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Functional analysis

A norm inequality for positive block matrices

Une inégalité de norme pour les matrices positives écrites par blocs

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ABSTRACT

Any positive matrix $M=(M_{i,j})_{i,j=1}^m$ with each block $M_{i,j}$ square satisfies the symmetric norm inequality $\|M\| \le \|\sum_{i=1}^m M_{i,i} + \sum_{i=1}^{m-1} \omega_i I\|$, where ω_i $(i=1,\dots,m-1)$ are quantities involving the width of numerical ranges. This extends the main theorem of Bourin and Mhanna (2017) [4] to a higher number of blocks.

RÉSUMÉ

Toute matrice positive $M=(M_{i,j})_{i,j=1}^m$ écrite en blocs carrés $M_{i,j}$ satisfait $\|M\| \le \|\sum_{i=1}^m M_{i,i} + \sum_{i=1}^{m-1} \omega_i I\|$, où les quantités ω_i, $i=1,\dots,m-1$, font intervenir la largeur du domaine des valeurs numériques. Ceci étend le théorème principal de Bourin, Mhanna (2017) [4] aux matrices écrites avec un nombre de blocs arbitraire.

1. Introduction

Bourin and Mhanna recently obtained a novel norm inequality for positive block matrices.

Theorem 1.1. *[4] Let* $M=\begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{1,2} & M_{2,2} \end{pmatrix}$ *be a positive matrix with each block square. Then for all symmetric norms*

$$\|M\| \le \|M_{1,1}+M_{2,2}+\omega I\|,$$

where ω is the width of the numerical range of $M_{1,2}$.

The numerical range (or the field of values [7]) is a convex set on the complex plane. By the width of a numerical range, we mean the smallest possible ω such that the numerical range is contained in a strip of width ω. In particular, if the

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numerical range of $M_{1,2}$ is a line segment (this happens, for example, when $M_{1,2}$ is Hermitian or skew-Hermitian), then the previous theorem gives (see [8, Theorem 2.6])

$$\|M\| \le \|M_{1,1}+M_{2,2}\|. \tag{1}$$

To the author's best knowledge, Mhanna's study [8] provides the first example for (1) to be true without the PPT (i.e. positive partial transpose) condition. We refer to [6,2] for some motivational background.

Bourin and Mhanna's proof of Theorem 1.1 makes use of a useful decomposition for 2×2 positive block matrices [1, Lemma 3.4]. Their approach seems difficult for an extension to a higher number of blocks, as remarked in their paper [4]. It is the purpose of the present paper to provide such an extension. Before closing this section, we fix some notation. The set of $m\times n$ complex matrices is denoted by $\mathbb{M}_{m\times n}$ and we use $\mathbb{M}_n$ for $\mathbb{M}_{n\times n}$. The $n\times n$ identity matrix is denoted by I. The Hermitian part of $A\in\mathbb{M}_n$ is $\Re A := (A+A^*)/2$. For two Hermitian matrices $A, B\in\mathbb{M}_n$, we write $A\ge B$ to mean $A-B$ is positive semidefinite. The numerical range of A is denoted by $W(A)$. If $A, B\in\mathbb{M}_n$, then we write $W(A\pm B)$ to mean $W(A+B)$ and $W(A-B)$. It is useful to notice that if the width of $W(A)$ is ω, then one can find a $\theta\in[0,\pi]$ such that

$$rI \le \Re(e^{i\theta}A) \le (r+\omega)I$$

for some $r\in\mathbb{R}$. We refer the reader to Chapter 1 of [7] for basic properties of the numerical range for matrices.

2. Main result

Our extension of Theorem 1.1 to a higher number of blocks is as follows.

Theorem 2.1. *Let $M=(M_{i,j})_{i,j=1}^m$ be a positive matrix with each block $M_{i,j}\in\mathbb{M}_n$. Then for all symmetric norms,*

$$\|M\| \le \Big\|\sum_{i=1}^m M_{i,i} + \sum_{i=1}^{m-1}\omega_i I\Big\|,$$

where ω_i ($i=1,\dots,m-1$) is the average of the widths of $W(M_{i,i+1}\pm M_{i,i+2}\pm\cdots\pm M_{i,m})$.

Proof. By Fan's dominance theorem [7, p. 206], it suffices to show that the inequality is true for the Ky Fan norms $\|\cdot\|_k$, $k=1,\dots,n$. The proof is by induction. The base case $m=2$, i.e. Theorem 1.1 was treated in [4]. We include a proof for completeness. The presentation is slightly different from that in [4]. As M is positive, we may write $M=\begin{pmatrix}X^*\\ Y^*\end{pmatrix}\begin{pmatrix}X & Y\end{pmatrix}$ for some $X, Y\in\mathbb{M}_{2n\times n}$ so that $M_{1,1}=X^*X$, $M_{1,2}=X^*Y$, $M_{2,2}=Y^*Y$. Clearly, $\|M\|_k=\|XX^*+YY^*\|_k$. As the norm of M is invariant if we replace Y with $e^{i\theta}Y$, we may assume that $rI\le\Re(X^*Y)\le(r+\omega)I$ for some $r\in\mathbb{R}$ and that ω is the width of $W(M_{1,2})$. Compute

$$\begin{aligned}
\|M\|_k &= \frac{1}{2}\|(X+Y)(X+Y)^*+(X-Y)(X-Y)^*\|_k\\
&\le \frac{1}{2}\Big(\|(X+Y)(X+Y)^*\|_k+\|(X-Y)(X-Y)^*\|_k\Big)\\
&= \frac{1}{2}\Big(\|(X+Y)^*(X+Y)\|_k+\|(X-Y)^*(X-Y)\|_k\Big)\\
&\le \frac{1}{2}\Big(\|X^*X+Y^*Y+2(r+\omega)I\|_k+\|X^*X+Y^*Y-2rI\|_k\Big)\\
&= \|X^*X+Y^*Y+\omega I\|_k = \|M_{1,1}+M_{2,2}+\omega I\|_k.
\end{aligned}$$

This completes the proof of the base case. Suppose the asserted inequality is true for $m=\ell$ for some $\ell>2$. Then we consider the $m=\ell+1$ case. In this case, M could be written in the form

$$M=\begin{pmatrix}
X_1^*X_1 & \cdots & X_1^*X_\ell & X_1^*X_{\ell+1}\\
\vdots & & \vdots & \vdots\\
X_\ell^*X_1 & \cdots & X_\ell^*X_\ell & X_\ell^*X_{\ell+1}\\
X_{\ell+1}^*X_1 & \cdots & X_{\ell+1}^*X_\ell & X_{\ell+1}^*X_{\ell+1}
\end{pmatrix}=\begin{pmatrix}X_1^*\\ \vdots\\ X_\ell^*\\ X_{\ell+1}^*\end{pmatrix}\begin{pmatrix}X_1 & \cdots & X_\ell & X_{\ell+1}\end{pmatrix}$$

for some $X, Y\in\mathbb{M}_{(\ell+1)n\times n}$. Again, we assume (by multiplying $X_{\ell+1}$ with a rotation unit) that

$$sI\le\Re(X_\ell^*X_{\ell+1})\le(s+\omega_\ell)I \tag{2}$$

for some $s\in\mathbb{R}$.

Consider the following $\ell \times \ell$ block positive matrices

$$M_1 = \begin{pmatrix} X_1^* \\ \vdots \\ X_{\ell-1}^* \\ (X_\ell + X_{\ell+1})^* \end{pmatrix} \begin{pmatrix} X_1 & \cdots & X_{\ell-1} & X_\ell + X_{\ell+1} \end{pmatrix}$$

and

$$M_2 = \begin{pmatrix} X_1^* \\ \vdots \\ X_{\ell-1}^* \\ (X_\ell - X_{\ell+1})^* \end{pmatrix} \begin{pmatrix} X_1 & \cdots & X_{\ell-1} & X_\ell - X_{\ell+1} \end{pmatrix}.$$

Let α_i $(i = 1, \dots, \ell-2)$ be the average of the widths of

$$W(X_i^* X_{i+1} \pm \cdots \pm X_i^* X_{\ell-1} \pm X_i^*(X_\ell + X_{\ell+1})),$$

and let $\alpha_{\ell-1}$ be the width of $W(X_{\ell-1}^*(X_\ell + X_{\ell+1}))$. Similarly, let β_i $(i = 1, \dots, \ell-2)$ be the average of the widths of

$$W(X_i^* X_{i+1} \pm \cdots \pm X_i^* X_{\ell-1} \pm X_i^*(X_\ell - X_{\ell+1})),$$

and let $\beta_{\ell-1}$ be the width of $W(X_{\ell-1}^*(X_\ell - X_{\ell+1}))$. We observe that

$$\omega_i = \frac{\alpha_i + \beta_i}{2}, \qquad i = 1, \dots, \ell-1.$$

By the inductive hypothesis,

$$\|M_1\|_k \le \| \sum_{i=1}^{\ell-1} X_i^* X_i + (X_\ell + X_{\ell+1})^*(X_\ell + X_{\ell+1}) + \sum_{i=1}^{\ell-1} \alpha_i I \|_k,$$
$$\|M_2\|_k \le \| \sum_{i=1}^{\ell-1} X_i^* X_i + (X_\ell - X_{\ell+1})^*(X_\ell - X_{\ell+1}) + \sum_{i=1}^{\ell-1} \beta_i I \|_k.$$

Furthermore, by (2), we have

$$\|M_1\|_k \le \| \sum_{i=1}^{\ell+1} X_i^* X_i + 2(s + \omega_\ell) I + \sum_{i=1}^{\ell-1} \alpha_i I \|_k,$$
$$\|M_2\|_k \le \| \sum_{i=1}^{\ell+1} X_i^* X_i - 2sI + \sum_{i=1}^{\ell-1} \beta_i I \|_k.$$

Now we proceed to estimating the norm of M,

$$\begin{aligned}
\|M\|_k &= \|X_1 X_1^* + \cdots + X_{\ell-1} X_{\ell-1}^* + X_\ell X_\ell^* + X_{\ell+1} X_{\ell+1}^*\|_k \\
&= \left\| X_1 X_1^* + \cdots + X_{\ell-1} X_{\ell-1}^* + \frac{1}{2}(X_\ell + X_{\ell+1})(X_\ell + X_{\ell+1})^* + \frac{1}{2}(X_\ell - X_{\ell+1})(X_\ell - X_{\ell+1})^* \right\|_k \\
&\le \frac{1}{2} \|X_1 X_1^* + \cdots + X_{\ell-1} X_{\ell-1}^* + (X_\ell + X_{\ell+1})(X_\ell + X_{\ell+1})^*\|_k \\
&\quad + \frac{1}{2} \|X_1 X_1^* + \cdots + X_{\ell-1} X_{\ell-1}^* + (X_\ell - X_{\ell+1})(X_\ell - X_{\ell+1})^*\|_k \\
&= \frac{1}{2}(\|M_1\|_k + \|M_2\|_k) \\
&\le \frac{1}{2} \left\| \sum_{i=1}^{\ell+1} X_i^* X_i + 2(s + \omega_\ell) I + \sum_{i=1}^{\ell-1} \alpha_i I \right\|_k + \frac{1}{2} \left\| \sum_{i=1}^{\ell+1} X_i^* X_i - 2sI + \sum_{i=1}^{\ell-1} \beta_i I \right\|_k \\
&= \left\| \sum_{i=1}^{\ell+1} X_i^* X_i + \sum_{i=1}^{\ell} \omega_i I \right\|_k = \left\| \sum_{i=1}^{\ell+1} M_{i,i} + \sum_{i=1}^{\ell} \omega_i I \right\|_k .
\end{aligned}$$

Thus the asserted inequality is true for $m = \ell + 1$, so the proof of induction step is complete. □

The previous theorem contains an important special case of Hiroshima's theorem [2, Corollary 2.2] when all off-diagonal blocks are Hermitian.

Corollary 2.2. *Let $M = (M_{i,j})_{i,j=1}^m$ be a positive matrix with each block $M_{i,j} \in \mathbb{M}_n$. If all off-diagonal blocks are Hermitian or all off-diagonal blocks are skew-Hermitian, then for all symmetric norms,*

$$\|M\| \le \|\sum_{i=1}^m M_{i,i}\|.$$

For general $X, Y \in \mathbb{M}_n$, it is clear that $W(X+Y) \subset W(X) + W(Y)$, but there is no simple comparison relation between the width of $W(X+Y)$ and the widths of $W(X)$ and $W(Y)$, for example, when X is Hermitian and Y is skew-Hermitian. Therefore, it seems interesting to capture a special case of Theorem 2.1 for block tridiagonal matrices.

Corollary 2.3. *Let* $A = \begin{pmatrix} A_1 & X_1 & 0 & \cdots & 0 \\ X_1^* & A_2 & X_2 & & \vdots \\ 0 & X_2^* & \ddots & \ddots & 0 \\ \vdots & & \ddots & A_{m-1} & X_{m-1} \\ 0 & \cdots & 0 & X_{m-1}^* & A_m \end{pmatrix}$ *be a positive matrix with each block in $\mathbb{M}_n$. Then for all symmetric norms*

$$\|A\| \le \|\sum_{i=1}^m A_i + \sum_{i=1}^{m-1} \omega_i I\|,$$

where ω_i are widths of $W(X_i)$, $i = 1, \dots, m-1$.

In [3], it was conjectured that, if $M_{1,2}$ in the positive 2×2 block matrix M is normal, then one still has (1). The conjecture was denied for $n \ge 3$; see [2] and [5]. Mhanna [8, Remark 2.7] pointed out, however, under the normality assumption (1) is true when $n = 2$. This is because the numerical range of a normal matrix coincides with the convex hull of its eigenvalues [7, Property 1.2.9], and so the width of the numerical range of a 2×2 normal matrix is zero. Since the sum of normal matrices is no longer normal in general, e.g., $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We have been unable to answer the following question.

Question 2.4. Let $M = (M_{i,j})_{i,j=1}^m$ be a positive matrix with each $M_{i,j} \in \mathbb{M}_2$ being normal. Is it true that for all symmetric norms

$$\|M\| \le \|\sum_{i=1}^m M_{i,i}\|?$$

Clearly, for Question 2.4, it suffices to show the inequality is true for the usual operator norm. On the other hand, we notice Question 2.4 would be implied by an affirmative answer to the following question, which is evidenced by Corollary 2.3.

Question 2.5. Let $M = (M_{i,j})_{i,j=1}^m$ be a positive matrix with each block $M_{i,j} \in \mathbb{M}_n$. Is it true that for all symmetric norms

$$\|M\| \le \|\sum_{i=1}^m M_{i,i} + \sum_{i<j} \omega_{i,j} I\|,$$

where $\omega_{i,j}$ are widths of $W(M_{i,j})$, $1 \le i < j \le m$?

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