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Combinatorics

Symmetries on plabic graphs and associated polytopes

Symétries dans les graphes plan bicolores et les polytopes associés

Xin Fang [a], Ghislain Fourier [b]

[a] *University of Cologne, Mathematical Institute, Weyertal 86–90, 50931 Cologne, Germany*
[b] *Leibniz Universität Hannover, Institute for Algebra, Number Theory and Discrete Mathematics, Welfengarten 1, 30167 Hannover, Germany*

A R T I C L E I N F O

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A B S T R A C T

For Grassmann varieties, we explain how the duality between the Gelfand–Tsetlin polytopes and the Feigin–Fourier–Littelmann–Vinberg polytopes arises from different positive structures.

R É S U M É

Nous expliquons, pour les variétés grasmanniennes, comment la dualité entre les polytopes de Gelfand–Tsetlin et les polytopes de Feigin–Fourier–Littelman–Vinberg émerge dans différentes structures positives.

1. Introduction

Plabic graphs (planar bicoloured graphs) were introduced by Postnikov [8] to parametrize cells in the totally non-negative (TNN) Grassmannians $(\mathrm{Gr}_{k,n}(\mathbb{R}))_{\geq 0}$. These graphs are drawn inside a disk with boundary vertices labelled by $1, 2, \dots, n$ in a fixed orientation and internal vertices coloured black and white. For a reduced plabic graph $\mathcal{G}$ corresponding to the top cell in the TNN-Grassmannian $(\mathrm{Gr}_{n-k,n}(\mathbb{R}))_{\geq 0}$, Rietsch and Williams [10] constructed a family of polytopes for positive integers r as Newton–Okounkov bodies [5,7] associated with the line bundle $r \in \mathbb{Z} \cong \mathrm{Pic}(\mathrm{Gr}_{n-k,n}(\mathbb{C}))$.

When the plabic graph $\mathcal{G} := \mathcal{G}^{\mathrm{rec}}_{k,n}$ is chosen as in [10] (see Section 4.2), the corresponding Newton–Okounkov body $\mathrm{NO}_{\mathcal{G}}$ is unimodularly equivalent to the Gelfand–Tsetlin polytope $\mathrm{GT}^1_{n-k,n}$.

The Newton–Okounkov body is by definition a closed convex hull of points; even when it is a polytope, to read off its defining inequalities is a hard problem. In [10], the authors used mirror symmetry of Grassmannians to obtain these inequalities from the tropicalization of the super-potential on an open set of the mirror Grassmannian arising from the Landau–Ginzburg model. By applying this symmetry, they give explicit defining inequalities of $\mathrm{NO}_{\mathcal{G}}$.

Lattice points in Gelfand–Tsetlin polytopes parametrize the bases of finite-dimensional irreducible representations of the Lie algebra $\mathfrak{sl}_n$. Motivated by a conjecture of Vinberg, another family of polytopes, called FFLV polytopes, is found by Feigin,

E-mail addresses: xinfang.math@gmail.com (X. Fang), fourier@math.uni-hannover.de (G. Fourier).

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the second author, and Littelmann [3], whose lattice points also parametrize the bases of finite- dimensional irreducible representations of $\mathfrak{sl}_n$.

For a plabic graph $\mathcal{G}$, its mirror $\mathcal{G}^\vee$ is defined by swapping the black/white colouring of internal vertices in $\mathcal{G}$. When the plabic graph $\mathcal{G}$ corresponds to the top cell in $(\mathrm{Gr}_{n-k,n}(\mathbb{R}))_{\geq 0}$, $\mathcal{G}^\vee$ parametrizes the top cell in $(\mathrm{Gr}_{k,n}(\mathbb{R}))_{\geq 0}$.

Theorem 1. *The Newton–Okounkov body $NO_{\mathcal{G}^\vee}$ is unimodularly equivalent to* $\mathrm{FFLV}^1_{k,n}$ *(see Section 4.1 for definition).*

Another way to relate Gelfand–Tsetlin polytopes to FFLV polytopes is via a connection between the corresponding clusters in different cluster algebras. Each reduced plabic graph $\mathcal{G}$ gives a cluster $\mathcal{C}$ consisting of Plücker coordinates $\Delta_{I_1}, \ldots, \Delta_{I_m}$ where $I_1, \ldots, I_m$ are some $(n-k)$-element subsets of $[n] = \{1, 2, \ldots, n\}$.

For $I \subset [n]$, let I^c denote its complement. Then the set $\mathcal{C}' = \{\Delta_{I_1^c}, \ldots, \Delta_{I_m^c}\}$ is a cluster for $\mathrm{Gr}_{k,n}(\mathbb{C})$, corresponding to a plabic graph $\mathcal{G}^\vee$.

Corollary 1. *The Newton–Okounkov body $NO_{\mathcal{G}^\vee}$ is unimodularly equivalent to* $\mathrm{FFLV}^1_{k,n}$.

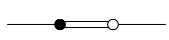
2. Plabic graphs

We recall the definition and basic properties of plabic graphs, following [8,10].

Definition 1. A *plabic graph* is an undirected planar graph $\mathcal{G}$ satisfying:

(1) $\mathcal{G}$ is embedded in a closed disk and considered up to homotopy;
(2) $\mathcal{G}$ has n vertices on the boundary of the disk, called *boundary vertices*, which are labelled clockwise by $1, 2, \ldots, n$;
(3) all other vertices of $\mathcal{G}$ are strictly inside the disk, they are called *internal vertices* and coloured in black and white;
(4) each boundary vertex is incident to a single edge.

In [8] (see also [10]), there are three *local moves* defined on plabic graphs: gluing two vertices of the same colour, removing redundant vertices, and mutating a square. For a plabic graph $\mathcal{G}$, let $\mathcal{F}(\mathcal{G})$ denote the set of its faces, which is invariant under the local moves.

Definition 2. A plabic graph $\mathcal{G}$ is called *reduced* if there are no parallel edges  after applying any sequences of local moves.

Definition 3. Let $\mathcal{G}$ be a reduced plabic graph. The *trip* T_i starting from a boundary vertex i is the path going through the edges of $\mathcal{G}$, obeying the following rules:

(1) at each internal black vertex, the path turns to the rightmost direction;
(2) at each internal white vertex, the path turns to the leftmost direction.

The trip T_i ends at a boundary vertex $\pi(i)$. We associate in this way a *trip permutation* $\pi_{\mathcal{G}} := (\pi(1), \ldots, \pi(n))$ with $\mathcal{G}$. Let $\pi_{k,n} = (n-k+1, n-k+2, \ldots, n, 1, 2, \ldots, n-k)$. The *face labelling* of $\mathcal{G}$ is the injective map $\lambda_{\mathcal{G}} : \mathcal{F}(\mathcal{G}) \to \binom{[n]}{k}$ (the set of k-element subsets of $\{1, \ldots, n\}$) defined as follows: for a face $F \in \mathcal{F}(\mathcal{G})$, $\lambda_{\mathcal{G}}(F)$ consists of those i such that F is to the left of the trip T_i. We set $\mathcal{V}_{\mathcal{G}} := \lambda_{\mathcal{G}}(\mathcal{F}(\mathcal{G}))$.

See Fig. 1 for an example.

3. Polytopes arising from plabic graphs

We associate polytopes with plabic graphs following [10]. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ be the base field.

3.1. Positive Grassmannians

For $0 < k < n$, let $\mathrm{Mat}_{k,n}$ denote the set of $k \times n$-matrices with entries in $\mathbb{K}$. For $J \in \binom{[n]}{k}$ and $A \in \mathrm{Mat}_{k,n}$, let $\Delta_J(A)$ denote the maximal minor of A corresponding to columns in J.

Let $\mathrm{Gr}_{k,n}$ be the Grassmann variety embedded into $\mathbb{P}^{N-1}$ via the Plücker embedding where $N = \binom{n}{k}$. The minors $\{\Delta_J \mid J \in \binom{[n]}{k}\}$ give the Plücker coordinates on $\mathrm{Gr}_{k,n}$. When the base field is $\mathbb{R}$, the *totally non-negative (resp. totally positive) Grassmannian* $(\mathrm{Gr}_{k,n}(\mathbb{R}))_{\geq 0}$ consists of those elements in $\mathrm{Gr}_{k,n}$ having non-negative (resp. positive) Plücker coordinates.

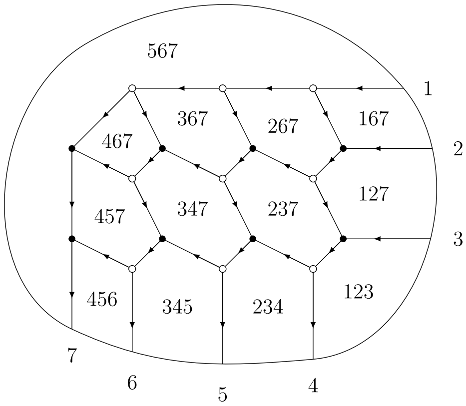


Fig. 1. Plabic graph $\mathcal{G}$ of trip permutation $\pi_{4,7}$ and face labelling $\lambda_{\mathcal{G}}$.

3.2. Perfect orientations

To study flow models on plabic graphs, we fix a perfect orientation $\mathcal{O}$ on $\mathcal{G}$. Such an orientation requires that, at each black (resp. white) internal vertex, there is exactly one edge going out (resp. going in). It is shown in [9] that each reduced plabic graph admits an acyclic perfect orientation. Once such an orientation is fixed, we denote the source set by $I_{\mathcal{O}} := \{i \in [n] \mid i$ is a boundary source of $\mathcal{O}\}$; its complement $I^c_{\mathcal{O}}$ is the set of boundary sinks.

For $I \in \binom{[n]}{k}$, let x_I be a variable. For $i \in I_{\mathcal{O}}$ and $j \in I^c_{\mathcal{O}}$, let $\mathcal{P}_{i,j}$ be the set of directed paths from i to j. For such a directed path γ, let $\mathcal{F}_\gamma(\mathcal{G})$ denote the set of faces to the left of γ. A flow $\mathfrak{F}$ from $I_{\mathcal{O}}$ to $J \in \binom{[n]}{k}$ is a collection of pairwise vertex-disjoint directed paths in $\mathcal{G}$ going from $I_{\mathcal{O}} \backslash (I_{\mathcal{O}} \cap J)$ to $J \backslash (I_{\mathcal{O}} \cap J)$.

For a directed path $\gamma \in \mathcal{P}_{i,j}$, we define the *weight* of γ in $\mathbb{C}[x_I \mid I \in \binom{[n]}{k}]$ by:

$$\mathrm{wt}(\gamma) := \prod_{F \in \mathcal{F}_\gamma(\mathcal{G})} x_{\lambda_{\mathcal{G}}(F)}.$$

The weight of a flow is the product of the weights of the paths it contains. For $J \in \binom{[n]}{k}$, we define P_J to be the sum of the weights of all flows from $I_{\mathcal{O}}$ to J.

For a reduced plabic graph $\mathcal{G}$ of trip permutation $\pi_{n-k,k}$ with perfect orientation $\mathcal{O}$, there exists only one face $F_\emptyset$ to the right of all directed paths with $\lambda_{\mathcal{G}}(F_\emptyset) = \{n-k+1, \cdots, n\}$. We set $\mathcal{V}^\circ_{\mathcal{G}} := \mathcal{V}_{\mathcal{G}} \backslash \{\lambda_{\mathcal{G}}(F_\emptyset)\}$, $\Delta_{\mathcal{G}} := \{x_I \mid I \in \mathcal{V}_{\mathcal{G}}\}$ and $\Delta^\circ_{\mathcal{G}} := \{x_I \mid I \in \mathcal{V}^\circ_{\mathcal{G}}\}$.

Theorem 2 *([8,12]). Let $\mathbb{X} := \mathrm{Gr}_{k,n}(\mathbb{C})$ and $\mathbb{C}(\mathbb{X})$ be the field of rational functions on $\mathbb{X}$. There exists an isomorphism of fields:*

$$\mathbb{C}(\mathbb{X}) \cong \mathbb{C}(x_I \mid x_I \in \Delta^\circ_{\mathcal{G}}), \quad \Delta_J \mapsto P_J.$$

The choice of the perfect orientation $\mathcal{O}$ will only change the formula of P_J by a scalar. We always assume that the choice $I_{\mathcal{O}} = \{1, 2, \cdots, k\}$ is made.

Let $<$ be a total order on $\Delta_{\mathcal{G}}$. It induces a term order $<$ on monomials in $\Delta_{\mathcal{G}}$ by taking the lexicographic order. Let f be a polynomial in Plücker coordinates of $\mathbb{X}$. By Theorem 2, f can be written as a polynomial in $\Delta^\circ_{\mathcal{G}}$:

$$f = \sum_{\mathbf{a} \in \mathbb{Z}^{\mathcal{V}^\circ_{\mathcal{G}}}} c_{\mathbf{a}} x^{\mathbf{a}}, \quad \text{where } x^{\mathbf{a}} = \prod_{I \in \mathcal{V}^\circ_{\mathcal{G}}} x_I^{a_I} \text{ if } \mathbf{a} = (a_I)_{I \in \mathcal{V}^\circ_{\mathcal{G}}}.$$

Let $\nu_{\mathcal{G}} : \mathbb{C}(\mathbb{X})^* \to \mathbb{Z}^{\mathcal{V}^\circ_{\mathcal{G}}}$ be the minimal term valuation on $\mathbb{C}(\mathbb{X})$ with respect to the above total order.

Let $\mathcal{L}_k$ denote the very ample line bundle on $\mathbb{X}$ generating $\mathrm{Pic}(\mathbb{X})$. It gives the Plücker embedding. The space of global sections $\mathrm{H}^0(\mathbb{X}, \mathcal{L}_k^r)$, as a representation of $\mathrm{GL}_n(\mathbb{C})$, is isomorphic to $V(r\varpi_k)^*$, where the latter is the dual of the finite-dimensional irreducible representation of highest weight $r\varpi_k$ (ϖ_k is the k-th fundamental weight). The homogeneous coordinate ring $\mathbb{C}[\mathbb{X}] := \bigoplus_{r \geq 0} \mathrm{H}^0(\mathbb{X}, \mathcal{L}_k^r)$ is embedded into $\mathbb{C}(\mathbb{X})$ by sending $s \in \mathrm{H}^0(\mathbb{X}, \mathcal{L}_k^r)$ to $s/\Delta^r_{[k]}$.

Definition 4. The Newton–Okounkov body associated with $\mathcal{L}_k$, $\nu_{\mathcal{G}}$ and the lexicographic order is defined by:

$$\mathrm{NO}_{\mathcal{G}} := \overline{\mathrm{conv}\left(\bigcup_{r \geq 1} \left\{\nu_{\mathcal{G}}(s)/r \mid s \in \mathrm{H}^0(\mathbb{X}, \mathcal{L}_k^r) \backslash \{0\}\right\}\right)}.$$

We set $\mathrm{NO}^1_{\mathcal{G}} := \mathrm{conv}(\{\nu_{\mathcal{G}}(s) \mid s \in \mathrm{H}^0(\mathbb{X}, \mathcal{L}_k)\backslash\{0\}\}) \subseteq \mathrm{NO}_{\mathcal{G}}$. For the issue on whether this inclusion is proper (i.e., whether $\mathrm{NO}_{\mathcal{G}}$ is integral), see [10, Theorem 15.17].

4. Duality between Newton–Okounkov bodies

4.1. Order polytopes and chain polytopes

Let $(P, \leq_P)$ be a poset with covering relation $\prec$. Stanley [11] associated two Ehrhart equivalent polytopes, the order polytope and the chain polytope, with this poset. We recall here a dilated version of them.

For $r \in \mathbb{N}_{>0}$, we denote the dilated order polytope $\mathcal{O}(P, r)$ to be the representation of the poset P on the interval $[0, r]$ with the order on real numbers:

$$\mathcal{O}(P, r) := \mathrm{Hom}_{\mathrm{Poset}}((P, \leq_P), ([0, r], \leq)) \subseteq \mathbb{R}^P.$$

The dilated chain polytope $\mathcal{C}(P, r) \subseteq \mathbb{R}^P$ has the following facets: for any $p \in P$, $x_p \geq 0$; for any maximal chain $p_1 \prec \cdots \prec p_s$, $x_{p_1} + \cdots + x_{p_s} \leq r$, where x_p is the coordinate of $p \in P$ in $\mathbb{R}^P$.

Stanley [11] showed that the integral points of the chain polytope $\mathcal{C}(P, 1)$ are given by the characteristic functions of the anti-chains in P. In particular, the element $p \in P$ gives an integral point χ_p in $\mathcal{C}(P, 1)$.

In the following, we fix $1 \leq k \leq n-1$, and let $(P_{k,n}, \leq)$ be the poset given by the elements $p_{i,j}$, where $1 \leq i \leq k$ and $k+1 \leq j \leq n$, with covering relations

$$p_{i+1,j} \prec p_{i,j} \text{ and } p_{i,j+1} \prec p_{i,j}.$$

The polytope $\mathcal{O}(P_{k,n}, r)$ is the Gelfand–Tsetlin polytope $\mathrm{GT}^r_{k,n}$ for the representation $V(r\varpi_k)$ of $\mathfrak{sl}_n$ ([4]); while $\mathcal{C}(P_{k,n}, r)$ is the Feigin–Fourier–Littelmann–Vinberg polytope $\mathrm{FFLV}^r_{k,n}$ ([1,3]) of the same representation.

For a polytope $Q \subset \mathbb{R}^m$, let $S(Q) := Q \cap \mathbb{Z}^m$ denote the set of integral points in it. The following integer decomposition properties hold: the r-fold Minkowski sum of $S(\mathcal{O}(P_{k,n}, 1))$ (resp. $S(\mathcal{C}(P_{k,n}, 1))$) coincides with $S(\mathcal{O}(P_{k,n}, r))$ (resp. $S(\mathcal{C}(P_{k,n}, r))$).

Moreover, if $\mathbf{a} = \{p_{i_1,j_1}, \ldots, p_{i_s,j_s}\}$ is an anti-chain in $P_{k,n}$, then one has, for the corresponding lattice points, $\chi_{\mathbf{a}} = \chi_{p_{i_1,j_1}} + \ldots + \chi_{p_{i_s,j_s}} \in \mathcal{C}(P_{k,n}, 1)$.

Proposition 1. *Suppose that Q is an integral polytope in $\mathbb{R}^{P_{k,n}}$ such that*

- $\#S(Q) = \#S(\mathrm{FFLV}^1_{k,n})$;
- *there is a parametrization of the lattice points in Q by anti-chains in $P_{k,n}$ sending an anti-chain $\mathbf{a}$ to $y_{\mathbf{a}} \in \mathbb{R}^{P_{k,n}}$ such that, for any anti-chain $\mathbf{a} = \{p_{i_1,j_1}, \ldots, p_{i_s,j_s}\}$, the relation $y_{\mathbf{a}} = y_{p_{i_1,j_1}} + \ldots + y_{p_{i_s,j_s}}$ holds;*
- *there is a linear map of determinant 1 expressing $y_{p_{i,j}}$ in terms of $\chi_{p_{i,j}}$.*

Then the assignment $\chi_{p_{i,j}} \mapsto y_{p_{i,j}}$ induces a unimodularly equivalence from $\mathrm{FFLV}^1_{k,n}$ to Q.

4.2. Duality of polytopes from positive structures

We refer the reader to [10, Section 7.1] for the definition of the rec-plabic graph $\mathcal{G}^{\mathrm{rec}}_{k,n}$. For example, the plabic graph in Fig. 1 is $\mathcal{G}^{\mathrm{rec}}_{4,7}$.

The following has been shown in [10, Lemma 15.2]:

Proposition 2. *The Newton–Okounkov body $\mathrm{NO}_{\mathcal{G}^{\mathrm{rec}}_{k,n}}$ is unimodularly equivalent to the Gelfand–Tsetlin polytope $\mathrm{GT}^1_{n-k,n}$.*

We define the dual rec-plabic graph $(\mathcal{G}^{\mathrm{rec}}_{k,n})^\vee$ by swapping the black/white colour of the internal vertices, reversing the perfect orientation and changing the boundary labelling $r \mapsto r + n - k \bmod n$. The dual rec-plabic graph is a plabic graph of trip permutation $\pi_{k,n}$ with a perfect orientation. The face labelling in $(\mathcal{G}^{\mathrm{rec}}_{k,n})^\vee$ of a face F in $\mathcal{G}^{\mathrm{rec}}_{k,n}$ is given by the complement:

$$\lambda_{(\mathcal{G}^{\mathrm{rec}}_{k,n})^\vee}(F) = (\lambda_{\mathcal{G}^{\mathrm{rec}}_{k,n}}(F))^c.$$

Notice that in $(\mathcal{G}^{\mathrm{rec}}_{k,n})^\vee$, for a boundary source i and a boundary sink j, the flow from i to j of strongly minimal weight (we borrow the notion of strongly minimal from [10, Definition 5.13]) is given by a "vertical" path starting from i followed by a "horizontal" path ending in j. We denote this path by $\gamma^{\min}_{i,j}$ (see Fig. 2 for an example for $\gamma^{\min}_{3,6}$).

Proposition 3. *In the dual rec-plabic graph $(\mathcal{G}^{\mathrm{rec}}_{k,n})^\vee$, let $\{i_1 < \ldots < i_r\}$ be a subset of the sources and $\{j_1 > \ldots > j_r\}$ be a subset of the sinks. Let $J = \{i_1, \ldots, i_r, j_1, \ldots, j_r\}$. Then the unique flow $\mathcal{F}(J)$ of strongly minimal weight is given by $\{\gamma^{\min}_{i_1,j_1}, \ldots, \gamma^{\min}_{i_r,j_r}\}$.*

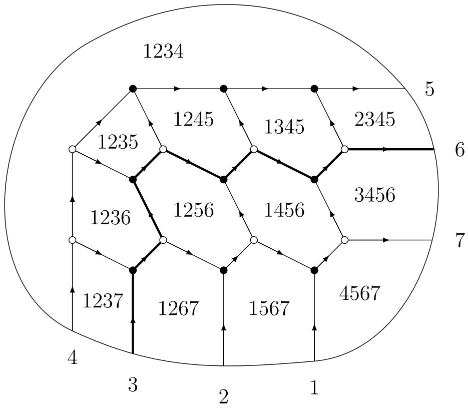


Fig. 2. Plabic graph $\mathcal{G}^\vee$, with a minimal path from 3 to 6.

Proof. Since the paths of strongly minimal weight do not intersect, the flow of minimal weight is given by the union of these paths. □

Theorem 3. *The Newton–Okounkov body* $\mathrm{NO}_{(\mathcal{G}^{\mathrm{rec}}_{k,n})^\vee}$ *is unimodularly equivalent to the FFLV polytope* $\mathrm{FFLV}^1_{k,n}$.

Proof. We first set $Q = \mathrm{NO}^1_{(\mathcal{G}^{\mathrm{rec}}_{k,n})^\vee}$ and verify the conditions in Proposition 1 to show that Q is unimodularly equivalent to $\mathrm{FFLV}^1_{k,n}$ by a linear map.

The polytope Q is a lattice polytope satisfying $\#S(Q) = \#S(\mathrm{FFLV}^1_{k,n})$ (the valuation images of the Plücker coordinates are different). Let $f_{i\times j} := \nu_{(\mathcal{G}^{\mathrm{rec}}_{k,n})^\vee}(\gamma^{\min}_{i,j})$. We define a linear map

$$\psi : \mathrm{FFLV}^1_{k,n} \longrightarrow Q,\ \chi_{p_{i,j}} \mapsto f_{i\times j}.$$

We label a basis on the right-hand side indexed by the faces of the plabic graph and a basis on the left-hand side indexed by the elements $p_{i,j}$. Using row operations, one can show straightforwardly, that the matrix of ψ corresponding to these bases has determinant 1.

Since ψ is linear, $\mathrm{NO}_{(\mathcal{G}^{\mathrm{rec}}_{k,n})^\vee}$ is unimodularly equivalent to $\mathrm{FFLV}^1_{k,n}$. □

Remark 1. We set $(\mathcal{G}^{\mathrm{rec}}_{k,n})_{w_0}$ to be the plabic graph obtained from $\mathcal{G}^{\mathrm{rec}}_{k,n}$ by replacing each $I = \{i_1, \ldots, i_{n-k}\}$ by $I_{w_0} = \{n+1-i_{n-k}, \ldots, n+1-i_1\}$. This is nothing but applying a maximal Green sequence of mutations [6] to the cluster variables in $\mathcal{G}^{\mathrm{rec}}_{k,n}$. Then one can show similarly to the theorem above, that the Newton–Okounkov body $\mathrm{NO}_{(\mathcal{G}^{\mathrm{rec}}_{k,n})_{w_0}}$ is unimodularly equivalent to $\mathrm{FFLV}^1_{n-k,n}$.

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