C. R. Acad. Sci. Paris, Ser. I 356 (2018) 843–845

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Group theory

Breaking points in centralizer lattices

Points de rupture des treillis de centralisateurs

Marius Tărnăuceanu

Faculty of Mathematics, "Al.I. Cuza" University, Iași, Romania

ARTICLE INFO

Article history:
Received 21 January 2018
Accepted after revision 26 June 2018
Available online 30 June 2018

Presented by the Editorial Board

ABSTRACT

In this note, we prove that the centralizer lattice $\mathfrak{C}(G)$ of a group G cannot be written as a union of two proper intervals. In particular, it follows that $\mathfrak{C}(G)$ has no breaking point. As an application, we show that the generalized quaternion 2-groups are not capable.

RÉSUMÉ

Dans cette note, nous montrons que le treillis des centralisateurs $\mathfrak{C}(G)$ d'un groupe G ne peut pas être écrit comme une union de deux intervalles appropriés. En particulier, il s'ensuit que $\mathfrak{C}(G)$ n'a pas de point de rupture. Comme application, nous montrons que les 2-groupes de quaternions généralisés ne sont pas capables.

1. Introduction

Let G be a finite group and $L(G)$ be the subgroup lattice of G. The starting point for our discussion is given by [2], where the proper nontrivial subgroups H of G satisfying the condition

$$\text{for every } X \in L(G) \text{ we have either } X \leq H \text{ or } H \leq X \tag{1}$$

have been studied. Such a subgroup is called a *breaking point* for the lattice $L(G)$, and a group G whose subgroup lattice possesses breaking points is called a *BP-group*. Clearly, all cyclic p-groups of order at least p^2 are BP-groups. Note that a complete classification of BP-groups can be found in [2]. Also, we observe that the condition (1) is equivalent to

$$L(G) = [1, H] \cup [H, G], \tag{2}$$

where for $X, Y \in L(G)$ with $X \subseteq Y$, we denote by $[X, Y]$ the interval in $L(G)$ between X and Y. A natural generalization of (2) has been suggested by Roland Schmidt, namely

E-mail address: tarnauc@uaic.ro.

https://doi.org/10.1016/j.crma.2018.06.006
1631-073X/

$$L(G) = [1, M] \cup [N, G] \text{ with } 1 < M, N < G, \tag{3}$$

and the abelian groups G satisfying (3) have been determined in [1].

The above concepts can be naturally extended to other remarkable posets of subgroups of G, and also to arbitrary posets. We recall here that the generalized quaternion 2-groups

$$Q_{2^n} = \langle a, b \mid a^{2^{n-2}} = b^2, a^{2^{n-1}} = 1, b^{-1}ab = a^{-1} \rangle, n \geq 3$$

can be characterized as being the unique finite non-cyclic groups whose posets of cyclic subgroups and of conjugacy classes of cyclic subgroups have breaking points (see [7] and [3], respectively).

In the current note, we will focus on the centralizer lattice

$$\mathfrak{C}(G) = \{C_G(H) \mid H \in L(G)\}$$

of G. Note that this is a complete meet-sublattice of $L(G)$ with least element $Z(G) = C_G(G)$ and greatest element $G = C_G(1)$. We will prove that there are no proper centralizers M and N such that $\mathfrak{C}(G) = [Z(G), M] \cup [N, G]$. This implies that $\mathfrak{C}(G)$ does not have breaking points. As an application, we show that Q_{2^n} is not a capable group, i.e. there is no group G with $G/Z(G) \cong Q_{2^n}$. Note that this result can be also derived from the more general Theorem 4.2 of [6].

Most of our notation is standard and will usually not be explained here. Elementary concepts and results on group theory can be found in [4]. For subgroup lattice notions, we refer the reader to [5].

2. Main results

Our main theorem is the following.

Theorem 1. *Let G be a group and $\mathfrak{C}(G)$ be the centralizer lattice of G. Then $\mathfrak{C}(G)$ cannot be written as $\mathfrak{C}(G) = [Z(G), M] \cup [N, G]$ with $M, N \neq Z(G), G$.*

Proof. Assume that there are two proper centralizers M and N such that $\mathfrak{C}(G) = [Z(G), M] \cup [N, G]$. Then, for every $x \in G$, we have either $C_G(x) \leq M$ or $N \leq C_G(x)$. In the first case, we infer that $x \in M$, while in the second one we get $x \in C_G(C_G(x)) \leq C_G(N)$, that is $x \in C_G(N)$. Thus, the group G is the union of its proper subgroups M and $C_G(N)$, a contradiction. □

Clearly, by taking $M = N$ in Theorem 1, we obtain the following corollary.

Corollary 2. *The centralizer lattice $\mathfrak{C}(G)$ of a group G has no breaking point.*

Next we remark that for an abelian group G we have $\mathfrak{C}(G) = \{G\}$, and also that there is no non-abelian group G with $\mathfrak{C}(G) = \{Z(G), G\}$ (i.e. $\mathfrak{C}(G)$ is not a chain of length 1). Since chains of length at least 2 have breaking points, Corollary 2 implies Corollary 3.

Corollary 3. *The centralizer lattice $\mathfrak{C}(G)$ of a group G cannot be a chain of length ≥ 1. Moreover, $\mathfrak{C}(G)$ is a chain if and only if G is abelian.*

Another consequence of Corollary 2 is Corollary 4.

Corollary 4. *The generalized quaternion 2-groups Q_{2^n}, $n \geq 3$, are not capable groups.*

Proof. Assume that there is a group G such that $G/Z(G) \cong Q_{2^n}$. Obviously, G is not abelian. Since Q_{2^n} has a unique subgroup of order 2, it follows that the lattice interval $[Z(G), G]$ of $L(G)$ contains a unique atom, say H. If $H \in \mathfrak{C}(G)$, then it is a breaking point of $\mathfrak{C}(G)$, contradicting Corollary 2. If $H \notin \mathfrak{C}(G)$, then it is (properly) contained in all minimal centralizers $M_1, M_2, \ldots, M_k$ of G, and so $H \subseteq \bigcap_{i=1}^k M_i$. Note that a intersection of centralizers is also a centralizer, that is $\bigcap_{i=1}^k M_i \in \mathfrak{C}(G)$. On the other hand, we have $k \geq 3$ because G is non-abelian. Then $\bigcap_{i=1}^k M_i < M_j$, for any $j = 1, 2, \ldots, k$, and therefore $\bigcap_{i=1}^k M_i = Z(G)$ by the minimality of M_j's. Consequently, $H \subseteq Z(G)$, a contradiction. □

Finally, we formulate an open problem concerning the above study.

Open problem. Let G be a group. Then $\mathfrak{C}'(G) = \{C_G(H) \mid H \trianglelefteq G)\}$ is also a complete meet-sublattice of $L(G)$ with the least element $Z(G) = C_G(G)$ and the greatest element $G = C_G(1)$. Which are the groups G such that $\mathfrak{C}'(G)$ has breaking points? (Note that this can happen, as for $G = S_3$.)

Acknowledgements

The author is grateful to the reviewer for his/her remarks, which improved the previous version of the paper.

References

[1] A. Breaz, G. Calugareanu, Abelian groups whose subgroup lattice is the union of two intervals, J. Aust. Math. Soc. 78 (1) (2005) 27–36.
[2] G. Călugăreanu, M. Deaconescu, Breaking points in subgroup lattices, in: Proceedings of Groups St. Andrews 2001 in Oxford, vol. 1, Cambridge University Press, Cambridge, UK, 2003, pp. 59–62.
[3] Y. Chen, G. Chen, A note on a characterization of generalized quaternion 2-groups, C. R. Acad. Sci. Paris, Ser. I 352 (6) (2014) 459–461.
[4] I.M. Isaacs, Finite Group Theory, American Mathematical Society, Providence, RI, USA, 2008.
[5] R. Schmidt, Subgroup Lattices of Groups, de Gruyter Expositions in Mathematics, vol. 14, de Gruyter, Berlin, 1994.
[6] S. Shahriari, On normal subgroups of capable groups, Arch. Math. 48 (3) (1987) 193–198.
[7] M. Tărnăuceanu, A characterization of generalized quaternion 2-groups, C. R. Acad. Sci. Paris, Ser. I 348 (13–14) (2010) 731–733.