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Mathematical problems in mechanics

Intrinsic formulation of the displacement-traction problem in linear shell theory

Formulation intrinsèque du problème en déplacement-traction en théorie linéaire des coques

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ABSTRACT

We recast the Dirichlet boundary conditions satisfied by the displacement field of the middle surface of a linearly elastic shell as boundary conditions satisfied by the corresponding linearized change of metric and of curvature tensor fields. This in turn allows us to give an intrinsic formulation of the linear shell model of W.T. Koiter with these two tensor fields as the sole unknowns.
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RÉSUMÉ

On reformule les conditions aux limites de Dirichlet satisfaites par le champ de déplacements de la surface moyenne d'une coque linéairement élastique comme des conditions aux limites satisfaites par les champs de tenseurs linéarisés de changement de métrique et de coubure correspondants. Ceci permet ensuite de donner une formulation intrinsèque du modèle linéaire de coques de W.T. Koiter avec ces deux champs de tenseurs comme seules inconnues.
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1. Preliminaries and main result

Latin, resp. Greek, indices vary in $\{1, 2, 3\}$, resp. in $\{1, 2\}$, and the summation convention with respect to repeated indices is used in conjunction with these rules.

The notations $\mathbb{E}^3$, $\mathbb{S}^2$, and $\mathbb{A}^3$, respectively designate the three-dimensional Euclidean space, the space of all real 2×2 symmetric matrices, and the space of all real 3×3 antisymmetric matrices. The inner product, vector product, and Euclidean

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norm in $\mathbb{E}^3$ are respectively denoted by $\cdot$, $\wedge$, and $|\cdot|$. A generic point in $\mathbb{R}^2$ is denoted $y = (y_\alpha)$ and partial derivatives of the first and second order are denoted $\partial_\alpha := \dfrac{\partial}{\partial y_\alpha}$ and $\partial_{\alpha\beta} := \dfrac{\partial^2}{\partial y^\alpha \partial y^\beta}$.

Let $\omega \subset \mathbb{R}^2$ be a non-empty connected open set whose boundary $\gamma := \partial\omega$ is of class $\mathcal{C}^2$ (in the sense of [8]), let $\gamma_0 \subset \gamma$ be a non-empty relatively open subset of γ, and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ be an immersion.

Consider a *linearly elastic shell* with middle surface

$$S := \boldsymbol{\theta}(\overline{\omega})$$

and thickness $2\varepsilon > 0$, made of an elastic material characterized by two Lamé constants $\lambda \geq 0$ and $\mu > 0$, subjected to applied forces whose density per unit area of its middle surface is a vector field $\boldsymbol{p}^\varepsilon \in L^2(\omega; \mathbb{E}^3)$, and subjected to a homogeneous boundary condition of place along the part of its lateral face whose mid-section is the curve

$$\boldsymbol{\theta}(\gamma_0).$$

Then, according to Koiter [7], the *classical formulation* of the two-dimensional displacement-traction problem for such a shell takes the form of the following quadratic minimization problem: the unknown *displacement field* of the middle surface of the shell is the unique minimizer of the quadratic functional $j : \boldsymbol{V}(\omega) \to \mathbb{R}$, where

$$\boldsymbol{V}(\omega) := \{\boldsymbol{\eta} := \eta_i \boldsymbol{a}^i \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3);\ \eta_i = \partial_\alpha \eta_3 = 0 \text{ on } \gamma_0\},$$
$$j(\boldsymbol{\eta}) := \int_\omega a^{\alpha\beta\sigma\varphi} \left(\frac{\varepsilon}{2} \gamma_{\sigma\varphi}(\boldsymbol{\eta}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{6} \rho_{\sigma\varphi}(\boldsymbol{\eta}) \rho_{\alpha\beta}(\boldsymbol{\eta}) \right) \sqrt{a}\, \mathrm{d}y - \int_\omega \boldsymbol{p}^\varepsilon \cdot \boldsymbol{\eta} \sqrt{a}\, \mathrm{d}y \text{ for all } \boldsymbol{\eta} \in \boldsymbol{V}(\omega).$$

Here and in the sequel, $\boldsymbol{a}^i \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$ denote the vector fields of the contravariant bases along the surface $\boldsymbol{\theta}(\overline{\omega})$, which are defined by the relations

$$\boldsymbol{a}^i \cdot \boldsymbol{a}_j = \delta^i_j \text{ in } \overline{\omega},$$

where

$$\boldsymbol{a}_\alpha := \partial_\alpha \boldsymbol{\theta} \text{ and } \boldsymbol{a}_3 := \frac{\boldsymbol{a}_1 \wedge \boldsymbol{a}_2}{|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|} \text{ in } \overline{\omega}$$

denote the vector fields of the covariant bases along the surface $\boldsymbol{\theta}(\overline{\omega})$,

$$a^{\alpha\beta\sigma\varphi} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\sigma\varphi} a^{\alpha\beta} + 2\mu \left(a^{\alpha\sigma} a^{\beta\varphi} + a^{\alpha\varphi} a^{\beta\sigma} \right) \in \mathcal{C}^2(\overline{\omega}), \quad \text{where } a^{\alpha\beta} := \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta \in \mathcal{C}^2(\overline{\omega}),$$

denote the contravariant components of the two-dimensional elasticity tensor of the shell,

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2} (\partial_\alpha \boldsymbol{\eta} \cdot \boldsymbol{a}_\beta + \partial_\beta \boldsymbol{\eta} \cdot \boldsymbol{a}_\alpha) \in \mathcal{C}^1(\overline{\omega}),$$

resp.

$$\rho_{\alpha\beta}(\boldsymbol{\eta}) := \left(\partial_{\alpha\beta} \boldsymbol{\eta} - \Gamma^\sigma_{\alpha\beta} \partial_\sigma \boldsymbol{\eta} \right) \cdot \boldsymbol{a}_3 \in \mathcal{C}^0(\overline{\omega}),$$

denote the covariant components of the linearized change of metric, resp. of curvature, tensor field associated with the displacement field $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i$ of the surface $\boldsymbol{\theta}(\overline{\omega})$,

$$\Gamma^\sigma_{\alpha\beta} := \partial_{\alpha\beta} \boldsymbol{\theta} \cdot \boldsymbol{a}^\sigma \in \mathcal{C}^1(\overline{\omega})$$

denote the Christoffel symbols of the second kind associated with the immersion $\boldsymbol{\theta}$, and

$$\sqrt{a}\, \mathrm{d}y, \text{ where } a := \det(a_{\alpha\beta}) \in \mathcal{C}^1(\overline{\omega}) \text{ and } a_{\alpha\beta} := \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta \in \mathcal{C}^1(\overline{\omega}),$$

denotes the area element along the surface $\boldsymbol{\theta}(\overline{\omega})$.

It is well known that the extension by continuity of the functional j to the larger space $\overline{\boldsymbol{V}}(\omega)$, defined as the completion of $\boldsymbol{V}(\omega)$ with respect to the natural norm associated with j, has a unique minimizer in $\overline{\boldsymbol{V}}(\omega)$; cf. [1] or [2] (see also [3]).

The objective of this paper is to provide an *intrinsic formulation* of the above displacement-traction problem for linearly elastic shells. This new formulation consists in replacing, in the above minimization problem, the unknown $\boldsymbol{\eta}$ by the *new unknowns* $(c_{\alpha\beta})$ and $(r_{\alpha\beta})$, where

$$c_{\alpha\beta} := \gamma_{\alpha\beta}(\boldsymbol{\eta}) \text{ and } r_{\alpha\beta} := \rho_{\alpha\beta}(\boldsymbol{\eta}).$$

This replacement is made possible thanks to a well-known *infinitesimal rigid displacement lemma*, asserting that the linear mapping

$$\mathcal{F} : \boldsymbol{\eta} \in \boldsymbol{V}(\omega) \to \mathcal{F}(\boldsymbol{\eta}) := \big((c_{\alpha\beta}), (r_{\alpha\beta})\big) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2) \times \mathcal{C}^0(\overline{\omega}; \mathbb{S}^2)$$

is injective; see, e.g., [3]. Thus the unknown $\boldsymbol{\eta}$ can be replaced in the classical formulation of Koiter's model in terms of the new unknowns $(c_{\alpha\beta})$ and $(r_{\alpha\beta})$ by

$$\boldsymbol{\eta} = \mathcal{G}\left((c_{\alpha\beta}, (r_{\alpha\beta})\right),$$

where $\mathcal{G}$ denotes the *inverse* of the bijective linear mapping $\boldsymbol{\eta} \in \boldsymbol{V}(\omega) \to \mathcal{F}(\boldsymbol{\eta}) \in \mathbb{V}(\omega)$, where

$$\mathbb{V}(\omega) := \{\mathcal{F}(\boldsymbol{\eta});\ \boldsymbol{\eta} \in \boldsymbol{V}(\omega)\}.$$

The *main result* of this Note is the following *explicit characterization of the space* $\mathbb{V}(\omega)$, to which the new unknowns $(c_{\alpha\beta})$ and $(r_{\alpha\beta})$ belong.

Theorem 1. *Let $\omega \subset \mathbb{R}^2$ be a non-empty, simply-connected open set whose boundary is of class $\mathcal{C}^2$, let $\gamma_0 \subset \partial\omega$ be a non-empty relatively open connected subset of the boundary $\partial\omega$ of ω, and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ be an immersion.*

Let the functions $\tau^\alpha \in \mathcal{C}^1(\gamma_0)$, $\nu^\alpha \in \mathcal{C}^1(\gamma_0)$, $\kappa_g \in \mathcal{C}^0(\gamma_0)$, $\kappa_n \in \mathcal{C}^0(\gamma_0)$, and $\tau_g \in \mathcal{C}^0(\gamma_0)$, respectively denote the contravariant components of the unit tangent vector, the contravariant components of the unit inner normal vector, the geodesic curvature, the normal curvature, and the geodesic torsion, along the curve $\boldsymbol{\theta}(\gamma_0)$ (cf. Section 2)*.*

Given any matrix fields $(c_{\alpha\beta}) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2)$ and $(r_{\alpha\beta}) \in \mathcal{C}^0(\overline{\omega}; \mathbb{S}^2)$, define the distributions

$$\begin{aligned} S_{\beta\alpha\sigma\varphi} &:= c_{\sigma\alpha|\beta\varphi} + c_{\varphi\beta|\alpha\sigma} - c_{\varphi\alpha|\beta\sigma} - c_{\sigma\beta|\alpha\varphi} + R^{\psi}_{\cdot\alpha\sigma\varphi} c_{\beta\psi} - R^{\psi}_{\cdot\beta\sigma\varphi} c_{\alpha\psi} - b_{\varphi\alpha} r_{\sigma\beta} - b_{\sigma\beta} r_{\varphi\alpha} \\ &\qquad + b_{\sigma\alpha} r_{\varphi\beta} + b_{\varphi\beta} r_{\sigma\alpha} \in \mathcal{D}'(\omega), \\ S_{3\alpha\sigma\varphi} &:= b^{\psi}_{\sigma}(c_{\alpha\psi|\varphi} + c_{\varphi\psi|\alpha} - c_{\varphi\alpha|\psi}) - b^{\psi}_{\varphi}(c_{\alpha\psi|\sigma} + c_{\sigma\psi|\alpha} - c_{\sigma\alpha|\psi}) - r_{\sigma\alpha|\varphi} + r_{\varphi\alpha|\sigma} \in \mathcal{D}'(\omega), \end{aligned}$$

where the functions

$$R^{\psi}_{\cdot\alpha\sigma\varphi} := \partial_\sigma \Gamma^{\psi}_{\alpha\varphi} - \partial_\varphi \Gamma^{\psi}_{\alpha\sigma} + \Gamma^{\beta}_{\alpha\varphi}\Gamma^{\psi}_{\beta\sigma} - \Gamma^{\beta}_{\alpha\sigma}\Gamma^{\psi}_{\beta\varphi} \in \mathcal{C}^0(\overline{\omega})$$

denote the mixed components of the Riemann curvature tensor associated with the immersion $\boldsymbol{\theta}$.

Then the space $\mathbb{V}(\omega)$ is given by

$$\begin{aligned} \mathbb{V}(\omega) = \Big\{ &((c_{\alpha\beta}), (r_{\alpha\beta})) \in \mathcal{C}^1(\overline{\omega}; \mathbb{S}^2) \times \mathcal{C}^0(\overline{\omega}; \mathbb{S}^2);\ S_{\beta\alpha\sigma\varphi} = 0 \text{ and } S_{3\alpha\sigma\varphi} = 0 \text{ in } \mathcal{D}'(\omega), \\ &c_{\alpha\beta}\tau^\alpha\tau^\beta = 0 \text{ and } c_{\alpha\beta|\sigma}\tau^\alpha\left(2\nu^\beta\tau^\sigma - \tau^\beta\nu^\sigma\right) + \kappa_g c_{\alpha\beta}\nu^\alpha\nu^\beta = 0 \text{ on } \gamma_0, \\ &r_{\alpha\beta}\tau^\alpha\tau^\beta = 0 \text{ and } r_{\alpha\beta}\tau^\alpha\nu^\beta - c_{\alpha\beta}\nu^\alpha\left(2\kappa_n\tau^\beta + \tau_g\nu^\beta\right) = 0 \text{ on } \gamma_0 \Big\}, \quad \square \end{aligned}$$

Theorem 1 is established by combining Theorem 2 (below) and Theorem 3 (Section 3). Since Theorem 2 is a simple consequence of Theorems 4.1 and 5.1 in [4], its proof is not given here.

Theorem 2. *Let $\omega \subset \mathbb{R}^2$ be a non-empty connected open set and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ be an immersion. Then:*

(a) *If $\boldsymbol{\eta} \in \mathcal{C}^2(\omega; \mathbb{E}^3)$, the functions $c_{\alpha\beta} := \gamma_{\alpha\beta}(\boldsymbol{\eta}) \in \mathcal{C}^1(\omega)$ and $r_{\alpha\beta} := \rho_{\alpha\beta}(\boldsymbol{\eta}) \in \mathcal{C}^0(\omega)$ satisfy the equations:*

$$S_{\beta\alpha\sigma\varphi} = 0 \text{ and } S_{3\alpha\sigma\varphi} = 0 \text{ in } \mathcal{D}'(\omega),$$

where the functions $S_{\beta\alpha\sigma\varphi}$ and $S_{3\alpha\sigma\varphi}$ are defined in terms of $c_{\alpha\beta}$ and $r_{\alpha\beta}$ as in Theorem 1.

(b) *If the functions $c_{\alpha\beta} \in \mathcal{C}^1(\omega)$ and $r_{\alpha\beta} \in \mathcal{C}^0(\omega)$ satisfy the equations*

$$S_{\beta\alpha\sigma\varphi} = 0 \text{ and } S_{3\alpha\sigma\varphi} = 0 \text{ in } \mathcal{D}'(\omega),$$

and if ω is simply-connected, then there exists a vector field $\boldsymbol{\eta} \in \mathcal{C}^2(\omega; \mathbb{E}^3)$ such that

$$c_{\alpha\beta} = \gamma_{\alpha\beta}(\boldsymbol{\eta}) \text{ and } r_{\alpha\beta} = \rho_{\alpha\beta}(\boldsymbol{\eta}) \text{ in } \omega.$$

(c) *If the boundary of ω is of class $\mathcal{C}^2$, the results of* (a) *and* (b) *hold "up to the boundary", in the sense that* (a) *holds for $\boldsymbol{\eta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$, in which case the corresponding functions $c_{\alpha\beta}$ and $r_{\alpha\beta}$ belong respectively to the spaces $\mathcal{C}^1(\overline{\omega})$ and $\mathcal{C}^0(\overline{\omega})$, and* (b) *holds for $c_{\alpha\beta} \in \mathcal{C}^1(\overline{\omega})$ and $r_{\alpha\beta} \in \mathcal{C}^0(\overline{\omega})$, in which case the corresponding vector field $\boldsymbol{\eta}$ belongs to the space $\mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$.* $\square$

2. Geometry of curves on a surface with boundary

More details about the geometry notions used in this section can be found in [9].

Let $\omega \subset \mathbb{R}^2$ be a nonempty connected open set whose boundary $\gamma := \partial\omega$ is of class $\mathcal{C}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ be an immersion. Then

$$S = \boldsymbol{\theta}(\overline{\omega})$$

is a *surface with boundary* in $\mathbb{E}^3$ and the *boundary* $\boldsymbol{\theta}(\gamma)$ *of* S is a curve, or a finite union of curves, of class $\mathcal{C}^2$. For definiteness, we consider this curve oriented by means of the *inner* normal vector field to the boundary of ω; this means that its unit tangent vector

$$\boldsymbol{\tau}(y) = \tau^\alpha(y)\boldsymbol{a}_\alpha(y)$$

at any point $\boldsymbol{\theta}(y)$, $y \in \partial\omega$, has the property that the vector $(-\tau^2(y), \tau^1(y)) \in \mathbb{R}^2$, which is normal to the curve γ at $y \in \gamma$, is oriented towards the interior of ω.

The covariant components of the *first and second fundamental forms* associated with the immersion $\boldsymbol{\theta}$ are respectively denoted and defined by

$$a_{\alpha\beta} := \partial_\alpha\boldsymbol{\theta} \cdot \partial_\beta\boldsymbol{\theta} \text{ and } b_{\alpha\beta} := \partial_{\alpha\beta}\boldsymbol{\theta} \cdot \frac{\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}}{|\partial_1\boldsymbol{\theta} \wedge \partial_2\boldsymbol{\theta}|}.$$

The *Darboux frame* at a point $\boldsymbol{\theta}(y)$, $y \in \gamma$, of the boundary of S is the orthogonal basis in $\mathbb{E}^3$ formed by the three vectors $\boldsymbol{\tau}(y)$, $\boldsymbol{\nu}(y)$, and $\boldsymbol{a}_3(y)$, where $\boldsymbol{\tau}(y)$ is the positively-oriented unit vector tangent to $\boldsymbol{\theta}(\gamma)$ defined above,

$$\boldsymbol{a}_3(y) := \frac{\partial_1\boldsymbol{\theta}(y) \wedge \partial_2\boldsymbol{\theta}(y)}{|\partial_1\boldsymbol{\theta}(y) \wedge \partial_2\boldsymbol{\theta}(y)|}$$

is a unit vector orthogonal to S at $\boldsymbol{\theta}(y)$, and

$$\boldsymbol{\nu}(y) := \boldsymbol{a}_3(y) \wedge \boldsymbol{\tau}(y)$$

is a unit vector in the tangent plan to S at $\boldsymbol{\theta}(y)$ that is orthogonal to the boundary of S at $\boldsymbol{\theta}(y)$.

The *geodesic curvature* $\kappa_g : \gamma \to \mathbb{R}$, the *normal curvature* $\kappa_n : \gamma \to \mathbb{R}$, and the *geodesic torsion* $\tau_g : \gamma \to \mathbb{R}$, of the curve $\boldsymbol{\theta}(\gamma)$ are respectively defined at each point $y \in \gamma$ as the scalars

$$\begin{aligned}
\kappa_g(y) &:= \partial_{\boldsymbol{\tau}}\boldsymbol{\tau}(y) \cdot \boldsymbol{\nu}(y) = -\boldsymbol{\tau}(y) \cdot \partial_{\boldsymbol{\tau}}\boldsymbol{\nu}(y),\\
\kappa_n(y) &:= \partial_{\boldsymbol{\tau}}\boldsymbol{\tau}(y) \cdot \boldsymbol{a}_3(y) = -\boldsymbol{\tau}(y) \cdot \partial_{\boldsymbol{\tau}}\boldsymbol{a}_3(y),\\
\tau_g(y) &:= \partial_{\boldsymbol{\tau}}\boldsymbol{\nu}(y) \cdot \boldsymbol{a}_3(y) = -\boldsymbol{\nu}(y) \cdot \partial_{\boldsymbol{\tau}}\boldsymbol{a}_3(y),
\end{aligned}$$

where the notation $\partial_{\boldsymbol{\tau}}\boldsymbol{\tau}(y)$ denotes the derivative at $\boldsymbol{\theta}(y)$ of the vector field $\boldsymbol{\tau}$ with respect to the arclength abscissa along the curve $\boldsymbol{\theta}(\gamma)$.

Note that, for any extension (still denoted τ^α) of τ^α of class $\mathcal{C}^1$ in a neighborhood of γ, we have

$$\partial_{\boldsymbol{\tau}}\boldsymbol{\tau}(y) = \tau^\alpha(y)\partial_\alpha\boldsymbol{\tau}(y) \text{ for all } y \in \gamma.$$

Let $\boldsymbol{\eta} = \eta^i\boldsymbol{a}_i \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$ denote a vector field on the surface $\mathcal{S} = \boldsymbol{\theta}(\overline{\omega})$. For each $t \in \mathbb{R}$, define the mapping $\boldsymbol{\theta}(t) \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$ by

$$\boldsymbol{\theta}(t) := \boldsymbol{\theta} + t\boldsymbol{\eta} \text{ in } \overline{\omega}.$$

Then a simple compactness argument shows that there exists $\delta > 0$ such that, for each $t \in [-\delta, \delta]$, the mapping $\boldsymbol{\theta}(t)$ is an immersion. For such t, the image

$$S(t) := \boldsymbol{\theta}(t)(\overline{\omega})$$

is a *surface with boundary* in $\mathbb{E}^3$ and the image $\boldsymbol{\theta}(t)(\gamma)$ by $\boldsymbol{\theta}(t)$ of $\gamma = \partial\omega$ is a curve, or a finite union of curves, on this surface. Hence the *geodesic curvature* $\kappa_g(t)(y)$, the *normal curvature* $\kappa_n(t)(y)$, and the *geodesic torsion* $\tau_g(t)(y)$, of the curve $\boldsymbol{\theta}(t)(\gamma)$ at $\boldsymbol{\theta}(t)(y)$, $y \in \gamma$, are defined as above in terms of the *Darboux frames* $\{\boldsymbol{\tau}(t)(y), \boldsymbol{\nu}(t)(y), \boldsymbol{a}_3(t)(y)\}$, $y \in \gamma$.

Then the *linearized change of length* $\Delta a_{\boldsymbol{\tau}}(\boldsymbol{\eta}) : \gamma \to \mathbb{R}$, the *linearized change of geodesic curvature* $\Delta\kappa_g(\boldsymbol{\eta}) : \gamma \to \mathbb{R}$, the *linearized change of normal curvature* $\Delta\kappa_n(\boldsymbol{\eta}) : \gamma \to \mathbb{R}$, and the *linearized change of geodesic torsion* $\Delta\tau_g(\boldsymbol{\eta}) : \gamma \to \mathbb{R}$, associated with the vector field $\boldsymbol{\eta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$ are respectively defined at each point $y \in \gamma$ as the scalars (note that $a_{\alpha\beta}\tau^\alpha\tau^\beta = |\boldsymbol{\tau}|^2 = 1$)

$$\Delta a_{\boldsymbol{\tau}}(\boldsymbol{\eta})(y) := \lim_{t\to 0}\frac{1}{2t}\left[a_{\alpha\beta}(t)\tau^\alpha\tau^\beta - a_{\alpha\beta}\tau^\alpha\tau^\beta\right](y) = \lim_{t\to 0}\frac{1}{2t}\left(a_{\alpha\beta}(t)(y)\tau^\alpha(y)\tau^\beta(y) - 1\right),$$
$$\Delta\kappa_g(\boldsymbol{\eta})(y) := \lim_{t\to 0}\frac{1}{t}\left(\kappa_g(t)(y) - \kappa_g(y)\right),$$
$$\Delta\kappa_n(\boldsymbol{\eta})(y) := \lim_{t\to 0}\frac{1}{t}\left(\kappa_n(t)(y) - \kappa_n(y)\right),$$
$$\Delta\tau_g(\boldsymbol{\eta})(y) := \lim_{t\to 0}\frac{1}{t}\left(\tau_g(t)(y) - \tau_g(y)\right).$$

Finally, the covariant components of the *first and second fundamental forms* associated with the immersion $\boldsymbol{\theta}(t)$ are respectively defined by

$$a_{\alpha\beta}(t) := \partial_\alpha\boldsymbol{\theta}(t)\cdot\partial_\beta\boldsymbol{\theta}(t) \text{ and } b_{\alpha\beta}(t) := \partial_{\alpha\beta}\boldsymbol{\theta}(t)\cdot\boldsymbol{a}_3(t) \text{ in } \overline{\omega},$$

and the *linear change of metric tensor field* $(\gamma_{\alpha\beta}(\boldsymbol{\eta})) : \overline{\omega} \to \mathbb{S}^2$ and the *linear change of curvature tensor field* $(\rho_{\alpha\beta}(\boldsymbol{\eta})) : \overline{\omega} \to \mathbb{S}^2$ associated with the displacement field $\boldsymbol{\eta}$ of the surface $S = \boldsymbol{\theta}(\overline{\omega})$ are respectively denoted and defined at each point $y \in \overline{\omega}$ by

$$\gamma_{\alpha\beta}(\boldsymbol{\eta})(y) := \lim_{t\to 0}\frac{1}{2t}(a_{\alpha\beta}(t)(y) - a_{\alpha\beta}(y)) \text{ and } \rho_{\alpha\beta}(\boldsymbol{\eta})(y) := \lim_{t\to 0}\frac{1}{t}(b_{\alpha\beta}(t)(y) - b_{\alpha\beta}(y)).$$

The following two lemmas play an essential role in the proof of Theorem 3 (Section 3), which itself play an essential role in the proof of our main result, Theorem 1 (Section 1).

Note that, as expected, the expressions found in Lemma 1 for the components $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ as defined above coincide with the classical ones; cf., e.g., [3].

Lemma 1. *Let $\omega \subset \mathbb{R}^2$ be a non-empty connected open set with a boundary γ of class $\mathcal{C}^2$, let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ be an immersion, and let $\boldsymbol{\eta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$ be a vector field. Then*

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \frac{1}{2}(\partial_\alpha\boldsymbol{\eta}\cdot\boldsymbol{a}_\beta + \partial_\beta\boldsymbol{\eta}\cdot\boldsymbol{a}_\alpha) \text{ and } \rho_{\alpha\beta}(\boldsymbol{\eta}) = \left(\partial_{\alpha\beta}\boldsymbol{\eta} - \Gamma^\sigma_{\alpha\beta}\partial_\sigma\boldsymbol{\eta}\right)\cdot\boldsymbol{a}_3 \text{ in } \overline{\omega},$$

and

$$\Delta a_{\boldsymbol{\tau}}(\boldsymbol{\eta}) = \partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{\tau} \text{ on } \gamma,$$
$$\Delta\kappa_g(\boldsymbol{\eta}) = \partial_{\boldsymbol{\tau}}(\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{\nu}) - \kappa_g(\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{\tau}) - \tau_g(\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{a}_3) + \kappa_n(\partial_{\boldsymbol{\nu}}\boldsymbol{\eta}\cdot\boldsymbol{a}_3) \text{ on } \gamma,$$
$$\Delta\kappa_n(\boldsymbol{\eta}) = \partial_{\boldsymbol{\tau}}(\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{a}_3) - \kappa_n(\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{\tau}) + \tau_g(\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{\nu}) - \kappa_g(\partial_{\boldsymbol{\nu}}\boldsymbol{\eta}\cdot\boldsymbol{a}_3) \text{ on } \gamma,$$
$$\Delta\tau_g(\boldsymbol{\eta}) = \partial_{\boldsymbol{\tau}}(\partial_{\boldsymbol{\nu}}\boldsymbol{\eta}\cdot\boldsymbol{a}_3) - \tau_g(\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{\tau}) - \kappa_n(\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{\nu}) + \kappa_g(\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{a}_3) \text{ on } \gamma,$$

where

$$\boldsymbol{\tau} = \tau^\alpha\boldsymbol{a}_\alpha, \quad \boldsymbol{\nu} = \nu^\alpha\boldsymbol{a}_\alpha, \text{ and } \boldsymbol{a}_3 := \frac{\boldsymbol{a}_1\wedge\boldsymbol{a}_2}{|\boldsymbol{a}_1\wedge\boldsymbol{a}_2|},$$

denote the vector fields that constitute the Darboux frames associated with the curve $\boldsymbol{\theta}(\gamma)$, and

$$\boldsymbol{a}_\alpha := \partial_\alpha\boldsymbol{\theta}, \quad \partial_{\boldsymbol{\tau}} := \tau^\alpha\partial_\alpha, \text{ and } \partial_{\boldsymbol{\nu}} := \nu^\alpha\partial_\alpha.$$

Furthermore, the following relations hold:

$$\Delta a_{\boldsymbol{\tau}}(\boldsymbol{\eta}) = \gamma_{\alpha\beta}(\boldsymbol{\eta})\tau^\alpha\tau^\beta \text{ on } \gamma,$$
$$\Delta\kappa_g(\boldsymbol{\eta}) = \gamma_{\alpha\beta}(\boldsymbol{\eta})|_\sigma\tau^\alpha\left(2\nu^\beta\tau^\sigma - \tau^\beta\nu^\sigma\right) + \kappa_g\gamma_{\alpha\beta}(\boldsymbol{\eta})\left(\nu^\alpha\nu^\beta - 2\tau^\alpha\tau^\beta\right) \text{ on } \gamma,$$
$$\Delta\kappa_n(\boldsymbol{\eta}) = \rho_{\alpha\beta}(\boldsymbol{\eta})\tau^\alpha\tau^\beta - 2\kappa_n\gamma_{\alpha\beta}(\boldsymbol{\eta})\tau^\alpha\tau^\beta \text{ on } \gamma,$$
$$\Delta\tau_g(\boldsymbol{\eta}) = \rho_{\alpha\beta}(\boldsymbol{\eta})\tau^\alpha\nu^\beta - \gamma_{\alpha\beta}(\boldsymbol{\eta})\left[2b^\alpha_\sigma\tau^\sigma\nu^\beta + \tau_g\left(\tau^\alpha\tau^\beta + \nu^\alpha\nu^\beta\right)\right] \text{ on } \gamma,$$

where

$$\gamma_{\alpha\beta}(\boldsymbol{\eta})|_\sigma := \partial_\sigma\gamma_{\alpha\beta}(\boldsymbol{\eta}) - \Gamma^\varphi_{\alpha\sigma}\gamma_{\varphi\beta}(\boldsymbol{\eta}) - \Gamma^\varphi_{\sigma\beta}\Gamma_{\alpha\varphi}(\boldsymbol{\eta}) \text{ in } \overline{\omega}$$

denotes the covariant derivatives of the components of the twice-covariant tensor $(\gamma_{\alpha\beta}(\boldsymbol{\eta}))$.

Proof. The first two relations are well-known in the theory of linearly elastic shells (see, e.g., [3]).

The next four relations follow from the definitions of the limits $\Delta a_{\boldsymbol{\tau}}(\boldsymbol{\eta})$, $\Delta\kappa_g(\boldsymbol{\eta})$, $\Delta\kappa_n(\boldsymbol{\eta})$, and $\Delta\tau_g(\boldsymbol{\eta})$, combined with the relations

$$\begin{aligned}
\lim_{t\to 0}\frac{1}{t}(\boldsymbol{\tau}(t)-\boldsymbol{\tau}) &= (\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{\nu})\boldsymbol{\nu}+(\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{a}_3)\boldsymbol{a}_3 \text{ on } \gamma,\\
\lim_{t\to 0}\frac{1}{t}(\boldsymbol{\nu}(t)-\boldsymbol{\nu}) &= -(\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{\nu})\boldsymbol{\tau}+(\partial_{\boldsymbol{\nu}}\boldsymbol{\eta}\cdot\boldsymbol{a}_3)\boldsymbol{a}_3 \text{ on } \gamma,\\
\lim_{t\to 0}\frac{1}{t}(\boldsymbol{a}_3(t)-\boldsymbol{a}_3) &= -(\partial_\sigma\boldsymbol{\eta}\cdot\boldsymbol{a}_3)\boldsymbol{a}^\sigma = -(\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{a}_3)\boldsymbol{\tau}-(\partial_{\boldsymbol{\nu}}\boldsymbol{\eta}\cdot\boldsymbol{a}_3)\boldsymbol{\nu} \text{ on } \overline{\omega},
\end{aligned}$$

where $(\boldsymbol{\tau},\boldsymbol{\nu},\boldsymbol{a}_3)$ denotes the Darboux frame associated with the curve $\boldsymbol{\theta}(\gamma)$, and $(\boldsymbol{\tau}(t),\boldsymbol{\nu}(t),\boldsymbol{a}_3(t))$ denotes the Darboux frame associated with the curve $\boldsymbol{\theta}(t)(\gamma)$.

Finally, the last four relations of Lemma 1 follow by combining the relations established above. □

Remark 1. The last four relations in the statement of Lemma 1 show that the *linearized change of length*, the *linearized change of geodesic curvature*, the *linearized change of normal curvature*, and the *linearized change of geodesic torsion*, associated with a vector field $\boldsymbol{\eta}$ depend on this vector field only by means of the *linear change of metric tensor field* and the *linear change of curvature tensor field* associated with $\boldsymbol{\eta}$.

Lemma 2. *Let $\omega\subset\mathbb{R}^2$ be a non-empty connected open set with a boundary γ of class $\mathcal{C}^2$, let $\boldsymbol{\theta}\in\mathcal{C}^3(\overline{\omega};\mathbb{E}^3)$ be an immersion, and let $\boldsymbol{\eta}\in\mathcal{C}^2(\overline{\omega};\mathbb{E}^3)$ be a vector field. Let $\gamma_0\subset\gamma$ be a non-empty and connected relatively open subset of the boundary of ω.*

Then

$$\Delta a_{\boldsymbol{\tau}}(\boldsymbol{\eta})=\Delta\kappa_g(\boldsymbol{\eta})=\Delta\kappa_n(\boldsymbol{\eta})=\Delta\tau_g(\boldsymbol{\eta})=0 \text{ on } \gamma_0$$

if and only if there exist two vectors $\boldsymbol{h}\in\mathbb{E}^3$ and $\boldsymbol{\kappa}\in\mathbb{E}^3$ such that the vector field $\boldsymbol{\eta}^\sharp\in\mathcal{C}^2(\overline{\omega};\mathbb{E}^3)$ defined by

$$\boldsymbol{\eta}^\sharp(y):=\boldsymbol{\eta}(y)-(\boldsymbol{h}+\boldsymbol{\kappa}\wedge\boldsymbol{\theta}(y)) \text{ for all } y\in\overline{\omega}$$

satisfies

$$\boldsymbol{\eta}^\sharp(y)=\boldsymbol{0} \text{ and } \partial_{\boldsymbol{\nu}}(\boldsymbol{\eta}^\sharp\cdot\boldsymbol{a}_3)(y)=0 \text{ for all } y\in\gamma_0.$$

Proof. In view of Lemma 1, the system of equations

$$\Delta a_{\boldsymbol{\tau}}(\boldsymbol{\eta})=\Delta\kappa_g(\boldsymbol{\eta})=\Delta\kappa_n(\boldsymbol{\eta})=\Delta\tau_g(\boldsymbol{\eta})=0 \text{ on } \gamma_0$$

is equivalent to the equations

$$\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{\tau}=0 \text{ and } \partial_{\boldsymbol{\tau}}\boldsymbol{F}(\boldsymbol{\eta})=\boldsymbol{A}\boldsymbol{F}(\boldsymbol{\eta}) \text{ on } \gamma_0,$$

where $\boldsymbol{F}(\boldsymbol{\eta}):\gamma\to\mathbb{M}^3$ denotes the column vector field whose components are $\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{\nu}$, $\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{a}_3$, and $\partial_{\boldsymbol{\nu}}\boldsymbol{\eta}\cdot\boldsymbol{a}_3$ (in this order), and $\boldsymbol{A}:\gamma\to\mathbb{A}^3$ denotes the matrix field defined by

$$\boldsymbol{A}:=\begin{pmatrix}0 & \tau_g & -\kappa_n\\ -\tau_g & 0 & \kappa_g\\ \kappa_n & -\kappa_g & 0\end{pmatrix}.$$

Given any point $y^0\in\gamma_0$, let

$$\begin{aligned}
\boldsymbol{k}&:=((\partial_{\boldsymbol{\nu}}\boldsymbol{\eta}\cdot\boldsymbol{a}_3)\,\boldsymbol{\tau}-(\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{a}_3)\,\boldsymbol{\nu}+(\partial_{\boldsymbol{\tau}}\boldsymbol{\eta}\cdot\boldsymbol{\nu})\,\boldsymbol{a}_3)\,(y^0)\in\mathbb{E}^3,\\
\boldsymbol{h}&:=\boldsymbol{\eta}(y^0)-\boldsymbol{k}\wedge\boldsymbol{\theta}(y^0)\in\mathbb{E}^3,\\
\boldsymbol{\xi}(y)&:=\boldsymbol{h}+\boldsymbol{k}\wedge\boldsymbol{\theta}(y),\ y\in\overline{\omega}.
\end{aligned}$$

Then one can show that

$$\partial_{\boldsymbol{\tau}}\boldsymbol{\xi}\cdot\boldsymbol{\tau}=0 \text{ and } \partial_{\boldsymbol{\tau}}\boldsymbol{F}(\boldsymbol{\xi})=\boldsymbol{A}\boldsymbol{F}(\boldsymbol{\xi}) \text{ on } \gamma,$$

and that

$$\boldsymbol{\xi}(y^0)=\boldsymbol{\eta}(y^0) \text{ and } \boldsymbol{F}(\boldsymbol{\xi})(y^0)=\boldsymbol{F}(\boldsymbol{\eta})(y^0).$$

Then the announced equivalence follows from the uniqueness of the solutions to the Cauchy problems

$$\begin{cases} \partial_{\boldsymbol{\tau}} \boldsymbol{F} = \boldsymbol{A}\boldsymbol{F} \text{ on } \gamma_0, \\ \boldsymbol{F}(y^0) = \boldsymbol{F}(\boldsymbol{\eta})(y^0), \end{cases}$$

and

$$\begin{cases} \partial_{\boldsymbol{\tau}} \boldsymbol{\eta} = \partial_{\boldsymbol{\tau}} \boldsymbol{\xi} \text{ on } \gamma_0, \\ \boldsymbol{\eta}(y^0) = \boldsymbol{\xi}(y^0). \quad \square \end{cases}$$

3. Intrinsic formulation of a homogeneous boundary condition of place for linearly elastic shells

By definition, a displacement field $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$ of the middle surface $\boldsymbol{\theta}(\overline{\omega})$ of a shell satisfies a *homogeneous boundary condition of place* along a portion $\boldsymbol{\theta}(\gamma_0)$ of its boundary, where γ_0 is a non-empty relatively open subset of the boundary of ω, if

$$\eta_i = \partial_\alpha \eta_3 = 0 \text{ on } \gamma_0.$$

An *intrinsic formulation* of this boundary condition is one that is expressed only by means of the functions $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$, as defined in the previous section, i.e. as the covariant components of the linearized change of metric tensor field and linearized change of curvature tensor field between the surfaces $\boldsymbol{\theta}(\overline{\omega})$ and $(\boldsymbol{\theta} + \boldsymbol{\eta})(\overline{\omega})$.

The next theorem, which is the crucial ingredient of our main result (Theorem 1 in Section 1), provides such an intrinsic formulation of a homogeneous boundary condition of place. Note that the substraction of an *infinitesimal rigid displacement* in part (b) of Theorem 3 below is necessary, since

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) = \gamma_{\alpha\beta}(\boldsymbol{\eta} + \boldsymbol{\xi}) \text{ and } \rho_{\alpha\beta}(\boldsymbol{\eta}) = \rho_{\alpha\beta}(\boldsymbol{\eta} + \boldsymbol{\xi}) \text{ on } \overline{\omega}$$

for all $\boldsymbol{\eta} \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$ and for all vector fields $\boldsymbol{\xi} : \overline{\omega} \to \mathbb{E}^3$ that are of the form

$$\boldsymbol{\xi}(y) = \boldsymbol{h} + \boldsymbol{k} \wedge \boldsymbol{\theta}(y), \ y \in \overline{\omega},$$

for some vectors $\boldsymbol{h} \in \mathbb{E}^3$ and $\boldsymbol{k} \in \mathbb{E}^3$; such vector fields $\boldsymbol{\xi}$ are called "infinitesimal rigid displacements" of the surface $\boldsymbol{\theta}(\overline{\omega})$.

Theorem 3. *Let $\omega \subset \mathbb{R}^2$ be a non-empty connected open set whose boundary is of class $\mathcal{C}^2$, let γ_0 be a non-empty relatively open subset of the boundary of ω, and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ be an immersion.*

(a) *If a vector field $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$ satisfies the boundary conditions*

$$\eta_i = \partial_\alpha \eta_3 = 0 \text{ on } \gamma_0,$$

then the functions $c_{\alpha\beta} := \gamma_{\alpha\beta}(\boldsymbol{\eta}) \in \mathcal{C}^1(\overline{\omega})$ and $r_{\alpha\beta} := \rho_{\alpha\beta}(\boldsymbol{\eta}) \in \mathcal{C}^0(\overline{\omega})$ satisfy the boundary conditions

$$c_{\alpha\beta}\tau^\alpha\tau^\beta = 0 \text{ and } c_{\alpha\beta|\sigma}\tau^\alpha(2\nu^\beta\tau^\sigma - \tau^\beta\nu^\sigma) + \kappa_g c_{\alpha\beta}\nu^\alpha\nu^\beta = 0 \text{ on } \gamma_0,$$
$$r_{\alpha\beta}\tau^\alpha\tau^\beta = 0 \text{ and } r_{\alpha\beta}\tau^\alpha\nu^\beta - c_{\alpha\beta}\nu^\alpha(2\kappa_n\tau^\beta + \tau_g\nu^\beta) = 0 \text{ on } \gamma_0.$$

(b) *If the functions $c_{\alpha\beta} := \gamma_{\alpha\beta}(\boldsymbol{\eta}) \in \mathcal{C}^1(\overline{\omega})$ and $r_{\alpha\beta} := \rho_{\alpha\beta}(\boldsymbol{\eta}) \in \mathcal{C}^0(\overline{\omega})$ associated with a vector field $\boldsymbol{\eta} = \eta_i \boldsymbol{a}^i \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$ satisfy the boundary conditions*

$$c_{\alpha\beta}\tau^\alpha\tau^\beta = 0 \text{ and } c_{\alpha\beta|\sigma}\tau^\alpha(2\nu^\beta\tau^\sigma - \tau^\beta\nu^\sigma) + \kappa_g c_{\alpha\beta}\nu^\alpha\nu^\beta = 0 \text{ on } \gamma_0,$$
$$r_{\alpha\beta}\tau^\alpha\tau^\beta = 0 \text{ and } r_{\alpha\beta}\tau^\alpha\nu^\beta - c_{\alpha\beta}\nu^\alpha(2\kappa_n\tau^\beta + \tau_g\nu^\beta) = 0 \text{ on } \gamma_0,$$

and if γ_0 is connected, then there exists two vectors $\boldsymbol{h} \in \mathbb{E}^3$ and $\boldsymbol{k} \in \mathbb{E}^3$ such that the vector field $\boldsymbol{\eta}^\sharp = \eta_i^\sharp \boldsymbol{a}^i \in \mathcal{C}^2(\overline{\omega}; \mathbb{E}^3)$ defined by

$$\boldsymbol{\eta}^\sharp(y) = \boldsymbol{\eta}(y) - (\boldsymbol{h} + \boldsymbol{k} \wedge \boldsymbol{\theta}(y)) \text{ for all } y \in \overline{\omega},$$

satisfies the boundary conditions

$$\eta_i^\sharp = \partial_\alpha \eta_3^\sharp = 0 \text{ on } \gamma_0.$$

Proof. Theorem 3 is a straightforward consequence of Lemmas 1 and 2, combined with the relation

$$b_\sigma^\alpha \tau^\sigma := a^{\alpha\beta} b_{\beta\sigma} \tau^\sigma = -\tau^\sigma \partial_\sigma \boldsymbol{a}_3 \cdot \boldsymbol{a}^\alpha = -\partial_{\boldsymbol{\tau}} \boldsymbol{a}_3 \cdot (\tau^\alpha \boldsymbol{\tau} + \nu^\alpha \boldsymbol{\nu}) = \tau^\alpha \kappa_n + \nu^\alpha \tau_g. \quad \square$$

It is worthwhile pointing out that Theorem 3 generalizes to a general immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ a previous result of the authors (Theorem 4.1 in [5]; see also Section 4 in [6]) established for the particular immersion corresponding to the displacement-traction problem of a linearly elastic *plate*. To see this, let the immersion $\boldsymbol{\theta}$ in Theorem 3 be defined by

$$\boldsymbol{\theta}(y) = (y_1, y_2, 0) \in \mathbb{E}^3 \text{ for all } y = (y_1, y_2) \in \overline{\omega}.$$

Then

$$c_{\alpha\beta|\sigma} = \partial_\sigma c_{\alpha\beta} \text{ in } \overline{\omega},$$

and

$$\kappa_g = \kappa, \quad \kappa_n = 0, \text{ and } \tau_g = 0, \text{ on } \gamma_0,$$

where $\kappa := \nu^\alpha \tau^\beta \partial_\beta \tau_\alpha$ is the signed curvature of the planar curve γ_0. These relations in turn imply that the boundary conditions satisfied by the functions $c_{\alpha\beta}$ and $r_{\alpha\beta}$ in Theorem 3 are equivalent in this case to the boundary conditions

$$c_{\alpha\beta}\tau^\alpha\tau^\beta = 0 \text{ and } \partial_\sigma c_{\alpha\beta}\tau^\alpha(2\tau^\sigma\nu^\beta - \tau^\beta\nu^\sigma) + \kappa c_{\alpha\beta}\nu^\alpha\nu^\beta \text{ on } \gamma_0,$$
$$r_{\alpha\beta}\tau^\alpha\tau^\beta = 0 \text{ and } r_{\alpha\beta}\tau^\alpha\nu^\beta = 0 \text{ on } \gamma_0,$$

where $\nu^\alpha = \nu_\alpha$ are the Cartesian components of the inner unit normal vector field to the boundary of ω and $\tau^\alpha = \tau_\alpha$ are the Cartesian components of the positively-oriented (i.e. so that $\nu^1 = -\tau^2$ and $\nu^2 = \tau^1$) unit tangent vector field to the boundary of ω, which are precisely the boundary conditions found in Theorem 4.1 of [5].

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