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Partial differential equations

Chemotaxis effect vs. logistic damping on boundedness in the 2-D minimal Keller–Segel model

Effet chimiotaxique contre amortissement logistique pour borner les solutions du modèle de Keller–Segel minimal en dimension 2

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A R T I C L E   I N F O

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A B S T R A C T

We study the chemotaxis effect vs. logistic damping on boundedness for the well-known minimal Keller–Segel model with logistic source:

$$\begin{cases} u_t = \nabla\cdot(\nabla u - \chi u \nabla v) + u - \mu u^2, & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, t > 0 \end{cases}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^2$ with $\chi, \mu > 0$, nonnegative initial data u_0, v_0, and homogeneous Neumann boundary data. It is well known that this model allows only for global and uniform-in-time bounded solutions for any $\chi, \mu > 0$. Here, we carefully employ a simple and new method to regain its boundedness, with particular attention to how upper bounds of solutions qualitatively depend on χ and μ. More, precisely, it is shown that there exists $C = C(u_0, v_0, \Omega) > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C\Big[1 + \frac{1}{\mu} + \chi K(\chi,\mu) N(\chi,\mu)\Big]$$

and

$$\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C\Big[1 + \frac{1}{\mu} + \frac{\chi^{\frac{8}{3}}}{\mu} K^{\frac{8}{3}}(\chi,\mu)\Big] =: CN(\chi,\mu)$$

uniformly on $[0,\infty)$, where

$$K(\chi,\mu) = M(\chi,\mu)E(\chi,\mu), \quad M(\chi,\mu) = 1 + \frac{1}{\mu} + \sqrt{\chi}(1 + \frac{1}{\mu^2})$$

and

$$E(\chi,\mu) = \exp\Big[\frac{\chi C_{GN}^2}{2\min\{1,\frac{2}{\chi}\}}\Big(\frac{4}{\mu}\|u_0\|_{L^1(\Omega)} + \frac{13}{2\mu^2}|\Omega| + \|\nabla v_0\|^2_{L^2(\Omega)}\Big)\Big].$$

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We notice that these upper bounds are increasing in χ, decreasing in μ, and have only one singularity at $\mu = 0$, where the corresponding minimal model (removing the term $u - \mu u^2$ in the first equation) is widely known to possess blow-ups for large initial data.

R É S U M É

Nous étudions l'effet chimiotaxique versus l'amortissement logistique pour borner les solutions du modèle de Keller–Segel minimal bien connu avec source logistique :

$$\begin{cases} u_t = \nabla\cdot(\nabla u - \chi u\nabla v) + u - \mu u^2, & x\in\Omega, t>0, \\ v_t = \Delta v - v + u, & x\in\Omega, t>0 \end{cases}$$

dans un domaine borné, lisse $\Omega\subset\mathbf{R}^2$ avec $\chi,\mu>0$, des données initiales u_0, v_0 positives ou nulles et des données au bord de Neumann homogènes. Il est bien connu que ce modèle n'a que des solutions bornées globales et uniformes en temps, pour tout $\chi,\mu>0$. Nous utilisons ici une méthode nouvelle et simple pour retrouver ces bornes en portant une attention particulière à la dépendance en χ et μ des bornes supérieures des solutions. Plus précisément, nous montrons qu'il existe $C = C(u_0, v_0, \Omega) > 0$ tel que

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C\Big[1 + \frac{1}{\mu} + \chi K(\chi,\mu)N(\chi,\mu)\Big]$$

et

$$\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C\Big[1 + \frac{1}{\mu} + \frac{\chi^{\frac{8}{3}}}{\mu}K^{\frac{8}{3}}(\chi,\mu)\Big] =: CN(\chi,\mu)$$

uniformément sur $[0,\infty[$, où

$$K(\chi,\mu) = M(\chi,\mu)E(\chi,\mu), \quad M(\chi,\mu) = 1 + \frac{1}{\mu} + \sqrt{\chi}(1 + \frac{1}{\mu^2})$$

et

$$E(\chi,\mu) = \exp\Big[\frac{\chi C_{GN}^2}{2\min\{1,\frac{2}{\chi}\}}\Big(\frac{4}{\mu}\|u_0\|_{L^1(\Omega)} + \frac{13}{2\mu^2}|\Omega| + \|\nabla v_0\|^2_{L^2(\Omega)}\Big)\Big].$$

Nous observons que ces bornes supérieures croissent avec χ, décroissent avec μ et n'ont qu'une singularité en $\mu = 0$. Il est bien connu que le modèle minimal correspondant (en ôtant le terme $u - \mu u^2$ dans la première équation) a des solutions qui explosent pour les grandes données initiales.

1. Introduction and main results

In this work, we are concerned with the well-known and extensively explored Keller–Segel minimal chemotaxis model with logistic source:

$$\begin{cases} u_t = \nabla\cdot(\nabla u - \chi u\nabla v) + ru - \mu u^2 & x\in\Omega, t>0, \\ v_t = \Delta v - v + u, & x\in\Omega, t>0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, t>0, \\ u(x,0) = u_0(x) \ge 0, v(x,0) = v_0(x) \ge 0, & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n (n\ge 1)$ is a bounded smooth domain, $r\ge 0$, $\chi,\mu>0$ and u and v respectively denote the density of cells and the concentration of the chemical signal. The chemotactic flux $-\chi u\nabla v$ (defining term in chemotaxis models) models the directed movement that u moves towards the higher concentration of v. This is commonly termed as chemotactic movement, a biological phenomenon whereby biological individuals orient their movement in response to some external signaling substances that attract cells to aggregate.

Without a logistic source, i.e. $r = 0, \mu = 0$, the system (1.1) is known as the classical Keller–Segel minimal model [15], which and whose variants have been widely investigated since 1970. The striking feature of KS-type models is the possibility of blow-up of solutions in a finite/infinite time, which strongly depends on the space dimension. A finite/infinite time blow-up never occurs in 1-D [8,25,37], a critical mass blow-up occurs in 2-D: when the initial mass $\|u_0\|_{L^1} < 4\pi/\chi$, solutions exist globally and converge to a single equilibrium in large time, whereas, when $\|u_0\|_{L^1} > 4\pi/\chi$, there exist solutions blowing up in finite time, cf. [11,3,22,21,26], and even small initial mass can result in blow-ups in $\geq$ 3-D [31,33]. See [1,12] for more surveys on the classical KS model and its variants.

The logistic source was introduced by Mimura and Tsujikawa [20], where they study aggregating patterns based on the chemotaxis, diffusion, and growth of bacteria. First, this additional logistic term apparently destroys the conservation law of mass of the classical KS model. On the other hand, it exerts a certain growth-inhibiting influence on the global existence and boundedness of solutions to the corresponding Keller–Segel models. Indeed, in the case $n = 1, 2$, even arbitrarily small $\mu > 0$ will be enough to prevent blow-ups by guaranteeing all solutions to (1.1) are global-in-time and uniformly bounded [8,25,24,38]. This is even true for a 2-D simplified version of parabolic–elliptic (the second PDE in (1.1) is replaced with $0 = \Delta v - v + u$) chemotaxis system with singular sensitivity [5], whereas, in the case $n \geq 3$, the global existence and boundedness were first obtained for a parabolic–elliptic simplification of (1.1) under $\mu > \frac{(n-2)}{n}\chi$ [30]. Nowadays, this result has been improved to the borderline case $\mu \geq \frac{n-2}{n}\chi$ [10,14,40]. Moreover, with a very slow self-diffusion of cells, the u component can exceed the carrying capacity $\frac{r}{\mu}$ to an arbitrary extent at some intermediate time [18,35]. Coming back to our parabolic mode (1.1), for Ω being convex, Winkler first derived the boundedness and global existence provided that μ is beyond a certain number μ_0 not explicitly known [32]. A further progress in this regard was derived as long as $\mu > \theta_0\chi$ for some implicit positive constant θ_0 in [41]. An explicit lower bound for a 3-D chemotaxis-fluid system with logistic source, when applied to (1.1) with $\chi = 1$, which states that $\mu \geq 23$, is enough to ensure boundedness [28]. This bound μ_0 was further improved by Lin and Mu [19] in 3-D, wherein they replaced the logistic source in (1.1) by the damping term $u - \mu u^r$ with $r \geq 2$ to derive the boundedness under $\mu^{\frac{1}{r-1}} > 20\chi$. Very recently, for a full-parameter version of (1.1), we calculate out the explicit formula for μ_0 in terms of the involving parameters, which states that the explicit condition $\mu > \frac{9}{\sqrt{10}-2}\chi = (7.743416\cdots)\chi$ is sufficient to ensure the boundedness and global existence of solutions to (1.1) in 3-D [39]. Yet, it is a big open challenging problem, whether or not blow-up occurs in (1.1) for small $\mu > 0$, even though the existence of global weak solutions is available in convex 3-D domains for $\mu > 0$ [17]. Under further conditions on χ, μ or r, convergence of bounded solutions to the constant equilibrium $(\frac{r}{\mu}, \frac{r}{\mu})$ as well as its convergence rates are available [7,19,34,39]. It also needs to be mentioned that, for certain choices of the parameters, the solutions to (1.1) even may oscillate drastically in time, as numerically illustrated in [9], and that the solutions may undergo transient growth phenomena, as demonstrated in [18,35,36].

In contrast to the rich knowledge on boundedness, convergence and other dynamical properties for (1.1) and its variants, understanding the qualitative or quantitative properties, even of bounded solutions to chemotaxis problems, seems much less developed. In this direction, a work was considered by Tao and Winkler in [29] to show the mass persistence phenomenon for (1.1), i.e. for any supposedly given global classical and bounded nontrivial solution (u, v) to (1.1), there is $m_* > 0$ such that $\|u(t)\|_{L^1} \geq m_*$ for all $t > 0$. To our best knowledge, there seems no work on how boundedness or upper bounds of solutions to (1.1) depend on the system parameters, say, χ, μ or r. In this paper, we aim as a first step to study chemotaxis effect vs. logistic damping on boundedness for the minimal chemotaxis-logistic model (1.1) in 2-D. We do so partially because all solutions in 2-D are global and bounded by [24,38]. We are particularly interested in the dependence of upper bounds of solutions to (1.1) on the most interesting parameters χ and μ. We hope that this qualitative boundedness will stimulate new research directions, especially, the same problem in higher dimensions. Since the constant r does not affect us much in our derivation, we include it here. With this goal in mind, our main qualitative boundedness result reads as follows:

Theorem 1.1. *Let $\chi, \mu > 0, r \geq 0$, $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary and let the initial data $u_0 \in C(\bar{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ be nonnegative. Then the Keller–Segel chemotaxis-logistic model (1.1) has a unique global classical nonnegative solution (u, v) on $\Omega \times [0, \infty)$ for which*

$$\|u(t)\|_{L^\infty(\Omega)} \leq C\Big[1 + \frac{1}{\mu} + \chi K(\chi, \mu)N(\chi, \mu)\Big] =: CL(\chi, \mu) \tag{1.2}$$

and

$$\|v(t)\|_{W^{1,\infty}(\Omega)} \leq C\Big[1 + \frac{1}{\mu} + \frac{\chi^{\frac{8}{3}}}{\mu} K^{\frac{8}{3}}(\chi, \mu)\Big] =: CN(\chi, \mu) \tag{1.3}$$

uniformly on $[0, \infty)$ and for some C depending on u_0, v_0, r and $|\Omega|$, where

$$K(\chi, \mu) = M(\chi, \mu)E(\chi, \mu), \quad M(\chi, \mu) = 1 + \frac{1}{\mu} + \sqrt{\chi}(1 + \frac{1}{\mu^2}) \tag{1.4}$$

and

$$E(\chi,\mu)=e^{\frac{\chi C_{GN}^2}{2\min\{1,\frac{2}{\chi}\}}\left[\frac{(r+3)}{\mu}\|u_0\|_{L^1(\Omega)}+\frac{(r+1)^3}{4\mu^2}|\Omega|+\|\nabla v_0\|^2_{L^2(\Omega)}+\frac{(r+2)^2}{2\mu^2}|\Omega|\right]}, \tag{1.5}$$

where C_{GN} relates to the Gagliardo–Nirenberg constant in Lemma 2.1 as follows:

$$C_{\mathrm{GN}}:=2C^2_{\mathrm{GN}}(4,2,1,\Omega). \tag{1.6}$$

Up to a scaling constant, Theorem 1.1 provides explicit upper bounds for $\|u(t)\|_{L^\infty}$ and $\|v(t)\|_{W^{1,\infty}}$ in terms of the most interesting parameters χ and μ in 2-D.

The crucial point of the proof of Theorem 1.1 consists in deriving a uniform-in-time estimate for $\|u(t)\|_{L^2}$ rather than $\|(u+1)\ln(u+1)\|_{L^1}$ as in [24,38]; indeed, we obtain an explicit uniform-in-time bound for $\|u(t)\|^2_{L^2}$ as follows:

$$\begin{aligned}\|u(t)\|^2_{L^2}\le E(\chi,\mu)\Big\{&\|u_0\|^2_{L^2}+\frac{8\min\{1,\frac{2}{\chi}\}}{C^2_{\mathrm{GN}}}+\frac{(r+1)}{\mu}\|u_0\|_{L^1}\\&+\frac{3\chi C^2_{\mathrm{GN}}}{4}\Big[\|u_0\|_{L^1}+\frac{(r+1)^2}{4\mu}|\Omega|\Big]^4+\frac{(r+1)^3}{4\mu^2}|\Omega|+\frac{8r^3}{9\mu^2}|\Omega|\Big\},\end{aligned} \tag{1.7}$$

where E is defined by (1.5) and C_{GN} is defined by (1.6). After obtaining precise bounds on $\|u\|_{L^1}$, $\|\nabla v\|_{L^2}$ and space–time integrals on u^2 and $|\Delta v|^2$, cf. Lemmas 2.3 and 3.1, we can use the 2-D Gagliardo–Nirenberg interpolation inequality to derive a differential inequality for $y(t)=\|u(t)\|^2_{L^2}+a$ of the form:

$$y'(t)\le ky(t)z(t)+b,\quad z(t)=\|\Delta v(t)\|^2_{L^2}$$

for some $a,k,b>0$ and then solving this ODI successively and using the gained space–time bounds, we achieve the desired estimate (1.7). This is inspired by the ideas presented in [27, Lemma 3.4]. Thanks to the $L^{\frac{n}{2}+}$-boundedness criterion in [1,38], the uniform-in-time bound for $\|u(t)\|_{L^2}$ indeed implies the global existence and boundedness, while, to dig out the dependence of boundedness on χ and μ, we first use the established L^2-estimate of u together with a widely used 'reciprocal' lemma obtained from the v-equation, cf. Lemma 3.4 to bound $\|\nabla v\|_{L^q}$ for any $q\in(1,\infty)$, and then, we test the u-equation in (1.1) by u^2 to derive the L^3-estimate of u, and finally, we apply the variation-of-constants formula for u and v and use the well-known smoothing L^p–L^q type estimates for the Neumann heat semigroup in Ω, cf. [2,31] to conclude about the respective bounds for (u,v) in (1.2) and (1.3).

Remark 1.2. *From (1.2), (1.3), (1.4) and (1.5), one can see that L, M, N, K, E defined on $[0,\infty)\times(0,\infty)$ are decreasing in μ, increasing in χ and have only one singularity at $\mu=0$. Therefore, our obtained bounds for $\|u(t)\|_{L^\infty}$ and $\|v(t)\|_{W^{1,\infty}}$ enjoy these properties. It is worthwhile to observe that, when $\mu=0$, even $r=0$, the corresponding KS minimal model possesses blow-ups for large initial data [11,22,21,26], illustrating the reasonableness of adding a logistic source to the KS minimal model to prevent blow-up.*

2. Preliminaries

For convenience of reference, we only state the well-known 2-dimensional Gagliardo–Nirenberg interpolation inequality for direct use in the sequel.

Lemma 2.1. *(The 2-D Gagliardo–Nirenberg interpolation inequality [4,23]) Let $\Omega\subset\mathbb{R}^2$ be a bounded and smooth domain, $p\ge1$, $q\in(0,p)$ and $s>0$. Then, there exists $C_{\mathrm{GN}}(p,q,s,\Omega)>0$ depending on p,q,s and Ω such that*

$$\|w\|_{L^p(\Omega)}\le C_{\mathrm{GN}}(p,q,s,\Omega)\Big(\|\nabla w\|^{1-\frac{q}{p}}_{L^2(\Omega)}\|w\|^{\frac{q}{p}}_{L^q(\Omega)}+\|w\|_{L^s(\Omega)}\Big)$$

for all $w\in H^1(\Omega)\cap L^q(\Omega)$.

The basic result on local existence, uniqueness, and extendibility of classical solutions to the minimal KS system (1.1) can be found in [32, Lemma 1.1].

Lemma 2.2. *Let $\chi,\mu>0$, $r\ge0$, $\Omega\subset\mathbb{R}^n(n\ge1)$ be a bounded smooth domain and let the initial data $u_0\in C(\bar\Omega)$ and $v_0\in W^{1,\infty}(\Omega)$ be nonnegative. Then there is a unique, nonnegative, and classical maximal solution (u,v) to the IBVP (1.1) on some maximal interval $[0,T_{\mathrm{m}})$ with $0<T_{\mathrm{m}}\le\infty$ such that*

$$u\in C(\bar\Omega\times[0,T_{\mathrm{m}}))\cap C^{2,1}(\bar\Omega\times(0,T_{\mathrm{m}})),$$

$$v\in C(\bar\Omega\times[0,T_{\mathrm{m}}))\cap C^{2,1}(\bar\Omega\times(0,T_{\mathrm{m}}))\cap L^\infty_{\mathrm{loc}}([0,T_{\mathrm{m}});W^{1,s}(\Omega))$$

for any $s>n$. In particular, if $T_{\mathrm{m}}<\infty$, then

$$\|u(t)\|_{L^\infty}+\|v(t)\|_{W^{1,s}}\to\infty\qquad \text{as } t\to T^-_{\mathrm{m}}.$$

Lemma 2.3. *For any $t \in [0, T_{\mathrm{m}})$, the nonnegative solution (u, v) to (1.1) satisfies*

$$\|u\|_{L^1} \le \|u_0\|_{L^1} + \frac{(r+1)^2}{4\mu}|\Omega| =: k_1 \tag{2.1}$$

and

$$\|\nabla v\|_{L^2}^2 \le \frac{2}{\mu}\Big[\|u_0\|_{L^1} + \frac{\mu}{2}\|\nabla v_0\|_{L^2}^2 + \frac{(r+2)^2}{4\mu}|\Omega|\Big] =: k_2. \tag{2.2}$$

Proof. The nonnegativity of u, v follows from the maximum principle. Then, integrating the u-equation and using the homogeneous Neumann boundary conditions, we derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u = r\int_\Omega u - \mu\int_\Omega u^2 \le -\int_\Omega u + \frac{(r+1)^2}{4\mu}|\Omega|, \tag{2.3}$$

which yields the L^1-bound for u in (2.1).

Then, testing the v-equation in (1.1) against $-\Delta v$ and integrating by parts and using Young's inequality with epsilon, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega |\Delta v|^2 \le -\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega u^2, \tag{2.4}$$

which together with the reasoning leading to (2.3) gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u + \frac{\mu}{2}|\nabla v|^2) + 2\int_\Omega (u + \frac{\mu}{2}|\nabla v|^2) \le \frac{(r+2)^2}{2\mu}|\Omega|.$$

Solving this standard Gronwall inequality shows

$$\|u\|_{L^1} + \frac{\mu}{2}\|\nabla v\|_{L^2}^2 \le \|u_0\|_{L^1} + \frac{\mu}{2}\|\nabla v_0\|_{L^2}^2 + \frac{(r+2)^2}{4\mu}|\Omega|,$$

which directly leads to (2.2). $\square$

3. Chemotaxis vs. logistic on boundedness in 2-D

In 2-D, it is well known that any presence of a logistic source will be sufficient to suppress blow-up by ensuring that all solutions to (1.1) are global-in-time and uniformly bounded [24,38]. In this section, we carefully scrutinize a different method motivated from [27, Lemma 3.4] to regain its boundedness, with particular focus on the qualitative dependence of upper bounds of solutions to (1.1) on χ and μ, and thus accomplish the proof of Theorem 1.1.

Lemma 3.1. *Given $\tau \in (0, T_{\mathrm{m}})$, then, for any $t \in [0, T_{\mathrm{m}} - \tau)$, the solution (u, v) of the KS model (1.1) fulfills*

$$\int_t^{t+\tau}\int_\Omega u^2 \le \frac{(r+1)k_1}{\mu}\max\{\tau, 1\} =: k_3 \max\{\tau, 1\}, \tag{3.1}$$

$$\int_t^{t+\tau}\int_\Omega |\nabla v|^2 \le k_2 \max\{\tau, 1\} \tag{3.2}$$

and

$$\int_t^{t+\tau}\int_\Omega |\Delta v|^2 \le (k_3 + k_2)\max\{\tau, 1\} =: k_4 \max\{\tau, 1\}. \tag{3.3}$$

Proof. For any $t \in [0, T_{\mathrm{m}} - \tau)$, integrating the u-equation in (1.1) over $\Omega \times (t, t+\tau)$ and using Lemma 2.3, we deduce that

$$\mu\int_t^{t+\tau}\int_\Omega u^2 \le r\int_t^{t+\tau}\int_\Omega u + \int_\Omega u \le (r+1)k_1 \max\{\tau, 1\},$$

yielding the desired inequality (3.1).

The estimate (3.2) follows directly from (2.2). Next, an integration of (2.4) over $(t, t+\tau)$ and the use of (2.2) and (3.1) telescope

$$\int_t^{t+\tau}\int_\Omega |\Delta v|^2(s) \le \int_t^{t+\tau}\int_\Omega u^2(s) + \int_\Omega |\nabla v|^2(t) \le (k_3+k_2)\max\{\tau, 1\},$$

which is exactly (3.3). □

Here, with Lemma 3.1 at hand, in 2-D setting, we can make use of the Gagliardo–Nirenberg interpolation inequality in Lemma 2.1 to derive an ODI satisfied by $\|u\|_{L^2}^2$, which enables us to deduce an estimate for $\|u\|_{L^2}$. This is the key point for us to derive qualitative bounds for $\|u\|_{L^\infty}$ and $\|v\|_{W^{1,\infty}}$ later on.

Lemma 3.2. *Given $\tau \in (0, T_{\mathrm{m}})$, then the u-component of the solution (u, v) to the KS minimal model (1.1) satisfies the explicit uniform-in-time bound:*

$$\begin{aligned}\|u(t)\|_{L^2}^2 \le &\Big\{\|u_0\|_{L^2}^2 + \frac{8\min\{1,\frac{2}{\chi}\}}{C_{\mathrm{GN}}^2} + \frac{3\chi C_{\mathrm{GN}}^2}{4}\Big[\|u_0\|_{L^1} + \frac{(r+1)^2}{4\mu}|\Omega|\Big]^4 \\ &+ \frac{(r+1)}{\mu}\|u_0\|_{L^1} + \frac{(r+1)^3}{4\mu^2}|\Omega| + \frac{8r^3}{9\mu^2}|\Omega|\Big\}\max\{1,\tau,\frac{1}{\tau}\} \\ &\times e^{\frac{\chi C_{\mathrm{GN}}^2}{2\min\{1,\frac{2}{\chi}\}}\Big[\frac{(r+3)}{\mu}\|u_0\|_{L^1} + \frac{(r+1)^3}{4\mu^2}|\Omega| + \|\nabla v_0\|_{L^2}^2 + \frac{(r+2)^2}{2\mu^2}|\Omega|\Big]\max\{1,\tau\}},\end{aligned} \tag{3.4}$$

and so a uniform estimate for $\|u\|_{L^2}$ in terms of χ and μ follows:

$$\|u(t)\|_{L^2} \le C\Big[1 + \frac{1}{\mu} + \sqrt{\chi}(1+\frac{1}{\mu^2})\Big]\max\{\sqrt{\tau}, \frac{1}{\sqrt{\tau}}\}E^{\max\{1,\tau\}}(\chi,\mu) \tag{3.5}$$

for all $t \in (0, T_{\mathrm{m}})$ and for some $C = C(u_0, r, |\Omega|)$, where E is defined by (1.5) and C_{GN} is defined by (1.6).

Proof. We test the u-equation in (1.1) by u and integrate by parts to deduce from Hölder's inequality that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2 + \int_\Omega |\nabla u|^2 &= \frac{\chi}{2}\int_\Omega \nabla(u^2)\nabla v + \int_\Omega u^2(r-\mu u) \\ &= -\frac{\chi}{2}\int_\Omega u^2\Delta v + \int_\Omega u^2(r-\mu u) \\ &\le \frac{\chi}{2}\Big(\int_\Omega u^4\Big)^{\frac12}\Big(\int_\Omega |\Delta v|^2\Big)^{\frac12} + \int_\Omega u^2(r-\mu u).\end{aligned} \tag{3.6}$$

Applying the 2-D GN interpolation inequality in Lemma 2.1 and the boundedness of $\|u\|_{L^1}$ in (2.1) and recalling the definition of C_{GN} in (1.6), we estimate

$$\begin{aligned}\Big(\int_\Omega u^4\Big)^{\frac12} = \|u\|_{L^4}^2 &\le \Big[C_{\mathrm{GN}}(4,2,1,\Omega)\Big(\|\nabla u\|_{L^2}^{\frac12}\|u\|_{L^2}^{\frac12} + \|u\|_{L^1}\Big)\Big]^2 \\ &\le 2C_{\mathrm{GN}}^2(4,2,1,\Omega)\Big(\|\nabla u\|_{L^2}\|u\|_{L^2} + \|u\|_{L^1}^2\Big) \\ &\le 2C_{\mathrm{GN}}^2(4,2,1,\Omega)\Big(\|\nabla u\|_{L^2}\|u\|_{L^2} + k_1^2\Big) \\ &= C_{\mathrm{GN}}\Big(\|\nabla u\|_{L^2}\|u\|_{L^2} + k_1^2\Big).\end{aligned}$$

Hence, upon twice uses of Young's inequality with epsilon, for $\epsilon > 0$ to be fixed as in (3.8), it follows that

$$\begin{aligned}&\Big(\int_\Omega u^4\Big)^{\frac{1}{2}}\Big(\int_\Omega |\Delta v|^2\Big)^{\frac{1}{2}}\\&\le C_{\rm GN}\|\nabla u\|_{L^2}\|u\|_{L^2}\|\Delta v\|_{L^2}+k_1^2C_{\rm GN}\|\Delta v\|_{L^2}\\&\le \epsilon\|\nabla u\|_{L^2}^2+\frac{C_{\rm GN}^2}{4\epsilon}\|u\|_{L^2}^2\|\Delta v\|_{L^2}^2+\|\Delta v\|_{L^2}^2+\frac{k_1^4C_{\rm GN}^2}{4}.\end{aligned}\tag{3.7}$$

Inserting (3.7) into (3.6), we conclude that

$$\begin{aligned}&\frac{1}{2}\frac{\rm d}{{\rm d}t}\int_\Omega u^2+\int_\Omega|\nabla u|^2\\&\le\frac{\chi}{2}\Big[\epsilon\|\nabla u\|_{L^2}^2+\frac{C_{\rm GN}^2}{4\epsilon}\|u\|_{L^2}^2\|\Delta v\|_{L^2}^2+\|\Delta v\|_{L^2}^2+\frac{k_1^4C_{\rm GN}^2}{4}\Big]\\&\quad+\int_\Omega u^2(r-\mu u),\end{aligned}$$

from which, upon setting

$$\epsilon:=\min\{1,\frac{2}{\chi}\},\tag{3.8}$$

we deduce

$$\begin{aligned}\frac{\rm d}{{\rm d}t}\int_\Omega u^2&\le\frac{\chi C_{\rm GN}^2}{4\epsilon}\Big(\|u\|_{L^2}^2+\frac{4\epsilon}{C_{\rm GN}^2}\Big)\|\Delta v\|_{L^2}^2+\frac{\chi k_1^4C_{\rm GN}^2}{4}+\frac{8r^3}{27\mu^2}|\Omega|\\&=:k_5y(t)z(t)+k_6,\end{aligned}\tag{3.9}$$

where $k_5=\frac{\chi C_{\rm GN}^2}{4\epsilon}$, $k_6=\frac{\chi k_1^4C_{\rm GN}^2}{4}+\frac{8r^3}{27\mu^2}|\Omega|$ and

$$y(t)=\|u\|_{L^2}^2+\frac{4\epsilon}{C_{\rm GN}^2},\quad z(t)=\|\Delta v\|_{L^2}^2.\tag{3.10}$$

For any $s\ge0$ and any $t\ge s$, multiplying the integrating factor $\exp(-k_5\int_s^t z(\lambda){\rm d}\lambda)$ on both sides of (3.9), we deduce that

$$y(t)\le y(s)e^{k_5\int_s^t z(\sigma){\rm d}\sigma}+k_6\int_s^t e^{k_5\int_\xi^t z(\sigma){\rm d}\sigma}{\rm d}\xi,\quad\forall t\in[s,\infty)\cap[0,T_{\rm m}).\tag{3.11}$$

In view of (3.1) and (3.3) in Lemma 3.1 and the mean value theorem, one infers from the definitions of y and z in (3.10) that

$$y(s_i)=\frac{1}{\tau}\int_{i\tau}^{(i+1)\tau}y(s){\rm d}s\le(k_3+\frac{4\epsilon}{C_{\rm GN}^2})\max\{1,\frac{1}{\tau}\}=:k_7\max\{1,\frac{1}{\tau}\}\tag{3.12}$$

and

$$\int_{i\tau}^{(i+1)\tau}z(s){\rm d}s\le k_4\max\{\tau,1\}\tag{3.13}$$

for some $s_i\in[i\tau,(i+1)\tau]$ and any nonnegative integers $i<\frac{T_{\rm m}}{\tau}-1$.

First, for $t\in[0,\tau]$, we set $s=0$ in (3.11) and $i=0$ in (3.13) to infer

$$\begin{aligned}y(t)&\le y(0){\rm e}^{k_5\int_0^\tau z(\sigma){\rm d}\sigma}+k_6\int_0^\tau{\rm e}^{k_5\int_0^\tau z(\sigma){\rm d}\sigma}{\rm d}\xi\\&\le(y(0)+k_6)\max\{\tau,1\}{\rm e}^{k_5k_4\max\{\tau,1\}}.\end{aligned}\tag{3.14}$$

Next, for $t\in[\tau,2\tau]$, we will always assume that $t<T_{\rm m}$, we put $s=s_0\in[0,\tau]$ in (3.11) to deduce from (3.12) and (3.13) that

$$
\begin{aligned}
y(t) &\le y(s_0)\mathrm{e}^{k_5\int_{s_0}^t z(\sigma)\mathrm{d}\sigma} + k_6\int_{s_0}^{t} \mathrm{e}^{k_5\int_{s_0}^t z(\sigma)\mathrm{d}\sigma}\mathrm{d}\xi \\
&\le y(s_0)\mathrm{e}^{k_5\int_{0}^{2\tau} z(\sigma)\mathrm{d}\sigma} + k_6\int_{0}^{2\tau} \mathrm{e}^{k_5\int_{0}^{2\tau} z(\sigma)\mathrm{d}\sigma}\mathrm{d}\xi \\
&\le (k_7+2k_6)\max\{1,\tau,\frac{1}{\tau}\}\mathrm{e}^{2k_5k_4\max\{\tau,1\}}.
\end{aligned}
\tag{3.15}
$$

In general, for any $t\in(\tau,T_{\mathrm{m}})$, one first chooses $i\ge 0$ such that $t\in[(i+1)\tau,(i+2)\tau]$ and set $s=s_i\in[i\tau,(i+1)\tau]$ in (3.11), and then infers from (3.12) and (3.13) that

$$
\begin{aligned}
y(t) &\le y(s_i)\,\mathrm{e}^{k_5\int_{s_i}^t z(\sigma)\mathrm{d}\sigma} + k_6\int_{s_i}^{t} \mathrm{e}^{k_5\int_{s_i}^t z(\sigma)\mathrm{d}\sigma}\mathrm{d}\xi \\
&\le y(s_i)\,\mathrm{e}^{k_5\int_{i\tau}^{(i+2)\tau} z(\sigma)\mathrm{d}\sigma} + k_6\int_{i\tau}^{(i+2)\tau} \mathrm{e}^{k_5\int_{i\tau}^{(i+2)\tau} z(\sigma)\mathrm{d}\sigma}\mathrm{d}\xi \\
&\le (k_7+2k_6)\max\{1,\tau,\frac{1}{\tau}\}\,\mathrm{e}^{2k_5k_4\max\{\tau,1\}}.
\end{aligned}
\tag{3.16}
$$

Recalling from the definition of $y(t)$ in (3.10), we then conclude from (3.14), (3.15), and (3.16) the uniform L^2-estimate of u:

$$
\begin{aligned}
\|u\|_{L^2}^2+\frac{4\epsilon}{C_{\mathrm{GN}}^2} &\le (y(0)+k_7+3k_6)\max\{1,\tau,\frac{1}{\tau}\}\mathrm{e}^{2k_5k_4\max\{\tau,1\}} \\
&=\Big(\|u_0\|_{L^2}^2+\frac{8\min\{1,\frac{2}{\chi}\}}{C_{\mathrm{GN}}^2}+k_3+\frac{3\chi k_1^4C_{\mathrm{GN}}^2}{4}+\frac{8r^3}{9\mu^2}|\Omega|\Big) \\
&\quad\times\max\{1,\tau,\frac{1}{\tau}\}\mathrm{e}^{\frac{\chi C_{\mathrm{GN}}^2}{2\min\{1,\frac{2}{\chi}\}}(k_3+k_2)\max\{1,\tau\}},
\end{aligned}
$$

where we have substituted the definitions of k_4, k_5, k_6, k_7, and ϵ in (3.3), (3.9), (3.12) and (3.8). In the above inequality, a further substitution of k_1, k_2, k_3 as defined in (2.1), (2.2), and (3.1) yields the desired L^2-estimate of u in (3.4). □

Remark 3.3. *Another way to view the uniform L^2-norm of u could be arguing as follows: assume that $T_{\mathrm{m}}<\infty$. Then, for any given large natural number $N\gg 1$, we set $\tau=\frac{T_{\mathrm{m}}}{N}$ so that $N\tau=T_{\mathrm{m}}$. Then, arguing as above, we can obtain that $\|u(t)\|_{L^2}$ is uniformly bounded in $(0,T_{\mathrm{m}})$, which violates the $L^{\frac{n}{2}+}$-criterion in [1,38] with $n=2$. Hence, $T_{\mathrm{m}}=\infty$ and $\|u(t)\|_{L^\infty}$ is uniformly bounded on $(0,\infty)$. Furthermore, this energy method offers a simple proof for global-in-time boundedness in 2-D setting compared to existing literatures, cf. [24,38].*

In the sequel, we shall seek how the $(L^\infty,W^{1,\infty})$-bound of (u,v) depends on χ and μ. Since the solution (u,v) is global in time by Remark 3.3, we will set $\tau=1$ to simplify our calculations. To get higher-order regularity of u, we control the $W^{1,q}$-bounds of v in terms of L^p-norms of u. For this purpose, we shall utilize the widely known smoothing L^p–L^q properties of the Neumann heat semigroup $\{\mathrm{e}^{t\Delta}\}_{t\ge 0}$ in Ω, see, e.g., [31,2] for instance. Applying these heat Neumann semigroup estimates to the v-equation in (1.1), we have the following widely known 'reciprocal' lemma, cf. [13, Lemma 4.1], [16, Lemma 1], [38, Lemma 3.5] for instance.

Lemma 3.4. *For $p\ge 1$, let*

$$
\begin{cases} q\in[1,\frac{np}{n-p}), & \text{if } p\le n,\\ q\in[1,\infty], & \text{if } p>n. \end{cases}
\tag{3.17}
$$

Then there exists $C=C(p,q,v_0,\Omega)>0$ such that the unique global-in-time classical solution (u,v) to (1.1) satisfies

$$
\|v(t)\|_{W^{1,q}}\le C(1+\sup_{s\in(0,t)}\|u(s)\|_{L^p}).
\tag{3.18}
$$

Proof. Indeed, the variation-of-constants formula applied to the v in (1.1) gives

$$v(t) = \mathrm{e}^{t(\Delta-1)}v_0 + \int_0^t \mathrm{e}^{(t-s)(\Delta-1)}u(s)\mathrm{d}s. \tag{3.19}$$

Now, the well-known L^p–L^q estimate for the heat Neumann semigroup guarantees, cf. [2,31], that, for $1 \le q \le p \le \infty$, one can find $k_8, k_9, k_{10} > 0$ such that

$$\|\mathrm{e}^{t\Delta}w\|_{L^p} \le k_8\left(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right)\|w\|_{L^q}, \quad \forall t > 0 \tag{3.20}$$

and

$$\|\nabla \mathrm{e}^{t\Delta}w\|_{L^q} \le k_9\|\nabla w\|_{L^\infty}, \quad \forall t > 0 \tag{3.21}$$

as well as

$$\|\nabla \mathrm{e}^{t\Delta}w\|_{L^p} \le k_{10}\left(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right)\mathrm{e}^{-\lambda_1 t}\|w\|_{L^q}, \quad \forall t > 0. \tag{3.22}$$

Here, $\lambda_1 (> 0)$ is the first nonzero eigenvalue of $-\Delta$ under homogeneous boundary conditions. Then applying the properties (3.20), (3.21) and (3.22) to (3.19) and the exponent relation p, q in (3.17), one can easily infer (3.18). □

Lemma 3.5. *The u-component of the unique global-in-time classical solution to the KS minimal chemotaxis-logistic model (1.1) satisfies the uniform estimate*

$$\|u(t)\|_{L^3} \le C\Big[1 + \frac{1}{\mu} + \frac{\chi^{\frac{8}{3}}}{\mu}M^{\frac{8}{3}}(\chi,\mu)E^{\frac{8}{3}}(\chi,\mu)\Big], \tag{3.23}$$

for all $t \in (0,\infty)$ and for some C depending on u_0, v_0, r and $|\Omega|$, where M and E are defined by (1.4) and (1.5), respectively.

Proof. Based on the uniform L^2-bound of u in (3.5) with $\tau = 1$, it follows from Lemma 3.4 with $n = 2$ and $p = 2$, for any $1 < q < \infty$, that

$$\|\nabla v(t)\|_{L^q} \le CM(\chi,\mu)E(\chi,\mu). \tag{3.24}$$

Multiplying the u-equation in (1.1) by u^2, integrating by parts and using Young's inequality with epsilon, we arrive at

$$\begin{aligned}
&\frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^3 + 2\int_\Omega u|\nabla u|^2 \\
&= 2\chi\int_\Omega u^2\nabla u\nabla v + \int_\Omega (ru^3 - \mu u^4) \\
&\le 2\int_\Omega u|\nabla u|^2 + \frac{\chi^2}{2}\int_\Omega u^3|\nabla v|^2 + \int_\Omega (ru^3 - \mu u^4) \\
&\le 2\int_\Omega u|\nabla u|^2 + \frac{\mu}{2}\int_\Omega u^4 + \frac{3^3\chi^8}{2\cdot 4^4\mu^3}\int_\Omega |\nabla v|^8 + \int_\Omega (ru^3 - \mu u^4),
\end{aligned}$$

which along with the algebraic fact $ru^3 - \frac{\mu}{2}u^4 \le -\frac{1}{3}u^3 + \frac{3^3(r+\frac{1}{3})^4}{2^5\mu^3}$ shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^3 + \int_\Omega u^3 \le \frac{3^4\chi^8}{2\cdot 4^4\mu^3}\|\nabla v\|_{L^8}^8 + \frac{3^4(r+\frac{1}{3})^4}{2^5\mu^3}|\Omega|.$$

Solving this standard Gronwall differential inequality, we directly have

$$\|u\|_{L^3}^3 \le \|u_0\|_{L^3}^3 + \frac{3^4\chi^8}{2\cdot 4^4\mu^3}\sup_{t\in(0,\infty)}\|\nabla v(t)\|_{L^8}^8 + \frac{3^4(r+\frac{1}{3})^4}{2^5\mu^3}|\Omega|,$$

which, together with (3.24) with $q = 8$, yields the desired estimate (3.23). □

Proof of Theorem 1.1. The $W^{1,\infty}$-bound of v in (1.3) follows directly from the uniform L^3-estimate of u in (3.23) and Lemma 3.4 with $(n,p,q)=(2,3,\infty)$.

For the L^∞-bound of u, we first apply the variation-of-constants formula to the u-equation in (1.1) to represent u as

$$\begin{aligned} u(t) &= e^{t(\Delta-1)}u_0 - \chi\int_0^t e^{(t-s)(\Delta-1)}\nabla\cdot((u\nabla v)(s))ds \\ &\quad + \int_0^t e^{(t-s)(\Delta-1)}[(r+1)u(s)-\mu u^2(s)]ds \\ &=: u_1(t)+u_2(t)+u_3(t). \end{aligned} \tag{3.25}$$

Because u is nonnegative and smooth, we thus have

$$\|u(t)\|_{L^\infty} = \sup_{x\in\Omega} u(x,t) \le \sup_{x\in\Omega} u_1(x,t) + \sup_{x\in\Omega} u_2(x,t) + \sup_{x\in\Omega} u_3(x,t).$$

Thanks to the maximum principle, the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ is order preserving. This allows us to control u_1 and u_3 as follows:

$$\|u_1(t)\|_{L^\infty} = \|e^{t(\Delta-1)}u_0\|_{L^\infty} \le e^{-t}\|u_0\|_{L^\infty} \le \|u_0\|_{L^\infty}. \tag{3.26}$$

As well as

$$\begin{aligned} u_3(t) &= \int_0^t e^{-(t-s)}e^{(t-s)\Delta}[(r+1)u(s)-\mu u^2(s)]ds \\ &\le \int_0^t e^{-(t-s)}e^{(t-s)\Delta}\frac{(r+1)^2}{4\mu}ds \le \frac{(r+1)^2}{4\mu}. \end{aligned} \tag{3.27}$$

To estimate u_2, we recall one more property of the Neumann heat semigroup $e^{t\Delta}$, cf. [2,6]: for any $1<q\le p\le\infty$, there exists $k_{11}>0$ such that

$$\|e^{t\Delta}\nabla\cdot w\|_{L^p} \le k_{11}\left(1+t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right)e^{-\lambda_1 t}\|w\|_{L^q}, \forall t>0,\ w\in(W^{1,p})^n. \tag{3.28}$$

Using the definition of u_2 in (3.25), (3.28) with $n=2$ and Hölder's interpolation inequality, we deduce that

$$\begin{aligned} \|u_2(t)\|_{L^\infty} &\le \chi\int_0^t \|e^{(t-s)(\Delta-1)}\nabla\cdot(u(s)\nabla v(s))\|_{L^\infty}ds \\ &\le k_{11}\chi\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{2}{5}})e^{-(\lambda_1+1)(t-s)}\|u(s)\nabla v(s)\|_{L^{\frac{5}{2}}}ds \\ &\le k_{11}\chi\int_0^t (1+(t-s)^{-\frac{1}{2}-\frac{2}{5}})e^{-(\lambda_1+1)(t-s)}\|u(s)\|_{L^3}\|\nabla v(s)\|_{L^{15}}ds \\ &\le k_{11}\chi\sup_{s\in(0,\infty)}[\|u(s)\|_{L^3}\|\nabla v(s)\|_{L^{15}}]\int_0^\infty (1+\sigma^{-\frac{9}{10}})e^{-(\lambda_1+1)\sigma}d\sigma \\ &=: k_{12}\chi\sup_{s\in(0,\infty)}\|u(s)\|_{L^3}\sup_{s\in(0,\infty)}\|\nabla v(s)\|_{L^{15}}. \end{aligned}$$

This in conjunction with (3.23) and (3.24) with $q=15$ gives the estimate of u_2:

$$\|u_2(t)\|_{L^\infty} \le C\chi M(\chi,\mu)E(\chi,\mu)\Big[1+\frac{1}{\mu}+\frac{\chi^{\frac{8}{3}}}{\mu}M^{\frac{8}{3}}(\chi,\mu)E^{\frac{8}{3}}(\chi,\mu)\Big]. \tag{3.29}$$

A substitution of (3.26), (3.27), and (3.29) into (3.25) yields the desired uniform bound for $\|u(t)\|_{L^\infty}$, as stated in (1.2). □

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