C. R. Acad. Sci. Paris, Ser. I 356 (2018) 732–736

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Complex analysis/Functional analysis

On the boundary behaviour of derivatives of functions in the disc algebra

Sur le comportement au bord de dérivées de fonctions de l'algèbre du disque

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ARTICLE INFO

Article history:
Received 16 April 2018
Accepted after revision 22 May 2018
Available online 30 May 2018

Presented by the Editorial Board

ABSTRACT

We provide with a simple and explicit construction of a function in the disc algebra, whose derivatives enjoy a disjoint universal property near the boundary. The set of functions with such property is topologically generic, densely lineable, and spaceable.

RÉSUMÉ

On présente une construction simple et explicite d'une fonction de l'algèbre du disque dont les dérivés possèdent des propriétés d'universalité disjointe au bord. L'ensemble des fonctions ayant une telle propriété est topologiquement générique et contient un sous-espace dense et un sous-espace fermé de dimension infinie.

1. Introduction and statement of the main result

Let us denote by $\mathbb{T}$ the boundary of the unit disc $\mathbb{D} := \{z \in \mathbb{C},\ |z| < 1\}$ and by $H(\mathbb{D})$ the space of all holomorphic functions in $\mathbb{D}$, endowed with the Fréchet topology of uniform convergence on compacta. The behaviour of functions in $H(\mathbb{D})$ near $\mathbb{T}$ is of crucial interest in complex analysis. It was shown by Bagemihl in 1954 that, for any function φ measurable on $\mathbb{T}$, there exists f in $H(\mathbb{D})$ such that $f(r\zeta) \to \varphi(\zeta)$ as $r \to 1$, for almost every $\zeta \in \mathbb{T}$ [2]. Kahane and Katznelson [11] proved that such functions can have an arbitrary radial growth to the boundary. Later, functions $f \in H(\mathbb{D})$ enjoying the following *universal* property were exhibited [3,8]: given any measurable function φ on $\mathbb{T}$, there exists an increasing sequence $(r_n)_n$, $0 < r_n < 1$, converging to 1, such that, for any $z_0 \in \mathbb{D}$ and almost every $\zeta \in \mathbb{T}$,

$$\lim_{n\to\infty} f(r_n(\zeta - z_0) + z_0) = \varphi(\zeta).$$

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https://doi.org/10.1016/j.crma.2018.05.015

Bayart's result proves that the set of such functions is *generic*, *i.e.* is a countable intersection of open and dense subsets of $H(\mathbb{D})$ [3]. These objects echo other universal objects, such as *universal Taylor series*, which have been much studied during the last two decades: $f = \sum_k a_k z^k \in H(\mathbb{D})$ is a universal Taylor series if given any compact set $K \subset \mathbb{C} \setminus \mathbb{D}$ with connected complement, and any function g continuous on K and holomorphic in its interior, there exists an increasing sequence $(\lambda_n)_n$ of integers such that $\sum_{k=1}^{\lambda_n} a_k z^k \to g$ uniformly on K as $n \to \infty$. The existence of such functions was proven in [14]. Universal Taylor series were shown to enjoy non-tangential or radial universal properties as above, see [9] and the references therein.

In the known results, building functions with universal boundary behaviour relies on applying complex approximation theorems, like Mergelyan's theorem, and the constructions give functions with wild non-tangential behaviour near large subsets of $\mathbb{T}$. Since these subsets are not explicit, the methods do not permit to prescribe points on $\mathbb{T}$ and to build explicit functions with universal behaviour near these specific points. In this short note, we will provide with a very simple way to build functions in $H(\mathbb{D})$ having universal boundary behaviour near prescribed subsets of $\mathbb{T}$. We will not make use of either Runge's or Mergelyan's theorems, as our construction will be simply based on polynomial interpolation. In particular, it will provide us with explicit functions universal with respect to the prescribed subset of $\mathbb{T}$. Subsequently, the built functions will live in the disc algebra $A(\mathbb{D})$ – the set of analytic functions in $\mathbb{D}$ that are continuous on $\overline{\mathbb{D}}$ – and the universal approximation will be a property of their derivatives.

More precisely, the main result is as follows. We recall that a subset A of a complete metrizable topological vector space X is a dense G_δ-subset of X if it is a countable intersection of dense open sets. It is said to be *densely lineable* if it contains, except 0, a dense subspace in X, and it is called *spaceable* if it contains, except 0, an infinite dimensional closed subspace of X.

Main Theorem. *Let $(\zeta_k)_{k\in\mathbb{N}} \subset \mathbb{T}$ and $(z_n^k)_{n,k\in\mathbb{N}} \subset \mathbb{D}$ be sequences such that $z_n^k \to \zeta_k$ as $n \to \infty$, $1 \le k < \infty$. There exists a function $f \in A(\mathbb{D})$ with the following property: for any sequence $(w_k)_k \subset \mathbb{C}$, there exists an increasing sequence $(n_j)_{j\in\mathbb{N}} \subset \mathbb{N}$ such that, for any $k \ge 1$,*

$$f^{(k)}(z_{n_j}^k) \to w_k \text{ as } j \to \infty.$$

The set of such functions, denoted by $\mathcal{U}((\zeta_k), (z_n^k))$, is a dense G_δ-subset of $A(\mathbb{D})$, and is densely lineable and spaceable.

An important feature is that the sequence $(n_j)_j$ does not depend on k. Observe that the sequences of points converging to each ζ_k are arbitrary, and in particular possibly contained in a curve tangent to $\mathbb{T}$. By the Riemann mapping theorem and its refinement ([12,16]), the theorem easily extends to any bounded simply connected domain with piecewise C^∞ boundary. For such domains, it is an improvement of a result due to Siskaki [17], which asserts that, generically, any function of the disc algebra has unbounded derivatives on $\mathbb{D}$. Obviously, taking $\{\zeta_k, k \ge 1\}$ dense in $\mathbb{T}$, the elements of $\mathcal{U}((\zeta_k), (z_n^k))$ are extendable at no point of $\mathbb{T}$ and totally unbounded. Thus we recover some generic results given in [10,15] for bounded simply connected domain with smooth boundary.

Our main theorem has some operator-theoretic flavour. Indeed, if you denote by $(L_n^k)_n : A(\mathbb{D}) \mapsto \mathbb{C}$ the sequence of continuous linear maps defined by $L_n^k(f) := f^{(k)}(z_n^k)$, $k \ge 1$, this theorem can be reformulated into saying that the family $\{(L_n^k)_n, k \ge 1\}$ is *disjoint universal* in the sense of [5,7]. We may also add that the search for linear structures in sets of strange functions is a classical topic, *e.g.*, [1,6].

2. Preliminaries

The Main Theorem will be obtained as an application of general results from the theory of universality. Let Y be a separable complete metrizable topological vector space (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and Z a metrizable topological vector space (over $\mathbb{K}$), whose topologies are induced by translation-invariant metrics d_Y and ϱ, respectively. Let $L_n : Y \to Z$, $n \in \mathbb{N}$, be continuous linear mappings.

Definition 2.1. We say that $y \in Y$ is universal with respect to $(L_n)_n$ if

$$Z \subset \overline{\{L_n y : n \in \mathbb{N}\}}.$$

We denote by $\mathcal{U}(L_n)$ the set of such universal elements.

Most of the classical universality results can be viewed as applications of the following theorem.

Theorem 2.2. *(1) ([4, Theorems 26 and 27]) We assume that there exists a dense subset Y_0 of Y such that $(L_n y)_n$ converges to an element in Z for any $y \in Y_0$. Then the following are equivalent:*

(i) *$\mathcal{U}(L_n) \neq \emptyset$;*
(ii) *for any open subset $U \neq \emptyset$ of Y and any open subset $V \neq \emptyset$ of Z, there is some $n \in \mathbb{N}$ with $L_n(U) \cap V \neq \emptyset$;*

(iii) *for every $z \in Z$ and $\varepsilon > 0$, there exist $n \geq 0$ and $y \in Y$ such that*

$$\varrho\left(L_n y, z\right) < \varepsilon \text{ and } d_Y\left(y, 0\right) < \varepsilon;$$

(iv) *$\mathcal{U}\left(L_n\right)$ is a dense G_δ subset of Y.*

(2) ([4, Theorem 28 (1)]) If, for every increasing sequence $(\mu_n)_n \subset \mathbb{N}$, $\mathcal{U}\left(L_{\mu_n}\right)$ is non-empty, then $\mathcal{U}\left(L_n\right)$ contains, apart from 0, a dense subspace of Y.

Remark 2.3. When the sequence $(L_n)_n$ satisfies Condition (ii) in the above theorem, we say that $(L_n)_n$ is topologically transitive. If for every increasing sequence $(\mu_n)_n \subset \mathbb{N}$, $\mathcal{U}\left(L_{\mu_n}\right)$ is non-empty, then $(L_n)_n$ is *mixing* (see for example [4, Remark 27 (b)]).

The question of the spaceability of the set of universal elements has always been considered separately. Actually, it is more involved in general because the condition $\mathcal{U}\left(L_n\right) \neq \emptyset$ does not always imply that $\mathcal{U}\left(L_n\right)$ is spaceable. Yet some criterion of spaceability has been recently exhibited by Menet [13], under the assumption that, for every increasing sequence $(\mu_n)_n \subset \mathbb{N}$, $\mathcal{U}\left(L_{\mu_n}\right) \neq \emptyset$, as in Theorem 2.2 (2). Let us state it in a bit weaker form that will be enough to us.

Theorem 2.4. ([13, Theorem 1.11 and Remark 1.12]) *With the above notation, we assume that Y is a Fréchet space with a continuous norm. Let $(p_n)_n$ be a non-decreasing sequence of norms and $(q_n)_n$ a non-decreasing sequence of semi-norms defining the topologies of Y and Z, respectively. If $\mathcal{U}\left(L_{\mu_n}\right) \neq \emptyset$ for every increasing sequence $(\mu_n)_n \subset \mathbb{N}$, and if there exists a non-increasing sequence of infinite dimensional closed subspaces $\left(M_j\right)_j$ of Y such that for every continuous semi-norm q on Z, there exists a positive number C, an integer $k \geq 1$ and a continuous norm p on Y such that we have, for any $j \geq k$ and any $x \in M_j$,*

$$q\left(L_{n_j}(x)\right) \leq Cp(x),$$

then $\mathcal{U}\left(L_n\right)$ is spaceable.

3. Proof of the Main Theorem

The proof is based on Theorems 2.2 and 2.4. With the notations of the previous section, we set $Y = A(\mathbb{D})$, $Z = \mathbb{C}^{\mathbb{N}}$ and $Z_N = \mathbb{C}^N$, $N \geq 1$, where Z and Z_N are both endowed with the Cartesian topology. For any $n \geq 1$ and $N \geq 1$, we also set $L_n : Y \to Z$, $L_n(f) := (f^{(k)}(z_n^k))_{k \geq 1}$ and $L_n^N : Y \to Z$, $L_n^N(f) := (f^{(k)}(z_n^k))_{1 \leq k \leq N}$. By taking Y_0 as the set of all polynomials, we easily check that the assumption of Theorem 2.2 is satisfied by Y, Z, $(L_n)_n$, and Y, Z_N, $(L_n^N)_n$, $N \geq 1$ as well. We fix (ζ_k) and (z_n^k) as in the theorem and, for convenience, simply denote by $\mathcal{U}$ the set $\mathcal{U}((\zeta_k), (z_n^k))$.

We start with the topological genericity of $\mathcal{U}$. First of all, for $N \geq 1$, let $\mathcal{U}_N$ stand for the set of those functions f in $A(\mathbb{D})$ such that, for any $w_1, \ldots, w_N \in \mathbb{C}$, there exists a sequence $(n_j)_j$ such that, for any $1 \leq k \leq N$, $f^{(k)}(z_{n_j}^k) \to w_k$. Thus

$$\mathcal{U} = \bigcap_N \mathcal{U}_N.$$

Indeed, let us fix $f \in \bigcap_N \mathcal{U}_N$ and $(w_k)_k \subset \mathbb{C}$. We build by induction an increasing sequence of integers $(n_j)_j$ such that $f^{(k)}(z_{n_j}^k) \to w_k$ for any $k \geq 1$. Since $f \in \mathcal{U}_1$, there exists $n_1 \in \mathbb{N}$ such that $|f^{(1)}(z_{n_1}^1) - w_1| < 1$. We assume that $n_1, \ldots, n_j$ have been built. Since $f \in \mathcal{U}_{j+1}$, there exists $n_{j+1} \in \mathbb{N}$ so that $|f^{(k)}(z_{n_{j+1}}^k) - w_k| < \frac{1}{j+1}$ for every $1 \leq k \leq j+1$. It is now plain to check that $(n_j)_j$ is the sequence that we seek for.

Thus by the Baire Category Theorem, in order to prove the topological genericity of $\mathcal{U}$, we are reduced to prove that of $\mathcal{U}_N$ for every $N \geq 1$. We fix $(p_j^1, \ldots, p_j^N)_j$ a countable dense family in $\mathbb{C}^N$ and $\varepsilon > 0$. By Part (1) of Theorem 2.2, it is then enough to find $m_1, \ldots, m_N \in \mathbb{N}$, $c_1, \ldots, c_N \in \mathbb{C}$ and $n \in \mathbb{N}$ so that the polynomial $h := c_1 z^{m_1} + \ldots c_N z^{m_N}$ satisfies the following two:

(1) $\|h\|_\infty = \left\|c_1 z^{m_1} + \ldots c_N z^{m_N}\right\|_\infty < \varepsilon$;
(2) $\left|h^{(k)}(z_n^k) - p_j^k\right| < \frac{1}{s}$ for any $1 \leq k \leq N$.

Since $\zeta_k = \lim_n z_n^k \in \mathbb{T}$ and by continuity of $h^{(k)}$ on $\mathbb{C}$ for any $k \geq 1$, in order to satisfy (2), it is enough to find $m_1, \ldots, m_N \in \mathbb{N}$ and $c_1, \ldots, c_N \in \mathbb{C}$ such that

$$\left\{\begin{array}{rcl} h^{(1)}(\zeta_1) & = & p_j^1 \\ \vdots & & \vdots \\ h^{(N)}(\zeta_N) & = & p_j^N \end{array}\right. . \tag{3.1}$$

Since $h^{(k)}(\zeta_k) = c_1 m_1(m_1-1)\dots(m_1-k+1)\zeta_k^{m_1-k} + \dots + c_N m_N(m_N-1)\dots(m_N-k+1)\zeta_k^{m_N-k}$ (here we only consider $m_1,\dots,m_N$ each larger than N), we are thus able to solve (1) and (2) whenever we have shown that the system $M(c_1,\dots,c_N) = (b_1,\dots,b_N)$, with $b_k := p_j^k$, associated with the matrix

$$M := \begin{pmatrix} m_1\zeta_1^{m_1-1} & \dots & m_N\zeta_1^{m_N-1} \\ m_1(m_1-1)\zeta_2^{m_1-2} & \dots & m_N(m_N-1)\zeta_2^{m_N-2} \\ \vdots & & \vdots \\ m_1(m_1-1)\dots(m_1-N+1)\zeta_N^{m_1-N} & \dots & m_N(m_N-1)\dots(m_N-N+1)\zeta_N^{m_N-N} \end{pmatrix}$$

has a solution $(c_1,\dots,c_N)$ with $|c_k| < \frac{\varepsilon}{N}$. Now we have the following.

Claim. *Let $N \geq 2$ be an integer and let $(m_{k,n})_n$, $1 \leq k \leq N$, be N increasing sequences of integers such that $m_{1,n} \to \infty$ and $m_{k+1,n}/m_{k,n} \to \infty$ as $n \to \infty$ for any $1 \leq k \leq N-1$. We denote by M_n the matrix*

$$\begin{pmatrix} m_{1,n}\zeta_1^{m_{1,n}-1} & \dots & m_{N,n}\zeta_1^{m_{N,n}-1} \\ m_{1,n}(m_{1,n}-1)\zeta_2^{m_{1,n}-2} & \dots & m_{N,n}(m_{N,n}-1)\zeta_2^{m_{N,n}-2} \\ \vdots & & \vdots \\ m_{1,n}(m_{1,n}-1)\dots(m_{1,n}-N+1)\zeta_N^{m_{1,n}-N} & \dots & m_{N,n}(m_{N,n}-1)\dots(m_{N,n}-N+1)\zeta_N^{m_{N,n}-N} \end{pmatrix}.$$

Then for any $(b_1,\dots,b_N) \in \mathbb{C}^N$ and any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that the system $M_{n_0}(c_1,\dots,c_N) = (b_1,\dots,b_N)$ has a (unique) solution $(c_1,\dots,c_N)$ with $|c_k| < \frac{\varepsilon}{N}$.

Proof of the claim. Using that $|\zeta_k| = 1$, $1 \leq k \leq N$, and the assumptions on the sequences $(m_{k,n})_n$, $1 \leq k \leq N$, it is not difficult to see that the modulus of the determinant $|M_n|$ of the system $M_n(c_1,\dots,c_N) = (b_1,\dots,b_N)$ is equivalent to

$$\prod_{i=1}^{N} m_{i,n}^i$$

(this can be seen by induction on $N \geq 2$ and upon developing the determinant along the first row), which tends to ∞ as n goes to ∞. So, there exists $n_1 \in \mathbb{N}$ such that the matrix M_n is invertible for any $n \geq n_1$, *i.e.* such that, for such n, the system $M_n(c_1,\dots,c_N) = (b_1,\dots,b_N)$ has a unique solution $(c_1^{(n)},\dots,c_N^{(n)})$. This solution is given by Cramer's formulae:

$$c_k^{(n)} = \frac{\begin{vmatrix} m_{1,n}\zeta_1^{m_{1,n}-1} & \dots & b_1 & \dots & m_{N,n}\zeta_1^{m_{N,n}-1} \\ m_{1,n}(m_{1,n}-1)\zeta_2^{m_{1,n}-2} & \dots & b_2 & \dots & m_{N,n}(m_{N,n}-1)\zeta_2^{m_{N,n}-2} \\ \vdots & & \vdots & & \vdots \\ m_{1,n}(m_{1,n}-1)\dots(m_{1,n}-N+1)\zeta_N^{m_{1,n}-N} & \dots & b_N & \dots & m_{N,n}(m_{N,n}-1)\dots(m_{N,n}-N+1)\zeta_N^{m_{N,n}-N} \end{vmatrix}}{\det(M)},$$

$1 \leq k \leq N$, where at the numerator the k-th column of M have been substituted by ${}^t(b_1,\dots,b_N)$. As for $|M_n|$ (upon developing along the k-th column for example), it is easily seen that, under the assumptions of the claim, the modulus of the determinant at the numerator of c_k is equivalent as n goes to ∞ to

$$b_1 \prod_{i=1}^{k-1} m_{i,n}^{i+1} \prod_{i=k+1}^{N} m_{i,n}^i.$$

Therefore,

$$c_k^{(n)} \sim_{n\to\infty} \frac{b_1 \prod_{i=1}^{k-1} m_{i,n}}{m_{k,n}^k} \to 0,\ n \to \infty.$$

We conclude that there exists $n_0 \geq n_1$ in $\mathbb{N}$ such that $(c_1,\dots,c_N) := (c_1^{(n_0)},\dots,c_N^{(n_0)})$ satisfies the conclusion of the claim. □

To finish the proof of the topological genericity of $\mathcal{U}_N$, we fix N increasing sequences of integers such that $m_{1,n} \to \infty$ and $m_{k+1,n}/m_{k,n} \to \infty$ as $n \to \infty$ for any $1 \leq k \leq N-1$, and we choose n_0 and $(c_1,\dots,c_N)$ as in the claim. We finally set $(m_1,\dots,m_N) := (m_{1,n_0},\dots,m_{N,n_0})$ and observe that the corresponding polynomial h well satisfies (1) and (2).

We have just shown that $\mathcal{U}((\zeta_k),(z_n^k))$ is non-empty for any sequence $(z_n)_n$ converging to 1, so in particular for any sequence $(z_{\lambda_n}^k)$. Thus the dense lineability of $\mathcal{U}((\zeta_k),(z_n^k))$ follows from Theorem 2.2, Part (2). As for the spaceability, we apply Theorem 2.4. With its notation, let

$$M_j = \bigcap_{l=0}^{n_j}\bigcap_{s=0}^{n_j} \mathrm{Ker}\left(f \mapsto f^{(l)}(z_s(l))\right).$$

$(M_j)_j$ is a non-increasing sequence of closed infinite dimensional subspaces of $A(\mathbb{D})$. It is now immediate to check that the last assumption of Theorem 2.4 is satisfied.

Acknowledgements

We would like to thank the referee for his careful reading and suggestions. The content of this note was partly discussed during a Research in Pairs at CIRM Luminy in May 2017. We would like to thank the CIRM for the very good conditions of work. The first author is partly supported by the grant ANR-17-CE40-0021 of the French National Research Agency ANR (project Front).

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