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Lie algebras

Action of Weyl group on zero-weight space

Action du groupe de Weyl sur l'espace de poids nul

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A R T I C L E   I N F O

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A B S T R A C T

For any simple complex Lie group, we classify irreducible finite-dimensional representations ρ for which the longest element w_0 of the Weyl group acts non-trivially on the zero-weight space. Among irreducible representations that have zero among their weights, w_0 acts by $\pm\mathrm{Id}$ if and only if the highest weight of ρ is a multiple of a fundamental weight, with a coefficient less than a bound that depends on the group and on the fundamental weight.

R É S U M É

Pour tout groupe de Lie complexe simple, nous classifions les représentations irréductibles ρ de dimension finie telles que le plus long mot w_0 du groupe de Weyl agisse non trivialement sur l'espace de poids nul. Parmi les représentations irréductibles dont zéro est un poids, w_0 agit par $\pm\mathrm{Id}$ si et seulement si le plus haut poids de ρ est un multiple d'un poids fondamental, avec un coefficient plus petit qu'une borne qui dépend du groupe et du poids fondamental.

1. Introduction and main theorem

Consider a reductive complex Lie algebra $\mathfrak{g}$. Let $\tilde{G}$ be the corresponding simply-connected Lie group.

We choose in $\mathfrak{g}$ a Cartan subalgebra $\mathfrak{h}$. Let Δ be the set of roots of $\mathfrak{g}$ in $\mathfrak{h}^*$. We call Λ the root lattice, i.e. the abelian subgroup of $\mathfrak{h}^*$ generated by Δ. We choose in Δ a system Δ^+ of positive roots; let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots in Δ^+. Let $\varpi_1, \dots, \varpi_r$ be the corresponding fundamental weights. Let $W := N_{\tilde{G}}(\mathfrak{h})/Z_{\tilde{G}}(\mathfrak{h})$ be the Weyl group, and let w_0 be its longest element (defined by $w_0(\Delta^+) = -\Delta^+$).

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For each simple Lie algebra, we call $(e_1, e_2, \dots)$ the vectors called $(\varepsilon_1, \varepsilon_2, \dots)$ in the appendix to [2], which form a convenient basis of a vector space containing $\mathfrak{h}^*$. Throughout the paper, we use the Bourbaki conventions [2] for the numbering of simple roots and their expressions in the coordinates e_i.

In the sequel, all representations are supposed to be complex and finite-dimensional. We call ρ_λ (resp. V_λ) the irreducible representation of $\mathfrak{g}$ with highest weight λ (resp. the space on which it acts). Given a representation (ρ, V) of $\mathfrak{g}$, we call V^λ the weight subspace of V corresponding to the weight λ.

Definition 1.1. We say that a weight $\lambda \in \mathfrak{h}^*$ is *radical* if $\lambda \in \Lambda$.

Remark 1. An irreducible representation (ρ, V) has non-trivial zero-weight space V^0 if and only if its highest weight is radical.

Definition 1.2. Let (ρ, V) be a representation of $\mathfrak{g}$. The action of $W = N_{\tilde{G}}(\mathfrak{h})/Z_{\tilde{G}}(\mathfrak{h})$ on V^0 is well-defined, since V^0 is by definition fixed by $\mathfrak{h}$, hence by $Z_{\tilde{G}}(\mathfrak{h})$. Thus w_0 induces a linear involution on V^0. Let p (resp. q) be the dimension of the subspace of V^0 fixed by w_0 (resp. by $-w_0$). We say that (p, q) is the w_0-*signature* of the representation ρ and that the representation is:

- w_0-*pure* if $pq = 0$ (*of sign* $+1$ if $q = 0$ and *of sign* -1 if $p = 0$);
- w_0-*mixed* if $pq > 0$.

Remark 2. Replacing $\tilde{G}$ by any other connected group G with Lie algebra $\mathfrak{g}$ (with a well-defined action on V) does not change the definition. Indeed the center of $\tilde{G}$ is contained in $Z_{\tilde{G}}(\mathfrak{h})$, so acts trivially on V^0.

Our interest in this property originates in the study of free affine groups acting properly discontinuously (see [7]). We prove the following complete classification. To the best of our knowledge, this specific question has not been studied before; see [4] for a survey of prior work on related, but distinct, questions about the action of the Weyl group on the zero-weight space.

Theorem 1.3. *Let $\mathfrak{g}$ be any simple complex Lie algebra; let r be its rank. For every index $1 \le i \le r$, we denote by p_i the smallest positive integer such that $p_i \varpi_i \in \Lambda$. For every such i, let the "maximal value" $m_i \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$ and the "sign" $\sigma_i \in \{\pm 1\}$ be as given in Table 1 on page 854.*

Let λ be a dominant weight.

(i) If $\lambda \notin \Lambda$, then the w_0-signature of the representation ρ_λ is $(0, 0)$.
(ii) If $\lambda = k p_i \varpi_i$ for some $1 \le i \le r$ and $0 \le k \le m_i$, then ρ_λ is w_0-pure of sign $(\sigma_i)^k$.
(iii) Finally, if $\lambda \in \Lambda$ but is not of the form $\lambda = k p_i \varpi_i$ for any $1 \le i \le r$ and $0 \le k \le m_i$, then ρ_λ is w_0-mixed.

Example 1. Any irreducible representation of $\mathrm{SL}(2, \mathbb{C})$ is isomorphic to $S^k\mathbb{C}^2$ (the k-th symmetric power of the standard representation) for some $k \in \mathbb{Z}_{\ge 0}$. Its w_0-signature is $(0, 0)$ if k is odd, $(1, 0)$ if k is divisible by 4 and $(0, 1)$ if k is 2 modulo 4. This confirms the A_1 entries $(p_1, m_1, \sigma_1) = (2, \infty, -1)$ of Table 1.

Table 1 also gives the values of p_i. These are not a new result; they are immediate to compute from the known descriptions of the simple roots and fundamental weights (given e.g. in [2]).

Point (i) is an immediate consequence of Remark 1.

For point (ii), we show in Section 3 that certain symmetric and antisymmetric powers of defining representations of classical groups are w_0-pure, and that almost all representations listed in point (ii) are sub-representations of these powers. The finitely many exceptions are treated by an algorithm described in Section 2.

For point (iii), we prove in Section 4 that the set of highest weights of w_0-mixed representations of a given group is an ideal of the monoid of dominant radical weights. For any fixed group, this reduces the problem to checking w_0-mixedness of finitely many representations. In Section 5, we immediately conclude for exceptional groups and for low-rank classical groups by the algorithm of Section 2; we proceed by induction on rank for the remaining classical groups.

2. An algorithm to compute explicitly the w_0-signature of a given representation

Proposition 2.1. *Any simple complex Lie group G admits a reductive subgroup S whose Lie algebra is isomorphic to $\mathfrak{sl}(2, \mathbb{C})^s \times \mathbb{C}^t$, where (t, s) is the w_0-signature of the adjoint representation of G, and whose w_0 element is compatible with that of G, in the sense that some representative of the w_0 element of S is a representative of the w_0 element of G. This subgroup S can be explicitly described.*

Note that $s + t = r$ (the rank of G) and that $t = 0$ except for A_n ($t = \lfloor \frac{n}{2} \rfloor$), D_{2n+1} ($t = 1$) and E_6 ($t = 2$).

Table 1
Values of (p_i, m_i, σ_i) for simple Lie algebras. Theorem 1.3 states that among irreducible representations with a highest weight λ that is radical, only those with λ of the form $kp_i\varpi_i$ with $k \le m_i$ are w_0-pure, with a sign given by σ_i^k. We write N.A. for σ_i sign entries that are not defined due to $m_i = 0$. Since $A_1 \simeq B_1 \simeq C_1$ and $B_2 \simeq C_2$ and $A_3 \simeq D_3$, the results match up to reordering simple roots (namely reordering $i = 1, \dots, r$).

	Values of i and r		p_i	m_i	σ_i
$A_{r\ge1}$	$i=1$ or r		$r+1$	∞	$(-1)^{\lfloor (r+1)/2\rfloor}$
	$1<i<r$	$r=3$	$\frac{r+1}{\gcd(i,r+1)}$	∞	$+1$
		$r>3$		0	N.A.
$B_{r\ge1}$	$i=1$	$r>1$	1	∞	$(-1)^{ri-\lfloor i/2\rfloor}$
	$i=2$	$r>2$	1	2	
	$2<i<r$		1	1	
	$i=r$	$r=1,2$	2	∞	
		$r>2$		1	
$C_{r\ge1}$	$i=1$		2	∞	-1
	$i=2$	$r=2$	1	∞	$+1$
		$r>2$		2	
	i odd >2	$i=r=3$	2	1	-1
		$r>3$		0	N.A.
	i even >2	$i=r=4$	1	2	$+1$
		$r>4$		1	
$D_{r\ge3}$ r odd	$i=1$		2	∞	$+1$
	$1<i<r-1$	i even	1	0	N.A.
		i odd	2		
	$i=r-1$ or r	$r=3$	4	∞	$+1$
		$r>3$		0	N.A.
$D_{r\ge4}$ r even	$i=1$		2	∞	$+1$
	$i=2$		1	2	-1
	$2<i<r-1$	i odd	2	0	N.A.
		i even	1	1	$(-1)^{i/2}$
	$i=r-1$ or r	$r=4$	2	∞	$(-1)^{r/2}$
		$r>4$		1	

	Values of i	p_i	m_i	σ_i
E_6	$i=1,3,5,6$	3	0	N.A.
	$i=2,4$	1	0	N.A.
E_7	$i=1$	1	2	-1
	$i=2,5$	2	0	N.A.
	$i=3,4$	1	0	N.A.
	$i=6$	1	1	$+1$
	$i=7$	2	1	-1
E_8	$i=1$	1	1	$+1$
	$1<i<8$	1	0	N.A.
	$i=8$	1	2	-1
F_4	$i=1$	1	2	-1
	$i=2,3$	1	0	N.A.
	$i=4$	1	2	$+1$
G_2	$i=1,2$	1	2	-1

Table 2
Sets of strongly orthogonal roots that span the vector space $(\mathfrak{h}^*)^{-w_0}$. We chose them among the positive roots.

A_n	$\{e_i - e_{n+2-i} \mid 1 \le i \le \lfloor (n+1)/2 \rfloor\}$	E_6	$\{-e_1+e_4,\ -e_2+e_3, \pm\frac{1}{2}(e_1+e_2+e_3+e_4)+\frac{1}{2}(e_5-e_6-e_7+e_8)\}$
B_{2n}	$\{e_{2i-1} \pm e_{2i} \mid 1 \le i \le n\}$	E_7	$\{\pm e_1+e_2,\ \pm e_3+e_4,\ \pm e_5+e_6,\ -e_7+e_8\}$
B_{2n+1}	$\{e_{2i-1} \pm e_{2i} \mid 1 \le i \le n\} \cup \{e_{2n+1}\}$	E_8	$\{\pm e_1+e_2,\ \pm e_3+e_4,\ \pm e_5+e_6,\ \pm e_7+e_8\}$
C_n	$\{2e_i \mid 1 \le i \le n\}$	F_4	$\{e_1 \pm e_2,\ e_3 \pm e_4\}$
D_n	$\{e_{2i-1} \pm e_{2i} \mid 1 \le i \le \lfloor n/2 \rfloor\}$	G_2	$\{e_1-e_2,\ -e_1-e_2+2e_3\}$

Proof. Let $(\mathfrak{h}^*)^{-w_0}$ be the -1 eigenspace of w_0. Recall that two roots α and β are called *strongly orthogonal* if $\langle \alpha, \beta \rangle = 0$ and neither $\alpha + \beta$ nor $\alpha - \beta$ is a root. Table 2 exhibits pairwise strongly orthogonal roots $\{\alpha_1, \dots, \alpha_s\} \subset \Delta$ spanning $(\mathfrak{h}^*)^{-w_0}$ as a vector space. (Our sets are conjugate to those of [1], but these authors did not need the elements w_0 to match.) We then set

$$\mathfrak{s} := \mathfrak{h} \oplus \bigoplus_{i=1}^{s} \left(\mathfrak{g}^{\alpha_i} \oplus \mathfrak{g}^{-\alpha_i} \right),$$

where $\mathfrak{g}^\alpha$ denotes the root space corresponding to α. This is a Lie subalgebra of $\mathfrak{g}$, as follows from $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha+\beta}$ and from strong orthogonality of the α_i. It is isomorphic to $\mathfrak{sl}(2,\mathbb{C})^s \times \mathbb{C}^t$, because it has Cartan subalgebra $\mathfrak{h}$ of dimension $r = s + t$ and a root system of type A_1^s. We define S to be the connected subgroup of G with algebra $\mathfrak{s}$.

Let $\overline{\sigma_i} := \exp[\frac{\pi}{2}(X_{\alpha_i} - Y_{\alpha_i})] \in S$, where for every α, X_α and Y_α denote the elements of $\mathfrak{g}$ introduced in [3, Theorem 7.19]. We claim that $\overline{\sigma} := \prod_i \overline{\sigma_i}$ is a representative of the w_0 element of S and of the w_0 element of G. By [3, Proposition 11.35], $\overline{\sigma_i}$ is a representative of the reflection s_{α_i}, which shows the first statement. Now since the α_i are orthogonal, the product of s_{α_i} acts by $-\mathrm{Id}$ on their span $(\mathfrak{h}^*)^{-w_0}$ and acts trivially on its orthogonal complement, like w_0. □

Then the w_0-signature of any representation ρ of G is equal to that of its restriction $\rho|_S$ to S. We use branching rules to decompose $\rho|_S = \oplus_i \rho_i$ into irreducible representations of S. The total w_0-signature is then the sum of those of the ρ_i.

Each ρ_i is a tensor product $\rho_{i,1}\otimes\cdots\otimes\rho_{i,s}\otimes\rho_{i,\mathrm{Ab}}$, where $\rho_{i,j}$ for $1\le j\le s$ is an irreducible representation of the factor $\mathfrak{s}_j\simeq\mathfrak{sl}(2,\mathbb{C})$, and $\rho_{i,\mathrm{Ab}}$ is an irreducible representation of the abelian factor isomorphic to $\mathbb{C}^t$. The w_0-signature of ρ_i is then the "product" of those of these factors, according to the rule $(p,q)\otimes(p',q')=(pp'+qq',pq'+qp')$. The w_0-signatures of all irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ have been described in Example 1; the w_0-signature of $\rho_{i,\mathrm{Ab}}$ is just $(1,0)$ if the representation is trivial and $(0,0)$ otherwise.

Branching rules are provided by several software packages. We implemented our algorithm separately in LiE [10] and in Sage [8]. In Sage, we used the Branching Rules module [9], largely written by Daniel Bump.

3. Proof of (ii): that some representations are w_0-pure

We must prove that representations of highest weight $\lambda=kp_i\varpi_i$, $k\le m_i$ are w_0-pure of sign σ_i^k (with data p_i, m_i, σ_i given in Table 1). We denote by $\square$ the defining representation of each classical group ($\mathbb{C}^{n+1}$ for A_n, $\mathbb{C}^{2n+1}$ for B_n, $\mathbb{C}^{2n}$ for C_n and D_n), and introduce a basis of it: for every $\varepsilon\in\{-1,0,1\}$ and i such that εe_i (or for A_n its orthogonal projection onto $\mathfrak{h}^*$) is a weight of $\square$, we call $h_{\varepsilon i}$ some nonzero vector in the corresponding weight space.

For exceptional groups, all m_i are finite, so the algorithm of Section 2 suffices; we also use it for the representations with highest weight $2\varpi_3$ of C_3 and $2\varpi_4$ of C_4.

Most other cases are subrepresentations of $S^m\square$ of A_n or D_{2n+1}, or one of $S^m\square$ or $\Lambda^m\square$ or $S^2(\Lambda^2\square)$ of B_n or C_n or D_{2n}, all of which will prove to be w_0-pure. Here $S^m\rho$ and $\Lambda^m\rho$ denote the symmetric and the antisymmetric tensor powers of a representation ρ. The remaining cases are mapped to these by the isomorphisms $B_2\simeq C_2$ and $A_3\simeq D_3$ and the outer automorphisms $\mathbb{Z}/2\mathbb{Z}$ of A_n and $\mathfrak{S}_3$ of D_4.

For $A_n=\mathfrak{sl}(n+1,\mathbb{C})$, the defining representation is $\square=\mathbb{C}^{n+1}=\mathrm{Span}\{h_1,\dots,h_{n+1}\}$. A representative $\overline{w_0}\in\mathrm{SL}(n+1,\mathbb{C})$ of w_0 acts on $\square$ by $h_j\mapsto h_{n+2-j}$ for $1\le j<n+1$ and by $h_{n+1}\mapsto\sigma_1h_1$ where $\sigma_1=(-1)^{\lfloor(n+1)/2\rfloor}$, the sign being such that $\det\overline{w_0}=+1$. We consider the representation $S^{k(n+1)}\square$. Its zero-weight space V^0 is spanned by symmetrized tensor products $h_{j_1}\otimes\cdots\otimes h_{j_{k(n+1)}}$ in which each h_j appears equally many times, namely k times. Hence, V^0 is one-dimensional (the representation is thus w_0-pure) and spanned by the symmetrization of $v=h_1^{\otimes k}\otimes h_2^{\otimes k}\otimes\cdots\otimes h_{n+1}^{\otimes k}$. We compute $\overline{w_0}\cdot v=h_{n+1}^{\otimes k}\otimes\cdots\otimes h_2^{\otimes k}\otimes(\sigma_1h_1)^{\otimes k}$, whose symmetrization is equal to σ_1^k times that of v; this gives the announced sign σ_1^k.

For $D_{2n+1}=\mathfrak{so}(4n+2,\mathbb{C})$, the defining representation is $\square=\mathbb{C}^{4n+2}=\mathrm{Span}\{h_{\pm j}\mid 1\le j\le 2n+1\}$ and $\overline{w_0}$ maps $h_{\pm j}\mapsto h_{\mp j}$ for $1\le j\le 2n$, but fixes $h_{\pm(2n+1)}$. The zero-weight space V^0 of $S^{2k}\square$ is spanned by symmetrizations of $h_{j_1}\otimes h_{-j_1}\otimes\cdots\otimes h_{j_k}\otimes h_{-j_k}$, each of which is fixed by $\overline{w_0}$. The representation is w_0-pure with $\sigma_1=+1$, as announced.

The cases of $B_n=\mathfrak{so}(2n+1,\mathbb{C})$, $C_n=\mathfrak{sp}(2n,\mathbb{C})$ and $D_{n\,\mathrm{even}}=\mathfrak{so}(2n,\mathbb{C})$ are treated together:

- B_n has $\square=\mathbb{C}^{2n+1}=\mathrm{Span}\{h_j\mid -n\le j\le n\}$ and $\overline{w_0}$ acts by $h_j\mapsto h_{-j}$ for $j\ne0$ and $h_0\mapsto(-1)^nh_0$;
- C_n has $\square=\mathbb{C}^{2n}=\mathrm{Span}\{h_{\pm j}\mid 1\le j\le n\}$ and $\overline{w_0}$ acts by $h_j\mapsto h_{-j}$ and $h_{-j}\mapsto -h_j$ for $j>0$;
- D_n has $\square=\mathbb{C}^{2n}=\mathrm{Span}\{h_{\pm j}\mid 1\le j\le n\}$ and, for n even, $\overline{w_0}$ acts by $h_j\mapsto h_{-j}$ for all j.

First consider $\Lambda^m\square$ and $S^m\square$. Their zero-weight spaces are spanned by (anti)symmetrizations of $h_{j_1}\otimes h_{-j_1}\otimes\cdots\otimes h_{j_k}\otimes h_{-j_k}\otimes h_0^{\otimes l}$, where $2k+l=m$. Each of these vectors is fixed by $\overline{w_0}$ up to a sign that only depends on the group, the representation, and on (k,l) or equivalently (l,m). For C_n and D_n we have $l=0$ so for each m the representation is w_0-pure, with a sign $(-1)^k$ for $S^{2k}\square$ of C_n and $\Lambda^{2k}\square$ of D_n, and no sign otherwise. For $\Lambda^m\square$ of B_n we note that $l\in\{0,1\}$ is fixed by the parity of m so the representation is w_0-pure; its sign is $(-1)^{nl+k}=(-1)^{nm+\lfloor m/2\rfloor}=\sigma_m$. For $S^m\square$ of B_n, only the parity of l is fixed, but the sign $(-1)^{nl}=(-1)^{nm}=\sigma_1^m$ still only depends on the representation; it confirms the data of Table 1. Finally, consider the representation $S^2(\Lambda^2\square)$. Its zero-weight space is spanned by symmetrizations of $(h_j\wedge h_{-j})\otimes(h_k\wedge h_{-k})$ and $(h_j\wedge h_k)\otimes(h_{-j}\wedge h_{-k})$ all of which are fixed by $\overline{w_0}$.

4. Cartan product: w_0-mixed representations form an ideal

Let G be a simply-connected simple complex Lie group and N a maximal unipotent subgroup of G. Define $\mathbb{C}[G/N]$ the space of regular (i.e. polynomial) functions on G/N. Pointwise multiplication of functions is G-equivariant and makes $\mathbb{C}[G/N]$ into a $\mathbb{C}$-algebra without zero divisors (because G/N is irreducible as an algebraic variety).

Theorem 4.1 *([6, (3.20)–(3.21)]). Each finite-dimensional representation of G (or equivalently of its Lie algebra $\mathfrak{g}$) occurs exactly once as a direct summand of the representation $\mathbb{C}[G/N]$. The $\mathbb{C}$-algebra $\mathbb{C}[G/N]$ is graded in two ways:*

- *by the highest weight λ, in the sense that the product of a vector in V_λ by a vector in V_μ lies in $V_{\lambda+\mu}$ (where V_λ stands here for the subrepresentation of $\mathbb{C}[G/N]$ with highest weight λ);*
- *by the actual weight λ, in the sense that the product of a weight vector with weight λ by a weight vector with weight μ is still a weight vector, with weight $\lambda+\mu$.*

For given λ and μ, we call *Cartan product* the induced bilinear map $\odot : V_\lambda \times V_\mu \to V_{\lambda+\mu}$. Given $u \in V_\lambda$ and $v \in V_\mu$, this defines $u \odot v \in V_{\lambda+\mu}$ as the projection of $u \otimes v \in V_\lambda \otimes V_\mu = V_{\lambda+\mu} \oplus \dots$. Since $\mathbb{C}[G/N]$ has no zero divisor, $u \odot v \neq 0$ whenever $u \neq 0$ and $v \neq 0$. We deduce the following.

Lemma 4.2. *The set of highest weights of w_0-mixed irreducible representations of $\mathfrak{g}$ is an ideal $\mathcal{I}_\mathfrak{g}$ of the additive monoid $\mathcal{M}$ of dominant elements of the root lattice.*

Proof. Consider a w_0-mixed representation V_λ and a representation V_μ whose highest weight is radical. We can choose u_+ and u_- in the zero-weight space of V_λ such that $w_0 \cdot u_+ = u_+$ and $w_0 \cdot u_- = -u_-$, and choose v in the zero-weight space of V_μ such that $w_0 \cdot v = \pm v$ for some sign. Then $u_+ \odot v$ and $u_- \odot v$ are non-zero elements of the zero-weight space of $V_{\lambda+\mu}$ on which w_0 acts by opposite signs. □

5. Proof of (iii): that other representations are w_0-mixed

Let $\mathcal{I}_\mathfrak{g}^{\text{Table}}$ be the set of dominant radical weights that are not of the form $\lambda = k p_i \varpi_i$, $k \le m_i$ (with data p_i, m_i given in Table 1). Observe that $\mathcal{I}_\mathfrak{g}^{\text{Table}}$ is an ideal of $\mathcal{M}$. In Section 3 we showed $\mathcal{I}_\mathfrak{g} \subset \mathcal{I}_\mathfrak{g}^{\text{Table}}$. We now show that $\mathcal{I}_\mathfrak{g}^{\text{Table}} \subset \mathcal{I}_\mathfrak{g}$, namely that V_λ is w_0-mixed for radical λ other than those described by Table 1. By Lemma 4.2, it is enough to show this for the basis of $\mathcal{I}_\mathfrak{g}^{\text{Table}}$. For any given group, $\mathcal{I}_\mathfrak{g}^{\text{Table}}$ has a finite basis, so we simply used the algorithm of Section 2 to conclude for $A_{\le 5}$, $B_{\le 4}$, $C_{\le 5}$, $D_{\le 6}$ and all exceptional groups.

Now let $\mathfrak{g}$ be one of $A_{>5}$, $B_{>4}$, $C_{>5}$, $D_{>6}$ and λ be in $\mathcal{I}_\mathfrak{g}^{\text{Table}}$. We proceed by induction on the rank of $\mathfrak{g}$.

Define as follows a reductive Lie subalgebra $\mathfrak{f} \times \mathfrak{g}' \subset \mathfrak{g}$:

- if $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, we choose $\mathfrak{f} \times \mathfrak{g}' \simeq \big(\mathfrak{gl}(1,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})\big) \times \mathfrak{sl}(n-2,\mathbb{C})$, where $\mathfrak{f}$ has the roots $\pm(e_1 - e_n)$ and $\mathfrak{g}'$ has the roots $\pm(e_i - e_j)$ for $1 < i < j < n$;
- if $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$, we choose $\mathfrak{f} \times \mathfrak{g}' \simeq \mathfrak{so}(4,\mathbb{C}) \times \mathfrak{so}(n-4,\mathbb{C})$, where $\mathfrak{f}$ has the roots $\pm e_1 \pm e_2$ and $\mathfrak{g}'$ has the roots $\pm e_i \pm e_j$ for $3 \le i < j \le n$;
- if $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, we choose $\mathfrak{f} \times \mathfrak{g}' \simeq \mathfrak{sp}(2,\mathbb{C}) \times \mathfrak{sp}(2n-2,\mathbb{C})$, where $\mathfrak{f}$ has the roots $\pm 2e_1$ and $\mathfrak{g}'$ has the roots $\pm e_i \pm e_j$ for $2 \le i < j \le n$ and $\pm 2e_i$ for $2 \le i \le n$.

In all three cases, $\mathfrak{f} \times \mathfrak{g}'$ and $\mathfrak{g}$ share their Cartan subalgebra, hence restricting a representation V of $\mathfrak{g}$ to $\mathfrak{f} \times \mathfrak{g}'$ does not change the zero-weight space V^0. Additionally, consider any connected Lie group G with Lie algebra $\mathfrak{g}$: then the w_0 elements of the connected subgroup of G with Lie algebra $\mathfrak{f} \times \mathfrak{g}'$ and of G itself coincide, or more precisely have a common representative in G, because the Lie algebras have the same Lie subalgebra $\mathfrak{s}$ defined in Proposition 2.1. It follows that a representation of $\mathfrak{g}$ is w_0-mixed if and only if its restriction to $\mathfrak{f} \times \mathfrak{g}'$ is.

Next, decompose $V_\lambda = \bigoplus_\iota (V_{\xi_\iota} \otimes V_{\mu_\iota})$ into irreducible representations of $\mathfrak{f} \times \mathfrak{g}'$, where ξ_ι and μ_ι are dominant weights of $\mathfrak{f}$ and $\mathfrak{g}'$, respectively. Consider the subspace

$$V_\lambda^{(0,\bullet)} := \bigoplus_\iota (V_{\xi_\iota}^0 \otimes V_{\mu_\iota}) \subset V_\lambda \tag{1}$$

fixed by the Cartan algebra of $\mathfrak{f}$. It is a representation of $\mathfrak{g}'$ whose zero-weight subspace coincides with that of V_λ. The direct sum obviously restricts to radical ξ_ι, and $\dim V_{\xi_\iota}^0 = 1$ because we chose $\mathfrak{f}$ to be a product of $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{gl}(1,\mathbb{C})$ factors. Thus the w_0 element of $\mathfrak{g}$ acts on $V_{\xi_\iota}^0 \otimes V_{\mu_\iota}$ in the same way, up to a sign, as the w_0 element of $\mathfrak{g}'$ acts on V_{μ_ι}. Lemma 5.2 shows that $V_\lambda^{(0,\bullet)}$ has an irreducible subrepresentation V_ν such that $\nu \in \mathcal{I}_{\mathfrak{g}'}^{\text{Table}}$. By the induction hypothesis, V_ν is then w_0-mixed hence w_0 has both eigenvalues ± 1 on the zero-weight space $V_\lambda^0 \subset V_\lambda^{(0,\bullet)}$, namely V_λ is w_0-mixed.

This concludes the proof of Theorem 1.3.

There remains to state and prove two lemmas. Let $\mathfrak{g}$ be A_{n-1}, B_n, C_n or D_n and let λ be a dominant radical weight of $\mathfrak{g}$. It can then be expressed in the standard basis $e_1, \dots, e_n$ as $\lambda = \sum_{i=1}^n \lambda_i e_i$ where $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n$ are integers subject to: for A_{n-1}, $\sum_i \lambda_i = 0$; for B_n, $\lambda_n \ge 0$; for C_n, $\lambda_n \ge 0$ and $\sum_i \lambda_i \in 2\mathbb{Z}$; for D_n, $\lambda_{n-1} \ge |\lambda_n|$ and $\sum_i \lambda_i \in 2\mathbb{Z}$. In addition, let $\mathfrak{f} \times \mathfrak{g}' \subset \mathfrak{g}$ be the subalgebra defined above. We identify weights of $\mathfrak{g}'$ with the corresponding weights of $\mathfrak{g}$ (acting trivially on the Cartan subalgebra of $\mathfrak{f}$). Note that this introduces a shift in their coordinates: the dual of the Cartan subalgebra of $\mathfrak{g}'$ is spanned by a subset of the vectors e_i (corresponding to $\mathfrak{g}$) that starts at e_2 or e_3, not at e_1 as expected.

Lemma 5.1. *Let μ be the dominant weight of $\mathfrak{g}'$ defined as follows:*

- *for A_{n-1}, $\mu = \big(\sum_{i=1}^{\ell-1} \lambda_i e_{i+1}\big) + \lambda_\ell e_\ell + \big(\sum_{i=\ell+1}^{n} \lambda_i e_{i-1}\big)$ where $1 < \ell < n$ is an index such that $\lambda_{\ell-1} + \lambda_\ell \ge 0 \ge \lambda_\ell + \lambda_{\ell+1}$ (when several ℓ obey this, μ does not depend on the choice);*
- *for B_n, $\mu = \sum_{i=1}^{n-2} \lambda_i e_{i+2}$;*

– *for* C_n, $\mu = \sum_{i=1}^{n-1} \lambda_i e_{i+1} - \eta e_n$ *where* $\eta \in \{0,1\}$ *obeys* $\eta \equiv \lambda_n \pmod 2$*;*
– *for* D_n, $\mu = \sum_{i=1}^{n-2} \lambda_i e_{i+2} - \eta e_n$ *where* $\eta \in \{0,1\}$ *obeys* $\eta \equiv \lambda_{n+1} + \lambda_n \pmod 2$*.*

Then V_μ *is a sub-representation of the space* $V_\lambda^{(0,\bullet)}$ *defined earlier.*

Proof for A_{n-1}. Let $\nu = \sum_{i=2}^{n-1} \nu_i e_i$ be a dominant radical weight of $\mathfrak{g}'$. The weight ν is among weights of $V_\lambda^{(0,\bullet)}$ if and only if it is among weights of V_λ. The condition is that $\langle \lambda - \tilde\nu, \varpi_k\rangle \ge 0$ for all k, where $\tilde\nu$ is the unique dominant weight of $\mathfrak{g}$ in the orbit of ν under the Weyl group of $\mathfrak{g}$.

Explicitly, $\tilde\nu = \left(\sum_{i=1}^{p-1} \nu_{i+1} e_i\right) + \sum_{i=p+2}^{n} \nu_{i-1} e_i$, where p is any index such that $\nu_p \ge 0 \ge \nu_{p+1}$. Then the condition is $\sum_{i=1}^{k} \lambda_i \ge \sum_{i=2}^{k+1} \nu_i$ for $1 \le k < p$ and $\sum_{i=1}^{p} \lambda_i \ge \sum_{i=2}^{p} \nu_i$ and $\sum_{i=1}^{k} \lambda_i \ge \sum_{i=2}^{k-1} \nu_i$ for $p < k < n$. Let us show that this is equivalent to

$$\sum_{i=2}^{k} \nu_i \le \min\left(\sum_{i=1}^{k-1} \lambda_i, \sum_{i=1}^{k+1} \lambda_i\right) \text{ for all } 2 \le k \le n-2. \tag{2}$$

In one direction, the only non-trivial statement is that $2\sum_{i=1}^{p} \lambda_i \ge \sum_{i=1}^{p-1} \lambda_i + \sum_{i=1}^{p+1} \lambda_i \ge 2\sum_{i=2}^{p} \nu_i$, where we used $2\lambda_p \ge \lambda_p + \lambda_{p+1}$. In the other direction, we check $\sum_{i=2}^{k} \nu_i \le \sum_{i=2}^{\min(p,k+2)} \nu_i \le \sum_{i=1}^{k+1} \lambda_i$ for $k \le p-1$ using $\nu_2 \ge \dots \ge \nu_p \ge 0$, and similarly for $p+1 \le k$ using $0 \ge \nu_{p+1} \ge \dots \ge \nu_{n-1}$.

Now, $\lambda_{\ell-1} + \lambda_\ell \ge 0 \ge \lambda_\ell + \lambda_{\ell+1}$ implies $\lambda_{\ell-2} \ge \lambda_{\ell-1} \ge \lambda_{\ell-1} + \lambda_\ell + \lambda_{\ell+1} \ge \lambda_{\ell+1} \ge \lambda_{\ell+2}$, so μ is a dominant weight of $\mathfrak{g}'$. It is radical because $\sum_{i=2}^{n-1} \mu_i = \sum_{i=1}^{n} \lambda_i = 0$. Furthermore, μ saturates all bounds (2) (with ν replaced by μ), as seen using $\lambda_k + \lambda_{k+1} \ge 0$ or ≤ 0 for $k < \ell$ or $k \ge \ell$ respectively. In particular, we deduce that μ is among the weights of $V_\lambda^{(0,\bullet)}$, hence of some irreducible summand $V_\nu \subset V_\lambda^{(0,\bullet)}$. The dominant radical weight ν of $\mathfrak{g}'$ must also obey (2), namely $\sum_{i=2}^{k} \nu_i \le \sum_{i=2}^{k} \mu_i$ (due to the aforementioned saturation). Since μ is dominant and among weights of V_ν, we must also have $\langle \nu - \mu, \varpi_k'\rangle \ge 0$ for all fundamental weights ϖ_k' of $\mathfrak{g}'$. This is precisely the reverse inequality $\sum_{i=2}^{k} \nu_i \ge \sum_{i=2}^{k} \mu_i$. We conclude that $\mu = \nu$. □

Proof for B_n, C_n, D_n. Let $\varepsilon = 1$ for C_n and otherwise $\varepsilon = 2$. Again, a dominant radical weight $\nu = \sum_{i=1+\varepsilon}^{n} (\nu_i e_i)$ of $\mathfrak{g}'$ is a weight of $V_\lambda^{(0,\bullet)}$ if and only if all $\langle \lambda - \tilde\nu, \varpi_k\rangle \ge 0$, where $\tilde\nu$ is the unique dominant weight of $\mathfrak{g}$ in the Weyl orbit of ν. In all three cases, $\tilde\nu = \sum_{i=1}^{n-\varepsilon} |\nu_{i+\varepsilon}| e_i$, where the absolute value is only useful for the ν_n component for D_n. The condition is worked out to be $\sum_{i=1}^{k} \lambda_i \ge \sum_{i=1}^{k} |\nu_{i+\varepsilon}|$ for $1 \le k \le n-\varepsilon$. It is easy to check that μ is a dominant radical weight of $\mathfrak{g}'$ and that it obeys these conditions.

Consider now an irreducible summand $V_\nu \subset V_\lambda^{(0,\bullet)}$ that has μ among its weights. On the one hand, $\sum_{i=1}^{k} \lambda_i \ge \sum_{i=1}^{k} |\nu_{i+\varepsilon}|$ for $1 \le k \le n-\varepsilon$, where the absolute value is only useful for ν_n for D_n. On the other hand, $\langle \nu - \mu, \varpi'\rangle \ge 0$ for all dominant weights ϖ' of $\mathfrak{g}'$ (in particular $e_{1+\varepsilon} + \dots + e_{k+\varepsilon}$), so $\sum_{i=1}^{k} \nu_{i+\varepsilon} \ge \sum_{i=1}^{k} \mu_{i+\varepsilon}$ for $1 \le k \le n-\varepsilon$. The two inequalities fix $\nu_i = \mu_i$ for all i, except $i = n$ when $\eta = 1$ for C_n and D_n: in these cases, we conclude by using $\sum_i \nu_i - \sum_i \mu_i \in 2\mathbb{Z}$, since both weights are radical. □

Lemma 5.2. *For any* $\lambda \in \mathcal{I}_{\mathfrak{g}}^{\text{Table}}$*, there exists* $\nu \in \mathcal{I}_{\mathfrak{g}'}^{\text{Table}}$ *such that the representation of* $\mathfrak{g}'$ *with highest weight* ν *is a subrepresentation of* $V_\lambda^{(\bullet,0)}$*.*

Proof for A_{n-1} with $n \ge 7$. If the weight μ defined by Lemma 5.1 is in $\mathcal{I}_{\mathfrak{g}'}^{\text{Table}}$, we are done. Otherwise, $\mu = m(n-2)\varpi_1'$ or $\mu = m(n-2)\varpi_{n-3}'$. By symmetry under $e_i \mapsto -e_{n+1-i}$, it is enough to consider the second case, so $\mu = \sum_{i=2}^{n-1} \mu_i e_i$ with $\mu_i = m$ for $2 \le i \le n-2$ and $\mu_{n-1} = -m(n-3)$. By the construction of μ in terms of λ, we know that there exists $1 < \ell < n$ such that $\mu_i = \lambda_{i-1} \ge 0$ for $1 < i < \ell$ and $\lambda_{\ell-1} \ge \mu_\ell = \lambda_{\ell-1} + \lambda_\ell + \lambda_{\ell+1} \ge \lambda_{\ell+1}$ and $\mu_i = \lambda_{i+1} \le 0$ for $\ell < i < n$. Since only $\mu_{n-1} \le 0$, the last constraint sets $\ell = n-2$ or $\ell = n-1$. In the first case, we learn that $\lambda_i = m$ for $1 \le i \le n-4$, but also that $m = \mu_{n-3} = \lambda_{n-4} \ge \lambda_{n-3} \ge \mu_{n-2} = m$ so $\lambda_{n-3} = m$, thus $\lambda_{n-2} + \lambda_{n-1} = \mu_{n-2} - \lambda_{n-3} = 0$, and we can change ℓ to $n-1$ (recall that the choice of ℓ such that $\lambda_{\ell-1} + \lambda_\ell \ge 0 \ge \lambda_\ell + \lambda_{\ell+1}$ does not affect μ). We are thus left with the case $\ell = n-1$, where $\lambda_i = m$ for $1 \le i \le n-3$, and where $\lambda_{n-2} + \lambda_{n-1} \ge 0$ and $m = \lambda_{n-3} \ge \lambda_{n-2}$.

We conclude that $\lambda = m\left(\sum_{i=1}^{n-3} e_i\right) + l e_{n-2} + k e_{n-1} - \left((n-3)m + l + k\right) e_n$ for integers $m \ge l \ge |k|$, with the exclusion of the case $k = l = m$ because of $\lambda \in \mathcal{I}_{\mathfrak{g}}^{\text{Table}}$. For these dominant weights, the particular irreducible summand $V_\mu \subset V_\lambda^{(0,\bullet)}$ of Lemma 5.1 is w_0-pure, but we now determine another summand that is w_0-mixed. The branching rules from $\mathfrak{g}$ to $\mathfrak{f} \times \mathfrak{g}'$ can easily be deduced from the classical branching rules from $\mathfrak{gl}(n,\mathbb{C})$ to $\mathfrak{gl}(n-1,\mathbb{C})$ (given for example in [5, Theorem 9.14]). Namely, consider the representation of $\mathfrak{gl}(n,\mathbb{C})$ on V_λ such that the diagonal $\mathfrak{gl}(1,\mathbb{C})$ acts by zero. Then $V_\lambda^{(0,\bullet)} \subset V_\lambda$ is the subspace on which all three $\mathfrak{gl}(1,\mathbb{C})$ factors of $\mathfrak{gl}(1,\mathbb{C}) \times \mathfrak{gl}(n-2,\mathbb{C}) \times \mathfrak{gl}(1,\mathbb{C}) \subset \mathfrak{gl}(n,\mathbb{C})$ act by zero. It decomposes into irreducible representations of $\mathfrak{g}' \simeq \mathfrak{sl}(n-2,\mathbb{C})$ with highest weights $\lambda'' = \sum_{i=2}^{n-1} \lambda_i'' e_i$ such that $\sum_i \lambda_i'' = 0$ and such that there exists $\lambda_1', \dots, \lambda_{n-1}'$ with $\sum_i \lambda_i' = 0$, and $\lambda_1 \ge \lambda_1' \ge \lambda_2 \ge \dots \ge \lambda_{n-1}' \ge \lambda_n$ and $\lambda_1' \ge \lambda_2'' \ge \lambda_2' \ge \dots \ge \lambda_{n-1}'' \ge \lambda_{n-1}'$. Concretely we focus on the summand where $(\lambda_i)_{i=1}^{n}$ and $(\lambda'_i)_{i=1}^{n-1}$ and $(\lambda''_i)_{i=2}^{n-1}$ all take the form $(m, \dots, m, l, k, -S)$ where S is the sum of all other entries, with a different number of m in each case. Given that we started in rank at least 6, the resulting weight λ'' cannot be a multiple of a fundamental weight, hence $\lambda'' \in \mathcal{I}_{\mathfrak{g}'}^{\text{Table}}$. □

Proof for B_n with $n \geq 5$, C_n with $n \geq 6$, D_n with $n \geq 7$. We recall $\varepsilon = 1$ for C_n and otherwise $\varepsilon = 2$. If the weight μ defined by Lemma 5.1 is in $\mathcal{I}_{\mathfrak{g}'}^{\text{Table}}$, we are done. Otherwise, μ can take a few possible forms because we took rank $\mathfrak{g}' = n - \varepsilon$ large enough to avoid special values listed in Table 1. Note that, by construction of $\mu = \sum_{i=1+\varepsilon}^{n} \mu_i e_i$, we have $\lambda_i = \mu_{i+\varepsilon}$ for $1 \leq i \leq n-3$ for D_n and $1 \leq i \leq n-2$ for B_n and C_n. The possible dominant radical weights not in $\mathcal{I}_{\mathfrak{g}'}^{\text{Table}}$ are as follows.

- First, $\mu = m\varpi'_1 = me_{1+\varepsilon}$, where additionally m is even for C_n and D_n. Then $\lambda_1 = \mu_{1+\varepsilon} = m$ and $\lambda_2 = \mu_{2+\varepsilon} = 0$ fix $\lambda = m\varpi_1$, which is not in $\mathcal{I}_{\mathfrak{g}}^{\text{Table}}$.
- Second, $\mu = 2\varpi'_2 = 2(e_{1+\varepsilon} + e_{2+\varepsilon})$, except for D_n with odd n. Then $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = 0$ fix $\lambda = 2\varpi_2$, which is not in $\mathcal{I}_{\mathfrak{g}}^{\text{Table}}$.
- Third, $\mu = \sum_{i=1}^{m} e_{i+\varepsilon}$ for some $m \geq 2$, except for D_n with odd n, and where additionally m is even for D_n with even n and for C_n. Since $\lambda_1 = \mu_{1+\varepsilon} = 1$ and λ is dominant, we deduce that either $\lambda_1 = \dots = \lambda_p = 1$ for some p and all other $\lambda_i = 0$, or (only in the D_n case) $\lambda_1 = \dots = \lambda_{n-1} = 1 = -\lambda_n$. These weights λ are not in $\mathcal{I}_{\mathfrak{g}}^{\text{Table}}$. Note, of course, that p and m are not independent; for example for $m \leq n-3$ one has $m = p$.
- Fourth, $\mu = \left(\sum_{i=1}^{n-3} e_{i+2}\right) - e_n$ for D_n with even n. This weight is not of the form of Lemma 5.1 because one would need $-1 = \lambda_{n-2} - \eta \geq -\eta \geq -1$; hence $\eta = 1$ and $\lambda_{n-2} = 0$, so $\lambda_{n-1} = \lambda_n = 0$ so $1 = \eta \equiv \lambda_{n-1} + \lambda_n = 0 \pmod 2$. □

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