Carrillo and Wittbold [1] extended the theory to L^1-data using the notion of renormalized solutions. If one assumes that the flux is genuinely nonlinear, it is possible to bypass the need of boundary entropies, since, in view of a result of Vasseur [12], entropy solutions admit in this case strong traces. The result of [12] was generalized by Panov [9]. Using [9], Andreianov and Sbihi [2] studied boundary value problems with a variety of boundary conditions.

We introduce next a notion of entropy solutions, which does not rely on any kind of strong traces, extends to inhomogeneous fluxes and yields a rather transparent uniqueness proof that does not require any boundary entropies.

For simplicity, we assume that $h \equiv 0$ and we refer to [6] for the general case. Moreover, if $g : \overline{U} \to \mathbb{R}$, $g(x)\delta_{\partial U}$ stands for the distribution, defined for all $\phi \in \mathcal{D}(\overline{U})$ by $\int_{\partial U} g(y)\phi(y) \, \mathrm{d}S(y)$.

Definition 3.1. A function $u \in C([0,T]; L^1(U)) \cap L^\infty(U \times (0,T))$ is an entropy sub- (resp. super-) solution to (16) with $h \equiv 0$ if, for every $k \in \mathbb{R}$,

$$(u \vee k)_t + \mathrm{div}_x \mathbf{f}(u \vee k) - \mathbf{1}_{\{u<k\}} \mathrm{div}_x \mathbf{f}(k,x) \le 0 \text{ in } \mathcal{D}'(U \times (0,T)),$$

(resp.

$$(u \wedge k)_t + \mathrm{div}_x \mathbf{f}(u \wedge k, x) - \mathbf{1}_{\{u>k\}} \mathrm{div}_x \mathbf{f}(k,x) \ge 0 \text{ in } \mathcal{D}'(U \times (0,T)),$$

and, for every $k \ge 0$ (resp. $k \le 0$),

$$(u \vee k)_t + \mathrm{div}_x \mathbf{f}(u \vee k) - \mathbf{1}_{\{u<k\}} \mathrm{div}_x \mathbf{f}(k,x) + \mathbf{f}(k,x) \cdot \nu(x) \delta_{\partial U} \le 0 \text{ in } \mathcal{D}'(\overline{U} \times (0,T)), \tag{17}$$

$$(\text{resp.} \quad (u \wedge k)_t + \mathrm{div}_x \mathbf{f}(u \wedge k) - \mathbf{1}_{\{u>k\}} \mathrm{div}_x \mathbf{f}(k,x) + \mathbf{f}(k,x) \cdot \nu(x) \delta_{\partial U} \ge 0 \text{ in } \mathcal{D}'(\overline{U} \times (0,T))). \tag{18}$$

A function $u \in C([0,T]; L^1(U)) \cap L^\infty(U \times (0,T))$ is an entropy solution to (16) with $h \equiv 0$ if it is both entropy sub- and super-solution.

The meaning of (17) (resp. (18)) is that, for every $k \ge 0$ (resp. $k \le 0$) and any non-negative and smooth ϕ, which is compactly supported in $\overline{U}$ and in $\mathcal{D}'((0,T))$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_U (u(x,t) \vee k) \phi(x) \, \mathrm{d}x - \int_U \mathbf{f}(u \vee k) \cdot \mathrm{D}\phi(x) \, \mathrm{d}x - \int_U \mathbf{1}_{\{u<k\}} \mathrm{div}_x \mathbf{f}(k,x) \phi(x) \, \mathrm{d}x$$
$$+ \int_{\partial U} \mathbf{f}(k,y) \cdot \nu(y) \phi(y) \, \mathrm{d}S(y) \le 0,$$

(resp.

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_U (u(x,t) \wedge k) \phi(x) \, \mathrm{d}x - \int_U \mathbf{f}(u \wedge k) \cdot \mathrm{D}\phi(x) \, \mathrm{d}x - \int_U \mathbf{1}_{\{u>k\}} \mathrm{div}_x \mathbf{f}(k,x) \phi(x) \, \mathrm{d}x$$
$$+ \int_{\partial U} \mathbf{f}(k,y) \cdot \nu(y) \phi(y), \mathrm{d}S(y) \ge 0.)$$

We remark that (17) (resp. (18)) is equivalent to saying that, for every $k \ge 0$ (resp. $k \le 0$)) and any non-negative and smooth ϕ, which is compactly supported in $\overline{U}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_U (u(x,t) - k)_+ \phi(x) \, \mathrm{d}x - \int_U [\mathbf{f}(u(x,t),x) - \mathbf{f}(k,x)] \cdot \mathrm{D}\phi(x) \mathbf{1}_{\{u \ge k\}} \, \mathrm{d}x + \int_U \mathbf{1}_{\{u \ge k\}} \, \mathrm{div}_x \mathbf{f}(k,x) \phi(x) \, \mathrm{d}x \le 0,$$

(resp.

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_U (k - u(x,t))_+ \phi(x) \, \mathrm{d}x - \int_U [\mathbf{f}(k,x) - \mathbf{f}(u(x,t),x)] \cdot \mathrm{D}\phi(x) \mathbf{1}_{\{u \le k\}} \, \mathrm{d}x$$
$$+ \int_U \mathbf{1}_{\{u \le k\}} \mathrm{div}_x \mathbf{f}(k,x) \phi(x) \, \mathrm{d}x \le 0).$$

Next we state the basic comparison and existence result for solutions in $C_t(L^1)) \cap L^\infty_{x,t}$. We discuss more general results in [6].

Theorem 3.2. *Assume (2) with U in place of $\mathbb{R}^d$ and let $u, v \in C([0,T]; L^1(U) \cap L^\infty(U \times (0,T))$ be respectively sub- and super-solutions to (16) with $h \equiv 0$. Then, for all $s, t \in (0,T)$ such that $s < t$,*

$$\int_U |u(x,t) - v(x,t)| \, \mathrm{d}x \le \int_U |u(x,s) - v(x,s)| \, \mathrm{d}x.$$

If, in addition, (8) and (11) with U in place of $\mathbb{R}^d$ hold and $u_0 \in (L^1 \cap L^\infty)(U)$, then (16) with $h \equiv 0$ has a solution in $C([0,T]; U) \cap L^\infty(U \times (0,T))$.

Sketch of the proof Theorem 3.2 for the problem (19). To simplify the argument but nevertheless give an idea of the arguments we discuss here the particular case $d = 1$, $U = (-\infty, 0)$ and homogeneous flux $f \in C(\mathbb{R})$, that is, the initial boundary value problem

$$u_t + f(u)_x = 0 \text{ in } (-\infty,0) \times (0,T) \quad u(0,t) = 0 \text{ in } (0,T) \quad u(\cdot,0) = u_0 \text{ in } (-\infty,0). \tag{19}$$

We begin the existence part and, for further simplification, we assume that $u_0 \in (L^1 \cap L^\infty \cap \mathrm{BV})((-\infty,0))$.

In view of the claimed contraction property, it is enough to assume that $u_0 \in (L^1 \cap L^\infty \cap BV \cap C^2)((-\infty,0))$, to consider the (smooth) solution $u^\epsilon \in C([0,T];L^1((-\infty,0))) \cap (L^\infty \cap BV)((-\infty,0) \times [0,T])$ of the regularized initial boundary value problem

$$u_t^\epsilon - \epsilon u_{xx}^\epsilon + f(u^\epsilon)_x = 0 \;\text{ in }\; (-\infty,0) \times (0,T) \quad u^\epsilon(0,t) = 0 \;\text{ in }\; (0,T) \quad u^\epsilon(\cdot,0) = u_0 \text{ in } (-\infty,0), \tag{20}$$

and to obtain, as already done in [3], independent of ϵ, L^∞- and BV-bounds. For the convenience of the reader here we sketch the proof of the BV-bound in x.

It is immediate from (20) that

$$|u_x^\epsilon|_t - \epsilon|u_x^\epsilon|_{xx} + (f'(u^\epsilon)|u_x^\epsilon|)_x \le 0 \;\text{ in }\; (-\infty,0) \times (0,T),$$

and, hence, for each $t \in (0,T)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{0} |u_x^\epsilon|(x,t)\,\mathrm{d}x \le \epsilon|u_x^\epsilon|_x(0,t) - f'(0)|u_x^\epsilon(0,t)|.$$

Using in the above inequality, that, in view the boundary condition $u^\epsilon(0,t) = 0$, $u_t^\epsilon(0,t) = 0$ and, hence, $-\epsilon u_{xx}^\epsilon(0,t) + f'(0)u_x^\epsilon(0,t) = 0$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{0} |u_x^\epsilon|(x,t)\,\mathrm{d}x \le 0,$$

which implies the claimed BV-bound.

The comparison and the contraction estimate follow from the general strategy of doubling the variables in the sub-solution and super-solution inequalities using $k = v(y,s)$ and $k = u(x,t)$ respectively. There is, however, a difficulty since the definition of the solution property up to the boundary requires the constant k to have a sign.

It is, therefore, necessary to introduce next a new property of the sub- and super-solutions, namely a type of weak trace, which is consequence of definition. Its proof is presented at the end of this section. We remark that although this weak trace property is stated for $d = 1$ and the domain $(-\infty,0)$, it actually holds in the general setting.

Lemma 3.3. *Let $\phi : (-\infty,0] \to [0,\infty)$ be piecewise C^1 and assume that, for some $R > 0$, $\phi \equiv 0$ in $(-\infty,-R]$. If $u \in C([0,T];L^1((-\infty,0))) \cap L^\infty((-\infty,0) \times (0,T))$ is a sub- (resp. super-) solution to (19), then, for every $k \ge 0$ (resp. $k \le 0$), $\epsilon > 0$, and $t \in [0,T]$,*

$$\int_0^t \mathrm{d}s \int_{-\infty}^{0} (f(u(x,s)) - f(k))\mathbf{1}_{\{u(x,s)\ge k\}} \frac{1}{\epsilon}\phi'(\frac{x}{\epsilon})\,\mathrm{d}x \ge -\epsilon R \|u(\cdot,0)\|_\infty \|\phi\|_\infty, \tag{21}$$

$$(\textit{resp.} \quad \int_0^t \mathrm{d}s \int_{-\infty}^{0} (f(k) - f(u(x,s)))\mathbf{1}_{\{k\ge u(x,s)\}} \frac{1}{\epsilon}\phi'(\frac{x}{\epsilon})\,\mathrm{d}x \ge -\epsilon R \|u(\cdot,0)\|_\infty \|\phi\|_\infty.) \tag{22}$$

We continue with the proof of the contraction property. As in the case of the whole space, letting $k = v(y,s)$ and $k = u(x,t)$ in the sub- and super-solution definition in $(-\infty,0) \times (0,T)$ leads to the distributional inequality

$$\begin{aligned} &\frac{\partial}{\partial t}(u(x,t) - v(y,t))_+ \\ &+ (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})(f(u(x,t)) - f(v(y,s)))\mathbf{1}_{\{u(x,t)\ge v(y,t)\}} \le 0 \;\text{ in }\; \mathcal{D}'((-\infty,0) \times (-\infty,0) \times (0,T)). \end{aligned} \tag{23}$$

Fix $\epsilon > 0$ and let $\rho_\epsilon(z) = \mathbf{1}_{\{|z|\le\epsilon/2\}}/\epsilon$ and $\chi_\epsilon(z) = \chi(z/\epsilon)$, where $\chi(z) = 1$ if $z \le -1$ and $\chi(z) = -z$ if $z \in [-1,0]$.

We use in (23) against the test function $\rho_\epsilon(x-y)\chi_\epsilon(x)\chi_\epsilon(y)$. Strictly speaking we need to use in place of ρ and χ smooth functions and, in the case of χ, compactly supported in $(-\infty,0)$. For the shake of simplicity we omit this technical but routine step.

A straightforward calculation which exploits the symmetry of ρ_ϵ in $x-y$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{0}\int_{-\infty}^{0}(u(x,t)-v(y,s))_+\rho_\epsilon(x-y)\chi_\epsilon(x)\chi_\epsilon(y)\,\mathrm{d}x\,\mathrm{d}y-\int_{-\infty}^{0}\int_{-\infty}^{0}(f(u(x,t))-f(v(y,s))) \\ \mathbf{1}_{\{u(x,t)\geq v(y,t)\}}\rho_\epsilon(x-y)\frac{1}{\epsilon}\left(\chi_\epsilon'(x)\chi_\epsilon(y)+\chi_\epsilon(x)\chi_\epsilon'(y)\right)\mathrm{d}x\,\mathrm{d}y\leq 0. \tag{24}$$

The claim is that there exists some $\overline{K}>0$ such that, for all $t\in(0,T)$,

$$-\int_0^t\int_{-\infty}^{0}\int_{-\infty}^{0}(f(u(x,s))-f(v(y,s)))\mathbf{1}_{\{u(x,s)\geq v(y,s)\}}\rho_\epsilon(x-y) \\ \frac{1}{\epsilon}\left(\chi_\epsilon'(x)\chi_\epsilon(y)+\chi_\epsilon(x)\chi_\epsilon'(y)\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\geq -\overline{K}(\|u(\cdot,0)\|_\infty+\|v(\cdot,0)\|_\infty)\epsilon. \tag{25}$$

It then follows, after letting $\epsilon\to 0$, that, for all $t\in(0,T)$,

$$\int_{-\infty}^{0}(u(x,t)-v(x,t))_+\leq\int_{-\infty}^{0}(u(x,0)-v(x,0))_+,$$

and, after replacing 0 by s, the claim.

It is clear that (25) should follow from Lemma 3.3. This, however, needs an extra step, which we describe next for the first of the two terms in (25). The argument for the other is identical.

First observe that, in view of the choice of χ_ϵ,

$$\begin{cases}-\int_0^t\int_{-\infty}^{0}\int_{-\infty}^{0}(f(u(x,s))-f(v(y,s)))\mathbf{1}_{\{u(x,s)\geq v(y,s)\}}\rho_\epsilon(x-y)\frac{1}{\epsilon}\chi_\epsilon'(x)\chi_\epsilon(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t= \\ \int_0^t\int_{-\infty}^{0}\int_{-\epsilon}^{0}(f(u(x,s))-f(v(y,s)))\mathbf{1}_{\{u(x,s)\geq v(y,s)\}}\rho_\epsilon(x-y)\frac{1}{\epsilon}\chi_\epsilon(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.\end{cases}$$

We show that, for some $\overline{K}>0$ and all $t\in(0,T)$,

$$\int_0^t\int_{-\infty}^{0}\int_{-\epsilon}^{0}(f(u(x,s))-f(v(y,s)))\mathbf{1}_{\{u(x,s)\geq v(y,s)\}}\rho_\epsilon(x-y)\frac{1}{\epsilon}\chi_\epsilon(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\geq-\overline{K}\|u(\cdot,0)\|_\infty\epsilon. \tag{26}$$

The crucial observation is the elementary remark that, for any $a,b\in\mathbb{R}$,

$$(f(a)-f(b))\mathbf{1}_{[a\geq b]}=(f(a)-f(\max(b,0))\mathbf{1}_{[a\geq\max(b,0)]}+(f(\min(a,0)-f(b))\mathbf{1}[\min(a,0)\geq b].$$

Then, the above and the choices of ρ_ϵ and χ_ϵ yield, for every $s\in(0,T)$,

$$-\int_{-\infty}^{0}\int_{-\infty}^{0}(f(u(x,s))-f(v(y,s)))\mathbf{1}_{\{u(x,s)\geq v(y,s)\}}\rho_\epsilon(x-y)\frac{1}{\epsilon}\chi_\epsilon'(x)\chi_\epsilon(y)\,\mathrm{d}x\,\mathrm{d}y=I_{1,\epsilon}(s)+I_{2,\epsilon}(s),$$

with

$$I_{1,\epsilon}(s):=\frac{1}{\epsilon}\int_{-\infty}^{0}\int_{-\epsilon}^{0}\frac{1}{\epsilon}(f(u(x,s))-f(\max(v(y,s),0)))\mathbf{1}_{\{|x-y|\leq\epsilon/2\}}\mathbf{1}_{\{u(x,s)\geq\max(v(y,s),0)\}}\,\mathrm{d}x\,\mathrm{d}y, \tag{27}$$

and

$$I_{2,\epsilon}(s):=\frac{1}{\epsilon}\int_{-\infty}^{0}\int_{-\epsilon}^{0}\frac{1}{\epsilon}(f(\min(u(x,s),0))-f(v(y,s)))\mathbf{1}_{\{|x-y|\leq\epsilon/2\}}\mathbf{1}_{\{\min(u(x,s),0)\geq v(y,s)\}}\,\mathrm{d}x\,\mathrm{d}y. \tag{28}$$

For $I_{1,\epsilon}(s)$ observe that, since $y\geq x-\epsilon/2\geq-3\epsilon/2$, for every $t\in(0,T)$,

$$\int_0^t I_{1,\epsilon}(s)\,\mathrm{d}s =$$

$$\frac{1}{\epsilon}\int_{-3\epsilon/2}^{0}\Big[\int_0^t\int_{-\epsilon}^{0}\frac{1}{\epsilon}(f(u(x,s))-f(\max(v(y,s),0)))\mathbf{1}_{\{|x-y|\le\epsilon/2\}}\mathbf{1}_{\{u(x,s)\ge\max(v(y,s),0)\}}\,\mathrm{d}x\,\mathrm{d}s\Big]\chi_\epsilon(y)\,\mathrm{d}y$$

$$=\int_{-3/2}^{0}\int_0^t\int_{-\epsilon}^{0}\frac{1}{\epsilon}(f(u(x,s))-f(\max(v(\epsilon z,s),0)))\mathbf{1}_{\{u(x,s)\ge\max(v(\epsilon z,s),0)\}}\phi'(\frac{x}{\epsilon})\,\mathrm{d}x\,\mathrm{d}s\,\chi(z)\,\mathrm{d}z,$$

where $\phi(\xi)=0$ for $\xi\le -1$ and $\phi'(\xi)=\mathbf{1}_{|\xi-z|\le 1/2}\chi(z)$.

It follows from Lemma 3.3 that there exists $\overline{K}>0$ such that, for all $t\in(0,T)$,

$$\int_0^t I_{1,\epsilon}(s)\,\mathrm{d}s \ge -\int_{-3/2}^{0}\chi(z)\,\mathrm{d}zK_1\|u(\cdot,0)\|_\infty\epsilon.$$

Arguing similarly, we find that, for all $t\in(0,T)$,

$$\int_0^t I_{2,\epsilon}(s)\,\mathrm{d}s=\int_{-1}^{0}\int_0^t\int_{-\infty}^{0}(f(\min(u(\epsilon z,s),0))-f(v(y,s)))\mathbf{1}_{\{\min(u(\epsilon z,s),0)\ge v(y,s)\}}\mathbf{1}_{|z-y/\epsilon|\le 1/2}\chi(\frac{y}{\epsilon})\frac{1}{\epsilon}\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}z.$$

We may now conclude applying again Lemma 3.3 to the super-solution v and the integral over (y,s) with $\phi(\eta)=0$ if $z\le -3/2$ and $\phi'(\eta)=\mathbf{1}_{|z-\eta|\le 1/2}\chi(z)$.

The proof of Lemma 3.3. We only show the argument for the sub-solution. Moreover, by approximation we may assume that $\phi\in C^1$.

The definition of the sub-solution gives that, for any $k\ge 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{0}(u(x,t)-k)_+\phi(x)\,\mathrm{d}x-\int_{-\infty}^{0}(f(u(x,t))-f(k))\mathbf{1}_{\{u(x,t)\ge k\}}\frac{1}{\epsilon}\phi'(\frac{x}{\epsilon})\,\mathrm{d}x\le 0,$$

and, hence,

$$\int_0^t\int_{-\infty}^{0}(f(u(x,s))-f(k))\mathbf{1}_{\{u(x,s)\ge k\}}\frac{1}{\epsilon}\phi'(\frac{x}{\epsilon})\,\mathrm{d}x\,\mathrm{d}s\ge$$

$$-\int_{-\infty}^{0}(u(x,0)-k)_+\phi(x)\,\mathrm{d}x=-\int_{-R\epsilon}^{0}(u(x,0)-k)_+\phi(x)\,\mathrm{d}x\ge -R\|u(\cdot,0)_+\|_\infty\|\phi\|_\infty.$$

The claim follows. □

4. Saturated solutions

We introduce the notion of saturated solutions, which turn out to be a kind of maximal solutions in some domains, and prove its well-posedness. For readers familiar with the theory of viscosity solutions, saturated solutions may be thought of as the analogues of the state constrained solutions.

To make the ideas accessible, here we discuss the simplified problem

$$u_t+\mathrm{div}_x\mathbf{f}(u)=0\ \text{ in }\ (-\infty,0)\times\mathbb{R}^{d-1}\times(0,T)\quad\text{and}\quad u(\cdot,0)=u_0\ \text{ in }\ (-\infty,0)\times\mathbb{R}^{d-1},\tag{29}$$

and, for $x\in(-\infty,0)\times\mathbb{R}^{d-1}$, we write $x=(x_1,x')$ with a $x_1\in(-\infty,0)$ and $x'\in\mathbb{R}^{d-1}$.

Definition 4.1. A function $u \in L^\infty_t(L^1_{x_1}(L^\infty_{x'}) \cap L^\infty_x) \cap C_t(L^1_{x_1,x'\mathrm{loc}})$ is a saturated entropy solution to (29), if, for every $k \in \mathbb{R}$,

$$(u \vee k)_t + \mathrm{div}_x \mathbf{f}(u \vee k) \le 0 \ \text{ in } \ \mathcal{D}'((-\infty,0) \times \mathbb{R}^{d-1} \times (0,T)), \tag{30}$$

and

$$(u \wedge k)_t + \mathrm{div}_x \mathbf{f}(u \wedge k) - f_1(k)\delta_0(x_1) \ge 0 \ \text{ in } \ \mathcal{D}'((-\infty,0] \times \mathbb{R}^{d-1} \times (0,T)). \tag{31}$$

We remark that (30) and (31) are respectively equivalent to that, for every $k \in \mathbb{R}$ and any non-negative and smooth ϕ, which is compactly supported in $(-\infty,0) \times \mathbb{R}^{d-1}$ and $(-\infty,0] \times \mathbb{R}^{d-1}$ respectively,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{0} \int_{\mathbb{R}^{d-1}} (u(x,t) - k)_+ \phi(x)\,\mathrm{d}x - \int_{-\infty}^{0} \int_{\mathbb{R}^{d-1}} (\mathbf{f}(u(x,t)) - \mathbf{f}(k)) \mathbf{1}_{\{u(x,t) \ge k\}} \cdot \mathrm{D}\phi(x)\,\mathrm{d}x \le 0, \tag{32}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{0} \int_{\mathbb{R}^{d-1}} (k - u(x,t))_+ \phi(x)\,\mathrm{d}x - \int_{-\infty}^{0} \int_{\mathbb{R}^{d-1}} (\mathbf{f}(k) - \mathbf{f}(u(x,t)) \mathbf{1}_{\{u(x,t) \le k\}} \cdot \mathrm{D}\phi(x)\,\mathrm{d}x \ge 0. \tag{33}$$

Observe that the difference between (17) and (18), and (30) and (31) is that the former restrict the sign of the admissible constants k while the latter must hold for all $k \in \mathbb{R}$.

To prove the well-posedness (existence and uniqueness) of the saturated solution, we impose an additional condition to the flux.

We assume that

$$\text{there exists a sequence } (\xi_n)_{n \in \mathbb{N}} \text{ such that } \lim_{n \to \infty} \xi_n = \infty \quad \text{and} \quad f_1(\xi) \ge f_1(\xi_n) \quad \text{for all } \ \xi \ge \xi_n. \tag{34}$$

Notice that (34) is clearly satisfied if f_1 is non-decreasing or $\lim_{\xi \to \infty} f_1(\xi) = \infty$.

We remark that the direction of the inequality in (34) is dictated by the fact that we are assuming that $x_1 \le 0$. If we study the problem in a general domain, the assumption is actually about the uniform "growth" of $-\mathbf{f}'(\xi) \cdot \nu$.

We state next the result about the well-posedness of the saturated solutions next. Recall that $B\mathcal{M}(V)$ denotes the set of bounded measures in V.

Theorem 4.2. *Assume that $\mathbf{f} \in (C^1(\mathbb{R}))^d$, $u_0 \in (L^1_{x_1}(L^\infty_{x'}) \cap L^\infty_x)$ and (34). There exists a unique saturated solution $u \in L^\infty_t(L^1_{x_1}(L^\infty_{x'}) \cap L^\infty_x) \cap C_t(L^1_{x_1,x'\mathrm{loc}})$ of (29). Moreover, $u \ge v$ where $v \in C([0,T]; (L^1((-\infty,0) \times \mathbb{R}^{d-1}) \cap L^\infty((-\infty,0) \times \mathbb{R}^{d-1}) \times (0,T))$ is an entropy sub-solution to (29) such that $v(\cdot,0) \le u_0$ almost everywhere in $(-\infty,0) \times \mathbb{R}^{d-1}$. If, in addition, $u_0 \in BV((-\infty,0) \times \mathbb{R}^{d-1}))$, then $u \in L^\infty((0,T); BV((-\infty,0) \times \mathbb{R}^{d-1}))$ and $u_t \in L^\infty((0,T); B\mathcal{M}((-\infty,0) \times \mathbb{R}^{d-1}))$.*

Proof. We only prove the existence and uniqueness. The other two claims are, then, immediate.

It follows from (34) that there exists $M > \|u_0\|_\infty$ such that

$$f_1(\xi) \ge f_1(M) \ \text{ for all } \ \xi \ge M. \tag{35}$$

Let $u^M \in C([0,T]; (L^1((-\infty,0) \times \mathbb{R}^{d-1}) \cap L^\infty((-\infty,0) \times \mathbb{R}^{d-1} \times (0,T))$ be the solution to

$$\begin{cases} u^M_t + \mathrm{div}_x \mathbf{f}(u^M) = 0 \ \text{ in } \ (-\infty,0) \times \mathbb{R}^{d-1} \times (0,T) \\ u^M = M \ \text{ on } \ \partial(-\infty,0) \times \mathbb{R}^{d-1} \times (0,T) \qquad u^M(\cdot,0) = u_0 \ \text{ on } \ (-\infty,0) \times \mathbb{R}^{d-1}. \end{cases} \tag{36}$$

The comparison principle of solutions for (16) yields that u^M is non-decreasing in M and bounded from below by u^0, the solution to (36) with zero Dirichlet data, and, in addition,

$$u^M \le M \ \text{ in } \ (-\infty,0) \times \mathbb{R}^{d-1} \times \ (0,T). \tag{37}$$

We show next that any such u^M is a saturated solution, which, in view of the uniqueness, implies that the u^M's are constant in M for all large enough M.

The subsolution property (30) in the interior is obviously satisfied since u^M solves (36). For the same reason, (31) is also satisfied for all $k \le M$. The definition in the previous section was stated for zero boundary condition, but the claim follows looking at $u^M - M$.

It remains to check (30) or equivalently (33) for $k > M$. For this, to keep the presentation and shorter, we assume that $d = 1$ and write f instead of f_1.

Recall (35) and (37), fix a non-negative, smooth and compactly supported in $(-\infty, 0]$ ϕ and observe that, since $k > M$, $(k - u^M)_+ = (k - u^M)$.

It follows that

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_{-\infty}^{0} (k - u^M(x,t))_+\phi(x)\,dx = \dfrac{d}{dt}\displaystyle\int_{-\infty}^{0} (k - u^M(x,t))\phi(x)\,dx \\ = \dfrac{d}{dt}\displaystyle\int_{-\infty}^{0} (M - u^M(x,t))\phi(x)\,dx = \dfrac{d}{dt}\displaystyle\int_{-\infty}^{0} (M - u^M(x,t))_+\phi(x)\,dx, \end{cases}$$

while

$$\begin{cases} -\displaystyle\int_{-\infty}^{0} (f(k) - f(u^M(x,t)))\mathbf{1}_{\{k \geq u^M(x,t)\}}\phi_x(x)\,dx = -\displaystyle\int_{-\infty}^{0} (f(k) - f(u^M(x,t)))\phi_x(x)\,dx \\ = -\displaystyle\int_{-\infty}^{0} (f(k) - f(M))\phi_x(x)\,dx - \displaystyle\int_{-\infty}^{0} (f(M) - f(u^M(x,t)))\phi_x(x)\,dx \\ = -\displaystyle\int_{-\infty}^{0} (f(k) - f(M))\phi_x(x)\,dx - \displaystyle\int_{-\infty}^{0} (f(M) - f(u^M(x,t)))\mathbf{1}_{\{M \geq u^M(x,t)\}}\phi_x(x)\,dx, \end{cases}$$

and

$$-\int_{-\infty}^{0} (f(k) - f(M))\phi_x(x)\,dx = -\int_{-\infty}^{0} (f(k) - f(M))\phi_x(x)\,dx = -(f(k) - f(M))\phi(0) \leq 0.$$

The comparison is an immediate consequence of the usual comparison argument, that is, the doubling of variables followed by the choice of an approximate identity $\rho_\delta(x_1 - y_1)$ such that, if $y_1 \leq 0$, then $x_1 < 0$. □

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