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Number theory

Many values of the Riemann zeta function at odd integers are irrational

Beaucoup de valeurs aux entiers impairs de la fonction zêta de Riemann sont irrationnelles

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A R T I C L E I N F O

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A B S T R A C T

In this note, we announce the following result: at least $2^{(1-\varepsilon)\frac{\log s}{\log\log s}}$ values of the Riemann zeta function at odd integers between 3 and s are irrational, where ε is any positive real number and s is large enough in terms of ε. This improves on the lower bound $\frac{1-\varepsilon}{1+\log 2}\log s$ that follows from the Ball–Rivoal theorem. We give the main ideas of the proof, which is based on an elimination process between several linear forms in odd zeta values with related coefficients.

R É S U M É

Dans cette note, on annonce le résultat suivant : au moins $2^{(1-\varepsilon)\frac{\log s}{\log\log s}}$ valeurs de la fonction zêta de Riemann aux entiers impairs compris entre 3 and s sont irrationnelles, où ε est un réel strictement positif et s un entier impair suffisamment grand en fonction de ε. Ceci améliore la borne $\frac{1-\varepsilon}{1+\log 2}\log s$ qui découle du théorème de Ball–Rivoal. On donne les idées principales de la preuve, qui est fondée sur un procédé d'élimination entre des formes linéaires en les valeurs de zêta aux entiers impairs dont les coefficients sont reliés.

Version française abrégée

Dans cette note, on annonce le résultat suivant (voir la version anglaise pour un bref historique). On ne donne que les idées principales de la preuve (voir [5] pour la preuve complète).

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Théorème 1. *Soient $\varepsilon > 0$ et $s \geq 3$ un entier impair suffisamment grand (en fonction de ε). Alors, parmi les nombres*

$$\zeta(3),\ \zeta(5),\ \zeta(7),\ \ldots,\zeta(s),$$

au moins

$$2^{(1-\varepsilon)\frac{\log s}{\log\log s}}$$

sont irrationnels.

Soient $\varepsilon > 0$ et s un entier impair suffisamment grand (en fonction de ε). On note D le produit de tous les nombres premiers inférieurs ou égaux à $(1-2\varepsilon)\log s$, de telle sorte qu'on a $D \leq s^{1-\varepsilon}$. En suivant essentiellement [10], on considère, pour tout entier n, la fraction rationnelle

$$R_n(t) = D^{3Dn}\, n!^{s+1-3D}\ \frac{\prod_{j=0}^{3Dn}(t-n+\frac{j}{D})}{\prod_{j=0}^{n}(t+j)^{s+1}}$$

et, pour tout $j \in \{1,\ldots,D\}$, on pose

$$r_{n,j} = \sum_{m=1}^{\infty} R_n\Big(m+\frac{j}{D}\Big).$$

Rappelons que la fonction zêta de Hurwitz est définie par $\zeta(i,\alpha) = \sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^i}$ pour $\alpha > 0$ et $i \geq 2$. En développant $R_n(t)$ en éléments simples, on démontre, comme dans [10], que

$$r_{n,j} = \rho_{0,j} + \sum_{\substack{3\leq i\leq s\\ i\text{ impair}}} \rho_i\,\zeta\Big(i,\frac{j}{D}\Big).$$

En outre, en notant d_n le plus petit commun multiple des n premiers entiers, on montre que d_{n+1}^{s+1} est un multiple commun des dénominateurs des nombres rationnels $\rho_{0,j}$ et ρ_i.

Le point crucial, comme dans [10], est l'identité suivante, valable pour tout diviseur d de D et pour tout entier $i \geq 2$:

$$\sum_{j=1}^{d}\zeta\Big(i,\frac{j\frac{D}{d}}{D}\Big) = \sum_{j=1}^{d}\zeta\Big(i,\frac{j}{d}\Big) = \sum_{n=0}^{\infty}\sum_{j=1}^{d}\frac{d^i}{(dn+j)^i} = d^i\zeta(i).$$

En effet, cette relation montre qu'en posant $\widehat{r}_{n,d} = \sum_{j=1}^{d} r_{n,j\frac{D}{d}}$, on a :

$$\widehat{r}_{n,d} = \sum_{j=1}^{d}\rho_{0,j\frac{D}{d}} + \sum_{\substack{3\leq i\leq s\\ i\text{ impair}}}\rho_i\, d^i\,\zeta(i).$$

Notons $\mathcal{D}$ l'ensemble des diviseurs de D. Pour tout $d \in \mathcal{D}$ et pour tout entier n, on a donc une combinaison linéaire $\widehat{r}_{n,d}$ de 1, $\zeta(3)$, $\zeta(5)$, …, $\zeta(s)$. Pour tout entier i impair compris entre 3 et s, le coefficient de $\zeta(i)$ est $\rho_i\, d^i$ (où le nombre rationnel ρ_i dépend implicitement de n, mais pas de d). Si des entiers w_d, $d \in \mathcal{D}$, vérifient $\sum_{d\in\mathcal{D}} w_d\, d^i = 0$ pour un certain i, alors la combinaison linéaire

$$\widetilde{r}_n = \sum_{d\in\mathcal{D}} w_d\,\widehat{r}_{n,d} \tag{0.1}$$

ne fait plus apparaître $\zeta(i)$. Le point central de ce procédé d'élimination (mis au point par le troisième auteur [12] pour $D = 2$, et généralisé par le deuxième auteur [10]) est que le coefficient d^i dépend de d, mais pas de n : on peut choisir les entiers w_d indépendamment de n.

Pour démontrer le théorème 1, on suppose que, parmi $\zeta(3)$, $\zeta(5)$, …, $\zeta(s)$, le nombre d'irrationnels est inférieur à $2^{(1-3\varepsilon)\frac{\log s}{\log\log s}}$; il est alors inférieur ou égal à $\delta - 1$, où $\delta = \mathrm{Card}\,\mathcal{D}$ est le nombre de diviseurs de D. Il existe donc des indices impairs $i_1 < i_2 < \ldots < i_{\delta-1}$ compris entre 3 et s tels que toute valeur irrationnelle $\zeta(i)$, avec i impair compris entre 3 et s, soit l'une des $\zeta(i_j)$. On peut choisir des entiers relatifs w_d non tous nuls (pour $d \in \mathcal{D}$) tels que

$$\sum_{d\in\mathcal{D}} w_d\, d^{i_j} = 0 \quad \text{pour tout } j \in \{1,\ldots,\delta-1\}. \tag{0.2}$$

La relation (0.1) définit alors une suite $(\widetilde{r}_n)_{n\geq 1}$ de nombres réels qui sont des combinaisons linéaires à coefficients rationnels de 1 et des $\zeta(i)$ pour i impair, $3 \leq i \leq s$. La propriété cruciale est que les $\zeta(i_j)$ ne figurent plus dans ces combinaisons linéaires, si bien que tous les $\zeta(i)$ qui y apparaissent avec un coefficient non nul sont, par hypothèse, des nombres rationnels. En multipliant par un dénominateur commun A de ces nombres rationnels, et aussi par d_{n+1}^{s+1} (qui est un dénominateur commun des coefficients ρ_i et $\rho_{0,j}$), on obtient une suite d'entiers relatifs. Une étude asymptotique des suites $(r_{n,j})_{n\geq 1}$ montre que cette suite d'entiers relatifs tend vers 0, donc est identiquement nulle à partir d'un certain rang ; précisément, on a, quand $n \to \infty$:

$$Ad_{n+1}^{s+1}\widetilde{r}_n = \left(\sum_{d\in\mathcal{D}} w_d\, d + o(1)\right) Ad_{n+1}^{s+1} r_{n,1}, \quad \text{avec } 0 < \lim_{n\to\infty} \left(Ad_{n+1}^{s+1} r_{n,1}\right)^{1/n} < 1$$

où $o(1)$ est une suite qui tend vers 0 quand n tend vers l'infini. Cela impose $\sum_{d\in\mathcal{D}} w_d\, d = 0$. Autrement dit, toute famille $(w_d)_{d\in\mathcal{D}}$ d'entiers relatifs vérifiant (0.2) devrait vérifier aussi $\sum_{d\in\mathcal{D}} w_d\, d = 0$. Ce n'est pas le cas, car la matrice $[d^{i_j}]_{d\in\mathcal{D},\, 0\leq j\leq\delta-1}$ (dans laquelle on pose $i_0 = 1$) est inversible : son déterminant est le produit d'un déterminant de Vandermonde et d'un polynôme de Schur, qui est un polynôme à coefficients entiers naturels évalué en la famille des diviseurs de D.

1. Introduction

When $s \geq 2$ is an even integer, the value $\zeta(s)$ of the Riemann zeta function is a non-zero rational multiple of π^s and, therefore, a transcendental number. On the other hand, very few results are known on the zeta values $\zeta(s)$ when $s \geq 3$ is odd, though we expect them all to be transcendental.

It was only in 1978 when Apéry astonished the mathematics community by his proof [1] of the irrationality of $\zeta(3)$, with the next breakthrough in this direction taken in 2000 by Ball and Rivoal [2,8], who proved the following theorem.

Theorem 1 *(Ball–Rivoal). Let $\varepsilon > 0$, and $s \geq 3$ be an odd integer sufficiently large with respect to ε. Then*

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}(1,\, \zeta(3),\, \zeta(5),\, \zeta(7),\, \ldots,\, \zeta(s)) \geq \frac{1-\varepsilon}{1+\log 2}\log s.$$

In spite of several refinements [2,11,4] for small s, the lower bound in Theorem 1 has never been improved for large values of s. The proof of Theorem 1 applies Nesterenko's linear independence criterion [7] to certain linear combinations of odd zeta values. To improve on this bound using the same strategy, one has to find linear combinations that are considerably smaller, with not too large coefficients — this comes out to be a rather difficult task.

The situation has drastically changed when the third author introduced [12] a new method, which has been generalized by the second author in [10]. For a given integer $D > 1$ and a certain rational function $R(t)$, the series

$$\sum_{j=1}^{d}\sum_{t=1}^{\infty} R\left(t + \frac{j}{d}\right), \quad \text{where } d \mid D,$$

gives $\mathbb{Q}$-linear combinations of $1, d^3\zeta(3), d^5\zeta(5), \ldots, d^s\zeta(s)$ with coefficients independent of d (except that of 1). This makes it possible to eliminate from the entire collection of these linear combinations essentially as many odd zeta values as the number of divisors of D. For applications of this idea, see [12,6,10,9].

In this note, we sketch the proof of the following result, building upon the approach in [12] and [10]. We refer the interested reader to the full version [5] of the paper for details.

Theorem 2. *Let $\varepsilon > 0$, and $s \geq 3$ be an odd integer sufficiently large with respect to ε. Then among the numbers*

$$\zeta(3),\ \zeta(5),\ \zeta(7),\ \ldots,\ \zeta(s),$$

at least

$$2^{(1-\varepsilon)\frac{\log s}{\log\log s}}$$

are irrational.

In comparison, Theorem 1 gives only $\frac{1-\varepsilon}{1+\log 2}\log s$ irrational odd zeta values, but they are linearly independent over the rationals, whereas Theorem 2 ends up only with their irrationality.

Our proof of Theorem 2 follows the above-mentioned strategy from [12] and [10]. The main new ingredient, compared to the proof in [10], is taking D large (about $s^{1-\varepsilon}$) and equal to the product of the first prime numbers (the so-called primorial) — such a number has asymptotically the largest possible number of divisors with respect to its size.

2. Small linear forms in values of the Hurwitz zeta function

Let s and D be positive integers such that s is odd and $s \geq 3D$. Let n be a positive integer, such that Dn is even. We consider the following rational function:

$$R_n(t) = D^{3Dn}\, n!^{s+1-3D}\, \frac{\prod_{j=0}^{3Dn}(t-n+\frac{j}{D})}{\prod_{j=0}^{n}(t+j)^{s+1}}$$

which, of course, depends also on s and D, and for $j \in \{1, \dots, D\}$ we let

$$r_{n,j} = \sum_{m=1}^{\infty} R_n\Big(m + \frac{j}{D}\Big).$$

We write d_n for $\mathrm{lcm}(1, 2, \dots, n)$. Expanding $R_n(t)$ into partial fractions yields the following $\mathbb{Q}$-linear combinations of values of the Hurwitz zeta function.

Lemma 1. *For each $j \in \{1, \dots, D\}$, we have*

$$r_{n,j} = \rho_{0,j} + \sum_{\substack{3\leq i\leq s \\ i \text{ odd}}} \rho_i\, \zeta\Big(i, \frac{j}{D}\Big)$$

with

$$d_n^{s+1-i}\rho_i \in \mathbb{Z} \quad \text{for } i \in \{3, 5, \dots, s\} \quad \text{and} \quad d_{n+1}^{s+1}\rho_{0,j} \in \mathbb{Z} \quad \text{for any } j \in \{1, \dots, D\}.$$

We point out that the coefficient ρ_i of $\zeta(i, \frac{j}{D})$ in the linear form $r_{n,j}$ does not depend on j. The expansion of $r_{n,j}$ is very classical; only odd zeta values appear because of the symmetry phenomenon from [2]. The proof that $d_n^{s+1-i}\rho_i \in \mathbb{Z}$ for odd i follows that of [3, Lemma 4.5]. The last assertion, namely $d_{n+1}^{s+1}\rho_{0,j} \in \mathbb{Z}$, is proved as in [10, Lemma 1.4].

An important feature of the linear forms $r_{n,j}$ is that they are simultaneously very small: even when multiplied by a common denominator of the rational coefficients, they still tend to 0.

Lemma 2. *Assume that $\frac{s}{D\log D}$ larger than some effectively computable absolute constant. Then we have*

$$\lim_{n\to\infty} r_{n,j}^{1/n} = g(x_0) < 3^{-(s+1)} \quad \text{and} \quad \lim_{n\to\infty} \frac{r_{n,j'}}{r_{n,j}} = 1 \quad \text{for any } j, j' \in \{1, \dots, D\},$$

where

$$g(x) = D^{3D}\, \frac{(x+3)^{3D}(x+1)^{s+1}}{(x+2)^{2(s+1)}}$$

and x_0 is the unique positive root of the polynomial $(X+3)^D(X+1)^{s+1} - X^D(X+2)^{s+1}$.

The following elementary lemma about the non-vanishing of generalized Vandermonde determinants is used to get rid of unwanted irrational odd zeta values (in §3 below).

Lemma 3. *For $t \geq 1$, let $x_1, \dots, x_t$ be pairwise distinct positive real numbers and $\alpha_1, \dots, \alpha_t$ pairwise distinct non-negative integers. Then the matrix $[x_j^{\alpha_i}]_{1\leq i,j\leq t}$ has non-zero determinant.*

3. Elimination of odd zeta values

Take $0 < \varepsilon < \frac{1}{3}$, and let s be odd and sufficiently large with respect to ε. We let D be the product of all primes less than or equal to $(1-2\varepsilon)\log s$, so that $D \leq s^{1-\varepsilon}$. Notice that D has $\delta = 2^{\pi((1-2\varepsilon)\log s)}$ divisors, with $\log\delta \geq (1-3\varepsilon)(\log 2)\frac{\log s}{\log\log s}$. Assume that the number of irrational odd zeta values between $\zeta(3)$ and $\zeta(s)$ is less than or equal to $\delta - 1$. Let $3 = i_1 < i_2 < \ldots < i_{\delta-1} \leq s$ be odd integers such that if $\zeta(i) \notin \mathbb{Q}$ and i is odd, $3 \leq i \leq s$, then $i = i_j$ for some j. Moreover, we let $i_0 = 1$, and consider the set $\mathcal{D}$ of all divisors of D, so that $\mathrm{Card}\,\mathcal{D} = \delta$. Lemma 3 shows that the matrix $[d^{i_j}]_{d\in\mathcal{D},0\leq j\leq\delta-1}$ is invertible. Therefore, there exist integers $w_d \in \mathbb{Z}$, where $d \in \mathcal{D}$, such that

$$\sum_{d\in\mathcal{D}} w_d\, d^{i_j} = 0 \quad \text{for any } j \in \{1, \dots, \delta-1\} \quad \text{and} \quad \sum_{d\in\mathcal{D}} w_d\, d \neq 0. \tag{3.1}$$

The crucial point (as in [10, §3]) is that for any $d \in \mathcal{D}$ and any $i \geq 2$,

$$\sum_{j=1}^{d} \zeta\left(i, \frac{j\frac{D}{d}}{D}\right) = \sum_{j=1}^{d} \zeta\left(i, \frac{j}{d}\right) = \sum_{n=0}^{\infty}\sum_{j=1}^{d} \frac{d^i}{(dn+j)^i} = d^i \zeta(i).$$

Lemma 1 implies that the quantities $\widehat{r}_{n,d} = \sum_{j=1}^{d} r_{n,j\frac{D}{d}}$ are linear forms in the odd zeta values:

$$\widehat{r}_{n,d} = \sum_{j=1}^{d} \rho_{0,j\frac{D}{d}} + \sum_{\substack{3 \leq i \leq s \\ i \text{ odd}}} \rho_i \, d^i \, \zeta(i), \tag{3.2}$$

while Lemma 2 leads to the following asymptotics:

$$\widehat{r}_{n,d} = (d + o(1)) r_{n,1} \quad \text{with } \lim_{n\to\infty} r_{n,1}^{1/n} = g(x_0). \tag{3.3}$$

We shall use now the integers w_d to eliminate $\delta - 1$ odd zeta values, including all irrational ones, as in [12] and [10]. For that, introduce $\widetilde{r}_n = \sum_{d\in\mathcal{D}} w_d \widehat{r}_{n,d}$. Eqns. (3.1)–(3.3) imply that

$$\widetilde{r}_n = \left(\sum_{d\in\mathcal{D}} w_d \sum_{j=1}^{d} \rho_{0,j\frac{D}{d}}\right) + \sum_{i\in I} \rho_i \left(\sum_{d\in\mathcal{D}} w_d \, d^i\right) \zeta(i) = \left(\sum_{d\in\mathcal{D}} w_d \, d + o(1)\right) r_{n,1},$$

where $I = \{3, 5, 7, \ldots, s\} \setminus \{i_1, \ldots, i_{\delta-1}\}$. In particular, no irrational zeta value $\zeta(i)$, where $3 \leq i \leq s$, appears in this linear combination, and $\lim_{n\to\infty} |\widetilde{r}_n|^{1/n} = g(x_0) < 3^{-(s+1)}$ since $\sum_{d\in\mathcal{D}} w_d \, d \neq 0$. Denoting by A a common denominator of the (rational) numbers $\zeta(i)$, $i \in I$, we deduce from Lemma 1 that $A d_{n+1}^{s+1} \widetilde{r}_n$ is an integer. Now the prime number theorem implies that $d_{n+1}^{1/n} \to e$ as $n \to \infty$, hence from Lemma 2 we conclude that the sequence of integers satisfies

$$0 < \lim_{n\to\infty} |A d_{n+1}^{s+1} \widetilde{r}_n|^{1/n} = e^{s+1} g(x_0) < \left(\frac{e}{3}\right)^{s+1} < 1.$$

This contradiction implies the truth of Theorem 2.

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