Complex analysis/Functional analysis

Analytic balayage of measures, Carathéodory domains, and badly approximable functions in L^p

Balayage analytique de mesures, de domaines de Carathéodory et de fonctions mal approximables dans L^p

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A R T I C L E   I N F O

A B S T R A C T

We give new formulae for analytic balayage of measures supported on subsets of Carathéodory compact sets in the complex plane, and consider a related problem of description of badly approximable functions in $L^p(\mathbb{T})$.

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R É S U M É

Nous donnons de nouvelles formules pour le balayage analytique des mesures supportées sur des sous-ensembles d'ensembles compacts de Carathéodory dans le plan complexe et considérons le problème connexe de la description des fonctions mal approximables dans $L^p(\mathbb{T})$.

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1. Analytic balayage and Carathéodory sets

Analytic balayage of measures, a notion introduced by D. Khavinson [12], is a useful tool for describing measures orthogonal to rational functions on compact sets in the complex plane. Certain formulae related to analytic balayage turned out to be helpful, for instance, in recent studies of uniform approximation of functions by polyanalytic polynomials (see, e.g., [2], [5], [6], [7]).

For a compact set X in the complex plane $\mathbb{C}$, let us denote by $C(X)$ the space of all continuous complex-valued functions on X endowed with the standard uniform norm $\|\cdot\|_X$. Moreover, let $R(X)$ be the closure in $C(X)$ of the subspace of all rational functions with poles lying outside X. Let us also remind several standard notations. In what follows, all measures

are assumed to be finite, complex-valued Borel measures in $\mathbb{C}$, symbols $\mathrm{Supp}(\mu)$ and $\|\mu\|$ will denote the (closed) support and the norm of a measure μ, respectively. Let μ be a measure, and let $\mathcal{F}$ be a subset of $C(X)$. One says that $\mu \perp \mathcal{F}$ (that is, μ is orthogonal to $\mathcal{F}$) if $\int f \, \mathrm{d}\mu = 0$ for each function $f \in \mathcal{F}$. As usual, the symbols E°, ∂E and $\overline{E}$ stand for the interior, the boundary and the closure of a given set E, respectively.

Definition 1.1. Let X be a compact set in $\mathbb{C}$, and let μ be a measure such that $\mathrm{Supp}(\mu) \subset X^{\circ}$. The measure ν on ∂X is called an *analytic balayage* of μ if $\mu - \nu \perp R(X)$, and for any measure $\widetilde{\nu}$ on ∂X such that $\mu - \widetilde{\nu} \perp R(X)$, the inequality $\|\widetilde{\nu}\| \geqslant \|\nu\|$ holds.

In all cases considered in this paper, the analytic balayage of a given measure is uniquely determined. Despite the above remark, this fact will be specially mentioned in almost all instances. So we will denote the analytic balayage of μ by $Ab(\mu) = Ab(\mu, \partial X)$.

This definition was given in [12, def. 2] for finitely connected compact sets with piecewise analytic boundaries, but it also makes sense for general compact sets. For measures μ supported on X (but not only on X°), the analytic balayage was defined in another way by means of a special implicit construction, namely, a weak-$*$ limit of analytic balayages of the initial measure to (piecewise analytic) boundaries of certain finitely connected compact sets approaching X [12, def. 3]. Such definition of the analytic balayage in the general case made it possible to obtain several interesting results. In particular, it follows from [12, theor. 2] that a point $a \in X$ is a peak-point for the algebra $R(X)$ if and only if the analytic balayage of the point-mass measure δ_a supported at the point a (which is treated in the sense of the construction just mentioned) is equal to δ_a.

In what follows, for a measure μ we denote by $\widehat{\mu}$ its Cauchy transform, i.e.

$$\widehat{\mu}(z) = \frac{1}{\pi} \int \frac{\mathrm{d}\mu(\zeta)}{z - \zeta},$$

and by $\mu|_B$ the restriction of μ onto a Borel set B, i.e. $\mu|_B(E) = \mu(E \cap B)$.

Let us see what an analytic balayage looks like in a simple case. Let G be a Jordan domain with piecewise analytic boundary Γ, and let μ be a measure such that $\mathrm{Supp}(\mu) \subset G$. As was shown in [12, prop. 2]

$$Ab(\mu) = \frac{1}{2\mathrm{i}} \widehat{\mu} \, \mathrm{d}z|_{\Gamma} - \frac{1}{2\mathrm{i}} g^* \, \mathrm{d}z|_{\Gamma}, \tag{1}$$

where $g^* \in R(\overline{G})$ is such that $\|\widehat{\mu} - g^*\|_{L^1(\Gamma, |\mathrm{d}z|)} = \inf \|\widehat{\mu} - g\|_{L^1(\Gamma, |\mathrm{d}z|)}$, the infimum being taken over all functions $g \in R(\overline{G})$. Notice that the analytic balayage is uniquely determined in this case; see [12, prop. 3]. It is also worth highlighting the role of the term $\frac{1}{2\mathrm{i}} \widehat{\mu} \, \mathrm{d}z|_{\Gamma}$ in (1). It follows from [7, lem. 4.1] that

$$\mu^* := \mu - \frac{1}{2\mathrm{i}} \widehat{\mu} \, \mathrm{d}z|_{\Gamma} \perp R(\overline{G}). \tag{2}$$

Furthermore, for any compact subset K of G and for any measure σ on $K \cup \Gamma$ such that $\sigma \perp R(\overline{G})$ we have

$$\sigma = (\sigma|_K)^* + h \, \mathrm{d}z|_{\Gamma}, \tag{3}$$

where h is a function belonging to the Smirnov class $E^1(G)$ (see, for instance, [15]).

Since the explicit expression for analytic balayage is known only for finitely connected compact sets with piecewise analytic boundaries, it would be interesting to find such formulae for a wider class of compact sets. The class of Carathéodory compact sets fits this problem most naturally. This is mainly due to natural analogues of formulae (2) and (3), which exist for the class of Carathéodory domains, but not for other known wider classes of domains in $\mathbb{C}$.

Recall that a compact set $X \subset \mathbb{C}$ is called a Carathéodory compact set if $\partial X = \partial \widehat{X}$, where $\widehat{X}$ is the union of X and of all bounded connected components of the set $\mathbb{C} \setminus X$. A bounded domain $G \subset \mathbb{C}$ is called a Carathéodory domain if $\partial G = \partial G_{\infty}$, where G_{∞} is the unbounded connected component of the set $\mathbb{C} \setminus \overline{G}$. It can be readily verified that any Carathéodory domain is simply connected and possesses the property $G = (\overline{G})^{\circ}$.

Let $\mathbb{D}$ and $\mathbb{T}$ stand for the unit disk $\{z : |z| < 1\}$ and the unit circle $\{z : |z| = 1\}$, respectively. Furthermore, let H^p, $1 \leqslant p \leqslant \infty$, denote the standard Hardy spaces (both in $\mathbb{D}$ and on $\mathbb{T}$), while $L^p = L^p(\mathbb{T})$ is the Lebesgue space with respect to the normalized Lebesgue measure $m(\cdot)$ on $\mathbb{T}$. For an open set $U \subset \mathbb{C}$, let us also denote by $H^{\infty}(U)$ the space of all bounded holomorphic functions in U. The norm of $h \in L^p$ will be denoted by $\|h\|_p$, while the norm of $f \in H^{\infty}(U)$ will be denoted by $\|f\|_{\infty, U}$.

To formulate our main results, we also need several special notations, whose detailed definition and explanation can be found, for instance, in [5, sect. 2]. For a function $h \in H^p$, we denote by $\mathscr{F}(h)$ its Fatou set, that is, the set of all point $\zeta \in \mathbb{T}$ where angular boundary values $h(\zeta)$ exist. $\partial_a U$ will denote the set of all accessible boundary points for an open set $U \subset \mathbb{C}$. We recall that $z \in \partial_a U$ if there exists a simple curve $\gamma \subset U \cup \{z\}$, such that z is one of its endpoints. Both $\mathscr{F}(\varphi)$ and $\partial_a U$ are known to be Borel sets. Let now G be a Carathéodory domain, and let f be some conformal mapping of $\mathbb{D}$ onto G. In this case, the functions f and f^{-1} can be extended to Borel measurable and mutually inverse functions

defined on the sets $\mathbb{D} \cup \mathscr{F}(f)$ and $G \cup \partial_a G$, respectively; see [5, cor. 1]. Furthermore, in the situation under consideration for every function $\varphi \in L^1$ the measure $f(\varphi \, dz|_{\mathbb{T}})$ is well defined and $f(\varphi \, dz|_{\mathbb{T}}) = (\varphi \circ f^{-1}) \omega$, where $\omega = f(dz|_{\mathbb{T}})$ is the complex-valued harmonic measure on ∂G (see [5, sect. 3] for details).

Theorem 1.2. *Let G be a Carathéodory domain, and let μ be a measure with* $\mathrm{Supp}(\mu) \subset G$.

1) The measure $Ab(\mu, \partial G)$ is concentrated on $\partial_a G$ and has the form

$$Ab(\mu, \partial G) = \frac{1}{2i}(\widehat{\eta} \circ f^{-1}) \omega - \frac{1}{2i}(h^* \circ f^{-1}) \omega,$$

where f is a conformal mapping of the disk $\mathbb{D}$ onto G, the measure η is defined as $\eta = f^{-1}(\mu)$, and the function $h^* \in H^1$ is the solution to the following extremal problem

$$\|\widehat{\eta} - h^*\|_1 = \inf_{h \in H^1} \|\widehat{\eta} - h\|_1. \tag{4}$$

2) We have $\| Ab(\mu, \partial G)\| = \dfrac{1}{\pi} \sup \left\{ \left| \displaystyle\int g \, d\mu \right| : g \in R(\overline{G}), \ \|g\|_{\overline{G}} \leqslant 1 \right\}$.

It follows from this theorem that the analytic balayage of μ in the case under consideration is uniquely determined (notice that the extremal problem (4) has a unique solution, see [10, ch. IV, sect. 1.2]).

In the general case of Carathéodory compact sets, the analytic balayage of measures with support on X° is described by the following statement.

Theorem 1.3. *Let X be a Carathéodory compact set, let μ be a measure such that $\mathrm{Supp}(\mu) \subset X^\circ$, and let $\{U_\alpha : \alpha \in A\}$ be the collection of all connected components of X° that intersect $\mathrm{Supp}(\mu)$. Then $Ab(\mu, \partial X) = \sum_{\alpha \in A} Ab(\mu|_{U_\alpha}, \partial U_\alpha)$.*

The proofs of Theorems 1.2 and 1.3 will appear in [9].

Let us revert once again to the construction used in [12] for the definition of the analytic balayage in the case when the measure μ under consideration is supported on X (of which no special assumptions are made), but not only on X°. Following [12, def. 3], a measure ν' on ∂X is said to *belong to the analytic balayage* of μ if there exists a decreasing sequence of finitely connected compact sets X_m, $m = 1, 2, \ldots$, with piecewise analytic boundaries such that the sequence $Ab(\mu, \partial X_m)$ converges to ν' in the sense of weak-$*$ convergence of measures.

Take an arbitrary Jordan domain G, and consider the conformal mapping f from $\mathbb{D}$ onto G such that $f(0) = z_0 \in G$ and $f'(0) > 0$. Let us construct a sequence of simple closed analytic curves Υ_n, $n = 1, 2, \ldots$, such that the domains D_n bounded by Υ_n converge to G in the sense of (Carathéodory) kernel convergence. Let f_n, $n = 1, 2, \ldots$, be the sequence of conformal mappings from $\mathbb{D}$ onto D_n such that $f_n(0) = z_0$ and $f_n'(0) > 0$. In this case, the measures $\omega = f(dz|_{\mathbb{T}})$ and $\omega_n = f_n(dz|_{\mathbb{T}})$, $n = 1, 2, \ldots$, are well defined. It can be readily verified that the sequence $(\widehat{\eta} \circ f_n^{-1}) \omega_n$ converges weak-$*$ to the measure $(\widehat{\eta} \circ f^{-1}) \omega$ as $n \to \infty$. This follows from the fact that under the given conditions, the sequence $f_n \circ f^{-1}$ converges uniformly on $\overline{G}$ to the identity function as $n \to \infty$, see [18, ch. II, sect. 2.3, corollary 1]. Moreover, for any function $h \in H^1$, the sequence of measures $(h \circ f_n^{-1}) \omega_n$ converges weak-$*$ to $(h \circ f^{-1}) \omega$. Therefore, the sequence $Ab(\mu, \Upsilon_n)$ converges weak-$*$ to $Ab(\mu, \partial G)$. Hence, in the case of Jordan domains with arbitrary boundaries, both approaches to the concept of an analytic balayage (introduced in [12, def. 3] and in Definition 1.1) lead to the same results.

2. Extremal problem (4) and badly approximable functions

The formula for analytic balayage in Theorem 1.2 has a simpler form in the case when the solution h^* to the extremal problem (4) is zero. Let us describe the measures for which it is the case. To this end, we use the concept of badly approximable functions. Take a number p, $1 \leqslant p < \infty$, and a function $\varphi \in L^p$. The function φ is called *p-badly approximable*, if only the function $g^* \equiv 0$ solves the extremal problem

$$\|\varphi - g^*\|_p = \inf_{g \in H^p} \|\varphi - g\|_p. \tag{5}$$

Recall that the solution to the extremal problem (5) is unique ([10, ch. IV, theor. 1.2]). The problem of describing p-badly approximable functions (in the above context as well as in certain related settings) is considered, for instance, in [1], [3], [4], [8], [13], [14], [17].

Recall that a function $\Theta \in H^\infty$ is said to be inner if $|\Theta(\zeta)| = 1$ for almost all $\zeta \in \mathbb{T}$. Moreover, a function Ψ is said to be an outer function for the class H^p if

$$\Psi(z) = e^{i\beta} \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \psi(t) \, dt \right\}, \quad \beta \in \mathbb{R}, \quad \psi \in L^p, \ \psi(t) \geqslant 0, \ \log \psi \in L^1.$$

A detailed account about inner and outer functions and inner-outer factorization of H^p-functions can be found, for instance, in [10, ch. II].

The following result was essentially obtained in the early 1950s by S.Ya. Khavinson in [11, theor. 1 and 2], where it was formulated in slightly different terms.

Theorem 2.1. *A function $\varphi \in L^1$ is 1-badly approximable if and only if it is of the form*

$$\varphi = \overline{\Theta}\, \Phi, \tag{6}$$

where Θ is an inner function, $\Theta(0) = 0$, and $\Phi \in L^1(\mathbb{T})$ is such that $\Phi \geqslant 0$.

Let $p > 1$. A function $\varphi \in L^p$ is p-badly approximable if and only if it has the form

$$\varphi = \overline{\Theta}\,\overline{\Psi}\,\Psi^{-1+2/p}, \tag{7}$$

where Θ is an inner function, $\Theta(0) = 0$, and Ψ is some outer function in H^2.

It turns out that the results of [11] were not widely known. Thus, for $p \geqslant 2$, the corresponding result was proved in [3, theor. 2.1] using the theory of Hankel operators. A similar (but slightly different) description of p-badly approximable function for $p > 1$ was obtained in [14]. One also ought to notice that the proofs given in [11] are lengthly and technically involved. Thus we deemed it appropriate to include a unified and readable proof of Theorem 2.1.

One more remark is needed before proving Theorem 2.1. In fact, the formula (7) makes sense also for $p = 1$, and leads to the same representation of 1-badly approximable functions as in (6), but the formula (6) is more explicit. Hence, it is stated in Theorem 2.1 separately.

Proof of Theorem 2.1. Let $1 \leqslant p < \infty$, and $p^{-1} + q^{-1} = 1$ (for $p = 1$ we put $q = \infty$). Take a function $\varphi \in L^p$. In view of [10, ch. IV, theor. 1.2] we have $\min_{g \in H^p} \|\varphi - g\|_p = \max \int F\varphi \, \mathrm{d}m$, where the maximum is taken over all $F \in H_0^q$ (i.e. $F \in H^q$ and $F(0) = 0$) with $\|F\|_q \leqslant 1$.

Suppose that φ is p-badly approximable. Applying [10, ch. IV, theor. 1.2] once again we conclude that there exists a unique function $F \in H_0^q$ with $\|F\|_q = 1$ such that $\|\varphi\|_p = \min_{g \in H^p} \|\varphi - g\|_p = \int F\varphi \, \mathrm{d}m$.

It follows from Hölder's inequality that

$$\|\varphi\|_p = \int F\varphi \, \mathrm{d}m \leqslant \int |F\varphi| \, \mathrm{d}m \leqslant \|\varphi\|_p \|F\|_q = \|\varphi\|_p.$$

Since all inequalities in this chain turn out to be equalities, the following is satisfied:

1°) for all $p \in [1, \infty)$ one has $\varphi F = |\varphi F|$ almost everywhere (a.e. in what follows) on $\mathbb{T}$;
2°) if $p = 1$, then $\varphi F = |\varphi|$ a.e. on $\mathbb{T}$;
3°) if $p > 1$, then $|\varphi|^p / \|\varphi\|_p^p = |F|^q$ a.e. on $\mathbb{T}$;

Let $p = 1$. It follows from 2° that $|F| = 1$ a.e. on $\mathbb{T}$, and hence F is an inner function. Therefore, $\varphi = \overline{F}|\varphi|$ a.e. on $\mathbb{T}$, which is the desired representation (where $\Theta = F$, and $\Phi = |\varphi|$).

Let now $p > 1$. It follows from 1° and 3° that $\varphi F = |\varphi F| = \|\varphi\|_p^{1-p}|\varphi|^p = \|\varphi\|_p |F|^q$ a.e. on $\mathbb{T}$. Put $c = 1/\|\varphi\|_p^p$, and take such outer function $\Psi \in H^2$ that $c|\Psi|^2 = |F|^q$. Thus $F = \Theta c^{1/q}\Psi^{2/q}$ for some inner function Θ with $\Theta(0) = 0$. Therefore, $\varphi = \|\varphi\|_p \overline{F}|F|^{q-2} = \overline{\Theta}\,\overline{\Psi}\,\Psi^{-1+2/p}$ a.e. on $\mathbb{T}$.

Let us verify the converse assertions. Let $p = 1$ and suppose $g^* \in H^1$ to be the best approximation element for the function $\varphi = \overline{\Theta}\,\Phi$ of the form (6). Therefore

$$\|\varphi - g^*\|_1 = \min_{g \in H^1} \|\varphi - g\|_1 = \max_{F \in H_0^\infty, \|F\|_\infty \leqslant 1} \left| \int F\overline{\Theta}\,\Phi \, \mathrm{d}m \right| \geqslant \int \Phi \, \mathrm{d}m = \|\Phi\|_1 = \|\overline{\Theta}\,\Phi\|_1,$$

and hence $g^* \equiv 0$. Thus, any function of the form (6) is 1-badly approximable.

Let, finally, $p > 1$. Take a function $\varphi = \overline{\Theta}\,\overline{\Psi}\,\Psi^{-1+2/p}$ of the form (7), and let $g^* \in H^p$ be the best approximation element for φ. Take $\widetilde{F} := \Theta c^{1/q}\Psi^{2/q}$, where $c = 1/\|\varphi\|_p^p$. Then $|\varphi|^p = |\Psi|^2 = c^{-1}|\widetilde{F}|^q$ a.e. on $\mathbb{T}$, and hence $\|\widetilde{F}\|_q = 1$. By definition of $\widetilde{F}$, we have $\widetilde{F}\varphi = c^{1/q}|\Theta|^2|\Psi|^2 = c^{1/q}|\varphi|^p$ a.e. on $\mathbb{T}$, and hence

$$\|\varphi - g^*\|_p = \min_{g \in H^p} \|\varphi - g\|_p = \max_{F \in H_0^q, \|F\|_q \leqslant 1} \left| \int \varphi F \, \mathrm{d}m \right| \geqslant \left| \int \varphi \widetilde{F} \, \mathrm{d}m \right| = c^{1/q}\|\varphi\|_p^p = \|\varphi\|_p.$$

Therefore, $g^* \equiv 0$ (as above) and φ is p-badly approximable, which completes the proof. □

In what follows let $\overline{\mathbb{C}}$ denote the standard one-point compactification of $\mathbb{C}$.

Corollary 2.2. *Let K be a compact subset of the unit disk $\mathbb{D}$, and let φ be a holomorphic function on $\overline{\mathbb{C}} \setminus K$. Then $\varphi|_{\mathbb{T}}$ is 1-badly approximable if and only if $\varphi = c\overline{B}$ on $\mathbb{T}$, with a constant c and a finite Blaschke product B such that $B(0) = 0$.*

Proof. As was shown in the proof of Theorem 2.1, $\varphi|_{\mathbb{T}}$ is 1-badly approximable if and only if $\varphi|_{\mathbb{T}} = \overline{\Theta}\,\Phi$ a.e. on $\mathbb{T}$, where Θ is some inner function, $\Theta(0) = 0$, and $\Phi = |\varphi|$. Since φ and $|\varphi|$ are continuous on $\mathbb{T}$, the function Θ (which is equal to $|\varphi|/\varphi$ a.e. on $\mathbb{T}$) can be extended to a continuous function on $\overline{\mathbb{D}}$, and hence Θ has neither a singular factor nor accumulation points of poles. Therefore $B := \Theta$ is a finite Blaschke product. Observe that $|\varphi| = \varphi B$ on $\mathbb{T}$, and hence φB takes positive real values on $\mathbb{T}$. So the function φB is holomorphic in some neighborhood of $\mathbb{T}$, extends meromorphically to $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and is positive on the unit circle. Using the Schwarz reflection principle, we conclude that φB is extendable to a meromorphic function in $\overline{\mathbb{C}}$ and hence is rational. Since the rational function φB is positive on the unit circle, it has the form

$$\varphi(z)B(z) = c \prod_{j=1}^{n} \frac{(z - a_j)(1 - \overline{a}_j z)}{(z - b_j)(1 - \overline{b}_j z)},$$

where a_j and b_j, $j = 1, \ldots, n$, are points in $\mathbb{D}$, and c is a positive constant. It readily follows from the argument principle and from the fact that the fraction $\big((z - a_j)(1 - \overline{a}_j z)\big)\big/\big((z - b_j)(1 - \overline{b}_j z)\big)$ takes positive values on $\mathbb{T}$ (see [16, p. 293]). Since φ is holomorphic outside the unit disk, the set $\{a_j, b_j\}$ is empty and hence $\varphi = c/B$, as required. $\quad\square$

The following proposition is an immediate consequence of Corollary 2.2.

Corollary 2.3. *Let G, μ, f and η be as in Theorem 1.2. Then $Ab(\mu, \partial G) = \frac{1}{2\mathrm{i}}(\widehat{\eta} \circ f^{-1})\,\omega$ if and only if μ is a finite sum of point-mass measures, one of which is supported at the point $f(0)$.*

Indeed, the solution h^* to the extremal problem (4) is zero if and only if the function $\widehat{\eta}$ coincides on $\mathbb{T}$ with the conjugation of some finite Blaschke product vanishing at 0. It means that η (and hence μ) is a finite sum of point-mass measures, one of which is supported at the origin (at the point $f(0)$, respectively).

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