C. R. Acad. Sci. Paris, Ser. I 356 (2018) 613–625

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Ordinary differential equations/Harmonic analysis

The Gelfand–Shilov smoothing effect for the radially symmetric homogeneous Landau equation with Shubin initial datum

L'effet de lissage de Gelfand–Shilov pour l'équation de Landau homogène à symétrie radiale, avec donnée initiale de Shubin

Hao-Guang Li

School of Mathematics and Statistics, South-Central University for Nationalities, 430074, Wuhan, PR China

ARTICLE INFO

Article history:
Received 26 November 2017
Accepted after revision 24 April 2018
Available online 27 April 2018

Presented by the Editorial Board

ABSTRACT

In this paper, we study the Cauchy problem associated with the radially symmetric spatially homogeneous non-cutoff Landau equation with Maxwellian molecules, while the initial datum belongs to negative-index Shubin space, which can be characterized by spectral decomposition of the harmonic oscillators. Based on this spectral decomposition, we construct the weak solution with Shubin's class initial datum, and then we prove the uniqueness and the Gelfand–Shilov smoothing effect of the solution to this Cauchy problem.

RÉSUMÉ

Dans cet article, nous étudions le problème de Cauchy associé à l'équation de Landau spatialement homogène à symétrie radiale et sans troncature angulaire avec des molécules de Maxwell, tandis que la donnée initiale appartient à un espace de Shubin d'indice négatif, qui peut être caractérisé à partir de la décomposition spectrale de l'oscillateur harmonique quantique. En utilisant cette décomposition spectrale, nous construisons une solution faible avec une donnée initiale dans un espace de Shubin, puis nous prouvons l'unicité et un effet de régularisation de Gelfand–Shilov de la solution à ce problème de Cauchy.

E-mail address: lihaoguang@mail.scuec.edu.cn.

https://doi.org/10.1016/j.crma.2018.04.022
1631-073X/

1. Introduction

In this work, we consider the spatially homogeneous Landau equation

$$\begin{cases} \partial_t f = Q_L(f, f), \\ f|_{t=0} = f_0 \geq 0, \end{cases} \tag{1.1}$$

where $f = f(t, v)$ is the density distribution function depending on the variables $v \in \mathbb{R}^3$ and the time $t \geq 0$. The Landau bilinear collision operator is given by

$$Q_L(g, f)(v) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} a(v - v_*)\big(g(v_*)(\nabla_v f)(v) - (\nabla_v g)(v_*) f(v)\big) \mathrm{d}v_* \right),$$

where $a = (a_{i,j})_{1 \leq i,j \leq 3}$ stands for the nonnegative symmetric matrix

$$a(v) = (|v|^2 \mathbf{I} - v \otimes v)|v|^\gamma \in M_3(\mathbb{R}), \quad -3 < \gamma < +\infty.$$

This equation is obtained as a limit of the Boltzmann equation, when all the collisions become grazing. See [13], [14] and [15] for more details and references on the subject.

In this paper, we only consider the Cauchy problem (1.1) for the radially symmetric homogeneous non-cutoff Landau equation under the Maxwellian molecules case $\gamma = 0$ with the initial datum $f_0 \geq 0$ satisfying

$$\int_{\mathbb{R}^3} f_0(v)\mathrm{d}v = 1, \int_{\mathbb{R}^3} v_j f_0(v)\mathrm{d}v = 0, \; j = 1, 2, 3, \int_{\mathbb{R}^3} |v|^2 f_0(v)\mathrm{d}v = 3. \tag{1.2}$$

See [14]. We shall study the linearization of the Landau equation (1.1). Considering the fluctuation of the density distribution function

$$f(t, v) = \mu(v) + \sqrt{\mu}(v) g(t, v)$$

near the absolute Maxwellian distribution

$$\mu(v) = (2\pi)^{-\frac{3}{2}} \mathrm{e}^{-\frac{|v|^2}{2}}.$$

Since $Q_L(\mu, \mu) = 0$, the Cauchy problem (1.1) is reduced to the Cauchy problem

$$\begin{cases} \partial_t g + \mathcal{L}(g) = \mathbf{\Gamma}(g, g), \quad t > 0, v \in \mathbb{R}^3, \\ g|_{t=0} = g^0, \end{cases} \tag{1.3}$$

with $g^0(v) = \mu^{-\frac{1}{2}} f_0(v) - \sqrt{\mu}(v)$, where

$$\mathbf{\Gamma}(g, g) = \mu^{-\frac{1}{2}} Q_L(\sqrt{\mu} g, \sqrt{\mu} g), \quad \mathcal{L}(g) = -\mu^{-\frac{1}{2}} \Big(Q_L(\sqrt{\mu} g, \mu) + Q_L(\mu, \sqrt{\mu} g) \Big).$$

The linear operator $\mathcal{L}$ is nonnegative ([5]) with the null space

$$\mathcal{N} = \operatorname{span}\left\{ \sqrt{\mu}, \sqrt{\mu} v_1, \sqrt{\mu} v_2, \sqrt{\mu} v_3, \sqrt{\mu}|v|^2 \right\}.$$

The projection function $\mathbf{P} : \mathcal{S}'(\mathbb{R}^3) \to \mathcal{N}$ is well defined. Under the assumption of the initial datum f_0 in (1.2), we have

$$\begin{cases} \displaystyle\int_{\mathbb{R}^3} \sqrt{\mu}(v) g^0(v)\mathrm{d}v = 0, \int_{\mathbb{R}^3} v_j \sqrt{\mu}(v) g^0(v)\mathrm{d}v = 0, \; j = 1, 2, 3, \\ \displaystyle\int_{\mathbb{R}^3} |v|^2 \sqrt{\mu}(v) g^0(v)\mathrm{d}v = 0. \end{cases} \tag{1.4}$$

This shows that $g^0 \in \mathcal{N}^\perp$.

It is well known that the angular singularity in the cross section leads to the regularity of the solution, see [8,9,15,16] and the references therein. We can also refer to [1,4] for the smoothing effect of the radially symmetric spatially homogeneous Kac equation. Regarding the linearized Cauchy problem (1.3), the global in-time smoothing effect of the solution to the Cauchy problem (1.3) has been shown, in [6] and in [2] with the small initial data in $L^2(\mathbb{R}^3)$. It proved that the solutions to the Cauchy problem (1.3) belong to the symmetric Gelfand–Shilov space $S^{\frac{1}{2s}}_{\frac{1}{2s}}(\mathbb{R}^3)$ for any positive time. Moreover, there exist positive constants $c > 0$ and $C > 0$, such that

$$\forall t > 0, \quad \|e^{ct\mathcal{H}^s} g(t)\|_{L^2} \leq C\|g_0\|_{L^2},$$

where $\mathcal{H}$ is the harmonic oscillator

$$\mathcal{H} = -\triangle_v + \frac{|v|^2}{4}.$$

The Gelfand–Shilov space $S^{\mu}_{\nu}(\mathbb{R}^3)$, with $\mu, \nu > 0$, $\mu + \nu \geq 1$, is the subspace of smooth functions satisfying:

$$\exists A > 0,\ C > 0,\ \sup_{v\in\mathbb{R}^3} |v^{\beta}\partial^{\alpha}_v f(v)| \leq CA^{|\alpha|+|\beta|}(\alpha!)^{\mu}(\beta!)^{\nu},\ \forall \alpha, \beta \in \mathbb{N}^3$$

so that Gelfand–Shilov class $S^{\mu}_{\nu}(\mathbb{R}^3)$ is Gevrey class $G^{\mu}(\mathbb{R}^3)$ with rapid decay at infinite.

In this paper, we will show that the Cauchy problem (1.3) for radially symmetric homogeneous non-cutoff Landau equation admits a unique weak solution, with the initial datum in the Shubin space $Q^{\alpha}(\mathbb{R}^3)$ for any $\alpha < -\frac{3}{2}$. This space contains the Sobolev space $H^{-\frac{3}{2}}(\mathbb{R}^3)$, so it is more singular than the measure-valued initial datum. Furthermore, we prove that the unique solution belongs to the Gelfand–Shilov space $S^{\frac{1}{2}}_{\frac{1}{2}}(\mathbb{R}^3)$ for any positive time. For $\beta \in \mathbb{R}$, Shubin introduce the following function spaces (see [12], Ch. IV, 25.3)

$$Q^{\beta}(\mathbb{R}^3) = \Big\{u \in \mathcal{S}'(\mathbb{R}^3);\ \|u\|_{Q^{\beta}(\mathbb{R}^3)} = \big\|\mathcal{H}^{\frac{\beta}{2}} u\big\|_{L^2(\mathbb{R}^3)} < +\infty\Big\}.$$

We denote by $Q^{\beta}_r(\mathbb{R}^3)$ the radially symmetric functions belonging to $Q^{\beta}(\mathbb{R}^3)$.

The main theorem of this paper is given below.

Theorem 1.1. *For any $\alpha < -\frac{3}{2}$ and let the initial datum $g^0 \in Q^{\alpha}_r(\mathbb{R}^3)$ with g^0 satisfying the assumption (1.4). The Cauchy problem (1.3) for a radially symmetric homogeneous non-cutoff Landau equation admits a unique global radial symmetric weak solution*

$$g \in L^{+\infty}([0, +\infty[;\ Q^{\alpha}_r(\mathbb{R}^3)) \cap C^1([0, +\infty[; \mathscr{S}'(\mathbb{R}^3)).$$

Moreover, we have that

$$\|e^{t\mathcal{L}}\mathcal{H}^{\frac{\alpha}{2}} g(t)\|_{L^2(\mathbb{R}^3)} \leq \|\mathcal{H}^{\frac{\alpha}{2}} g^0\|_{L^2(\mathbb{R}^3)} = \|g^0\|_{Q^{\alpha}(\mathbb{R}^3)}.$$

Remark 1.2. We remark that the general case is similar to the radially symmetric case, but the proofs are more technical, see [7]. However, compared with the result in [7], we prove the uniqueness properties for the weak solution, and no smallness condition is required for the initial data g^0 in Theorem 1.1.

Example 1.1. It is well known that the single Dirac mass $\delta_{\mathbf{0}}$ is a stationary solution to the Cauchy problem (1.1). However, the energy of the single Dirac mass is 0. The following example is interesting. Let

$$f_0 = \delta_{\mathbf{0}} - \left(\frac{3}{2} - \frac{|v|^2}{2}\right)\mu$$

be the initial datum of Cauchy problem (1.1), then $f_0 = \mu + \sqrt{\mu} g^0$ with

$$g^0 = \frac{1}{\sqrt{\mu}}\delta_{\mathbf{0}} - \left(\frac{5}{2} - \frac{|v|^2}{2}\right)\sqrt{\mu}. \tag{1.5}$$

We will prove in Section 2 that $g^0 \in Q^{\alpha}(\mathbb{R}^3) \cap \mathcal{N}^{\perp}$ for $\alpha < -\frac{3}{2}$. Then Theorem 1.1 implies that the Cauchy problem (1.1) admits a unique global solution

$$f = \mu + \sqrt{\mu} g \in L^{+\infty}([0, +\infty[;\ Q^{\alpha}(\mathbb{R}^3)) \cap C^0(]0, +\infty[;\ S^{\frac{1}{2}}_{\frac{1}{2}}(\mathbb{R}^3)).$$

The rest of the paper is arranged as follows: In Section 2, we introduce the spectral analysis of the linear and nonlinear Landau operators for the radially symmetric homogeneous Landau equation. In Section 3, we present the explicit solution to the Cauchy problem (1.3) by transforming the linearized Landau equation into an infinite system of ordinary differential equations that can be solved explicitly. The proof of the main Theorem 1.1 and the Example 1.1 will be presented in Section 4. In the Appendix 5, we present some identity properties of the Gelfand–Shilov spaces and of the Shubin spaces used in this paper.

2. Preliminary

Diagonalization of the linear operators. We first recall the spectral decomposition of the linear Landau operator.

$$\mathcal{L}(\varphi_n) = \lambda_n \varphi_n,$$

with

$$\lambda_0 = \lambda_1 = 0;\ \lambda_n = 4n, \quad \forall n \geq 2$$

and

$$\varphi_n(v) = \sqrt{\frac{n!}{4\sqrt{2}\pi\Gamma(n+\frac{3}{2})}} L_n^{(\frac{1}{2})}\left(\frac{|v|^2}{2}\right) \mathrm{e}^{-\frac{|v|^2}{4}},$$

where $\Gamma(\cdot)$ is the standard Gamma function, for any $x > 0$,

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} \mathrm{e}^{-t} \mathrm{d}t,$$

and the Laguerre polynomial $L_n^{(\alpha)}$ of order α, degree n reads

$$L_n^{(\alpha)}(x) = \sum_{r=0}^{n} (-1)^{n-r} \frac{\Gamma(\alpha+n+1)}{r!(n-r)!\Gamma(\alpha+n-r+1)} x^{n-r}.$$

Then $\{\varphi_n\}_{n\geq 0}$ constitute an orthonormal basis of $L^2_{\mathrm{rad}}(\mathbb{R}^3)$, the radially symmetric function space (see [6]). In particular,

$$\varphi_0(v) = (2\pi)^{-\frac{3}{4}} \mathrm{e}^{-\frac{|v|^2}{4}} = \sqrt{\mu},$$
$$\varphi_1(v) = \sqrt{\frac{2}{3}}\left(\frac{3}{2} - \frac{|v|^2}{2}\right)(2\pi)^{-\frac{3}{4}} \mathrm{e}^{-\frac{|v|^2}{4}} = \sqrt{\frac{2}{3}}\left(\frac{3}{2} - \frac{|v|^2}{2}\right)\sqrt{\mu}.$$

Furthermore, we have, for suitable radial symmetric function g,

$$\mathcal{H}(g) = \sum_{n=0}^{\infty} (2n + \frac{3}{2})\, g_n\, \varphi_n \quad \mathcal{L}(g) = \sum_{n=2}^{\infty} 4n\, g_n\, \varphi_n,$$

where $g_n = \langle g, \varphi_n \rangle$.

Triangular effect of the non-linear operators. We study now the algebra property of the nonlinear terms

$$\mathbf{\Gamma}(\varphi_n, \varphi_m).$$

In a similar fashion to the proof of Proposition 2.1 in [2] or Lemma 3.3 in [6], we can prove the following triangular effect for the nonlinear Landau operators on the basis $\{\varphi_n\}$.

Proposition 2.1. *The following algebraic identities hold:*

$$\mathbf{\Gamma}(\varphi_n, \varphi_m) = -4m\delta_{0,n}\varphi_m + \frac{4\sqrt{3(2m+3)(m+1)}}{3}\delta_{1,n}\varphi_{m+1}, \ \textit{for } n, m \in \mathbb{N}.$$

Proof. In all the proof of this proposition, we will set $\Psi_k : \mathbb{R} \to \mathbb{R}$

$$\Psi_n(\rho) = c_n L_n^{(\frac{1}{2})}(\rho)\, \mathrm{e}^{-\rho} \tag{2.1}$$

with

$$c_n = (2\pi)^{-\frac{3}{4}} \sqrt{\frac{n!}{4\sqrt{2}\pi\Gamma(n+\frac{3}{2})}}.$$

Therefore, recalling from the definition of $\varphi_n(v)$ that, for $v \in \mathbb{R}^3$,

$$\sqrt{\mu}(v)\varphi_n(v) = c_n L_n^{(\frac{1}{2})}\left(\frac{|v|^2}{2}\right) \mathrm{e}^{-\frac{|v|^2}{2}} = \Psi_n\left(\frac{|v|^2}{2}\right).$$

It follows that, for $m, n \in \mathbb{N}$

$$\nabla_{v_*}(\sqrt{\mu}\varphi_n)(v_*) = \Psi_n'\left(\frac{|v_*|^2}{2}\right)v_*;$$

$$\nabla_v(\sqrt{\mu}\varphi_m)(v) = \Psi_m'\left(\frac{|v|^2}{2}\right)v,$$

where we used the notation $\Psi_k(\rho)$ in (2.1) and $\Psi_k'(\rho) = \frac{d\Psi_k(\rho)}{d\rho}$ for $k \in \mathbb{N}$. Then

$$\begin{aligned}\mathbf{\Gamma}(\varphi_n, \varphi_m) = &\frac{1}{\sqrt{\mu(v)}} \sum_{1\le i,j\le 3} \partial_{v_i} \int_{\mathbb{R}^3} a_{i,j}(v - v_*) \\ &\times \left[\Psi_n\left(\frac{|v_*|^2}{2}\right)\Psi_m'\left(\frac{|v|^2}{2}\right)v_j - \Psi_n'\left(\frac{|v_*|^2}{2}\right)\Psi_m\left(\frac{|v|^2}{2}\right)v_j^*\right]dv_*,\end{aligned}$$

where we have written $v_* = (v_1^*, v_2^*, v_3^*) \in \mathbb{R}^3$ and

$$a_{i,i}(v - v_*) = \sum_{\substack{1\le k\le 3\\ k\neq i}} (v_k - v_k^*)^2;\ a_{i,j}(v - v_*) = -(v_i - v_i^*)(v_j - v_j^*) \text{ when } i \neq j.$$

It follows that

$$\mathbf{\Gamma}(\varphi_n, \varphi_m) = \mathbf{\Gamma}_1(\varphi_n, \varphi_m) - \mathbf{\Gamma}_2(\varphi_n, \varphi_m),$$

where

$$\begin{aligned}\mathbf{\Gamma}_1(\varphi_n, \varphi_m) = &\frac{1}{\sqrt{\mu(v)}} \sum_{\substack{1\le i,j\le 3\\ i\neq j}} \partial_{v_i} \int_{\mathbb{R}^3} (v_j - v_j^*)^2 \\ &\times \left[\Psi_n\left(\frac{|v_*|^2}{2}\right)\Psi_m'\left(\frac{|v|^2}{2}\right)v_i - \Psi_n'\left(\frac{|v_*|^2}{2}\right)\Psi_m\left(\frac{|v|^2}{2}\right)v_i^*\right]dv_*;\\ \mathbf{\Gamma}_2(\varphi_n, \varphi_m) = &\frac{1}{\sqrt{\mu(v)}} \sum_{\substack{1\le i,j\le 3\\ i\neq j}} \partial_{v_i} \int_{\mathbb{R}^3} (v_i - v_i^*)(v_j - v_j^*) \\ &\times \left[\Psi_n\left(\frac{|v_*|^2}{2}\right)\Psi_m'\left(\frac{|v|^2}{2}\right)v_j - \Psi_n'\left(\frac{|v_*|^2}{2}\right)\Psi_m\left(\frac{|v|^2}{2}\right)v_j^*\right]dv_*.\end{aligned}$$

Since $f(v_*) = \Psi_n\left(\frac{|v_*|^2}{2}\right), \Psi_n'\left(\frac{|v_*|^2}{2}\right)$ is symmetric with respect to v_*, one can verify that

$$\int_{\mathbb{R}^3} v_i^* f(v_*)dv_* = 0$$

$$\int_{\mathbb{R}^3} v_i^* v_j^* f(v_*)dv_* = 0, \quad \text{when } 1 \le i \neq j \le 3$$

$$\int_{\mathbb{R}^3} (v_i^*)^2 v_j^* f(v_*)dv_* = 0, \forall 1 \le i, j \le 3.$$

We can deduce that

$$\begin{aligned}&\mathbf{\Gamma}_1(\varphi_n, \varphi_m)\\ &= \frac{1}{\sqrt{\mu(v)}} \sum_{\substack{1\le i,j\le 3\\ i\neq j}} \partial_{v_i}\left[\int_{\mathbb{R}^3} \left[v_j^2 v_i + (v_j^*)^2 v_i\right]\Psi_n\left(\frac{|v_*|^2}{2}\right)dv_* \Psi_m'\left(\frac{|v|^2}{2}\right)\right];\end{aligned}$$

$$\begin{aligned}
&\mathbf{\Gamma}_2(\varphi_n,\varphi_m)\\
&=\frac{1}{\sqrt{\mu(v)}}\sum_{\substack{1\le i,j\le 3\\ i\neq j}}\partial_{v_i}\left[\int_{\mathbb{R}^3} v_i v_j^2\Psi_n\left(\frac{|v_*|^2}{2}\right)\mathrm{d}v_*\Psi_m'\left(\frac{|v|^2}{2}\right)\right]\\
&\quad+\frac{1}{\sqrt{\mu(v)}}\sum_{\substack{1\le i,j\le 3\\ i\neq j}}\partial_{v_i}\left[\int_{\mathbb{R}^3}(v_j^*)^2\Psi_n'\left(\frac{|v_*|^2}{2}\right)\mathrm{d}v_* v_i\Psi_m\left(\frac{|v|^2}{2}\right)\right].
\end{aligned}$$

Therefore, one can obtain that

$$\begin{aligned}
&\mathbf{\Gamma}(\varphi_n,\varphi_m)=\mathbf{\Gamma}_1(\varphi_n,\varphi_m)-\mathbf{\Gamma}_2(\varphi_n,\varphi_m)\\
&=\frac{1}{\sqrt{\mu(v)}}\sum_{\substack{1\le i,j\le 3\\ i\neq j}}\partial_{v_i}\left[v_i\Psi_m'\left(\frac{|v|^2}{2}\right)\right]\left(\int_{\mathbb{R}^3}(v_j^*)^2\Psi_n\left(\frac{|v_*|^2}{2}\right)\mathrm{d}v_*\right)\\
&\quad-\frac{1}{\sqrt{\mu(v)}}\sum_{\substack{1\le i,j\le 3\\ i\neq j}}\partial_{v_i}\left[v_i\Psi_m\left(\frac{|v|^2}{2}\right)\right]\left(\int_{\mathbb{R}^3}(v_j^*)^2\Psi_n'\left(\frac{|v_*|^2}{2}\right)\mathrm{d}v_*\right).
\end{aligned}$$

By using the symmetric of the coordinate axis and the elementary equality

$$|v_*|^2\sqrt{\mu}(v_*)=3\varphi_0-\sqrt{6}\varphi_1,$$

we deduce from $\Psi_n\left(\frac{|v_*|^2}{2}\right)=\sqrt{\mu}(v_*)\varphi_n(v_*)$ that

$$\begin{aligned}
&\sum_{\substack{1\le i,j\le 3\\ i\neq j}}\left(\int_{\mathbb{R}^3}(v_j^*)^2\Psi_n\left(\frac{|v_*|^2}{2}\right)\mathrm{d}v_*\right)\\
&=\frac{2}{3}\int_{\mathbb{R}^3}|v_*|^2\Psi_n\left(\frac{|v_*|^2}{2}\right)\mathrm{d}v_*\\
&=\frac{2}{3}\int_{\mathbb{R}^3}|v_*|^2\sqrt{\mu}(v_*)\varphi_n(v_*)\mathrm{d}v_*\\
&=2(\varphi_0,\varphi_n)_{L^2(\mathbb{R}^3)}-\frac{2\sqrt{6}}{3}(\varphi_1,\varphi_n)_{L^2(\mathbb{R}^3)}\\
&=2\delta_{0,n}-\frac{2\sqrt{6}}{3}\delta_{1,n}.
\end{aligned}$$

Similar to the above symmetric property, and integration by parts, we have

$$\begin{aligned}
&\sum_{\substack{1\le j\le 3\\ j\neq i}}\int_{\mathbb{R}^3}(v_j^*)^2\Psi_n'\left(\frac{|v_*|^2}{2}\right)\mathrm{d}v_*\\
&=\frac{2}{3}\int_{\mathbb{R}^3}|v_*|^2\Psi_n'\left(\frac{|v_*|^2}{2}\right)\mathrm{d}v_*\\
&=\frac{16\sqrt{2}\pi}{3}c_n\int_0^{+\infty}\rho^{\frac{3}{2}}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\mathrm{e}^{-\rho}L_n^{(\frac{1}{2})}(\rho)\right)\mathrm{d}\rho
\end{aligned}$$

$$= -8\sqrt{2}\pi c_n \int_0^{+\infty} \rho^{\frac{1}{2}} e^{-\rho} L_n^{(\frac{1}{2})}(\rho) d\rho$$

$$= -2 \int_{\mathbb{R}^3} \sqrt{\mu}(v_*)\varphi_n(v_*) dv_* = -2\delta_{0,n}.$$

One can verify that

$$\begin{aligned}\mathbf{\Gamma}(\varphi_n, \varphi_m) &= (2\delta_{0,n} - \frac{2\sqrt{6}}{3}\delta_{1,n}) \frac{1}{\sqrt{\mu(v)}} \sum_{1\le i\le 3} \partial_{v_i}\left(v_i \Psi_m'\left(\frac{|v|^2}{2}\right)\right) \\ &\quad + 2\delta_{0,n} \frac{1}{\sqrt{\mu(v)}} \sum_{1\le i\le 3} \partial_{v_i}\left(v_i \Psi_m\left(\frac{|v|^2}{2}\right)\right) \\ &= \frac{1}{\sqrt{\mu(v)}}\left[2\delta_{0,n}\mathbf{A}(v) - \frac{2\sqrt{6}}{3}\delta_{1,n}\mathbf{B}(v)\right], \end{aligned} \tag{2.2}$$

where

$$\mathbf{A}(v) = 3\Psi_m\left(\frac{|v|^2}{2}\right) + (3+|v|^2)\Psi_m'\left(\frac{|v|^2}{2}\right) + |v|^2\Psi_m''\left(\frac{|v|^2}{2}\right)$$

$$\mathbf{B}(v) = 3\Psi_m'\left(\frac{|v|^2}{2}\right) + |v|^2\Psi_m''\left(\frac{|v|^2}{2}\right).$$

Recalling the definition of $\Psi_m\left(\frac{|v|^2}{2}\right)$ in (2.1) and taking intermediate variable $\rho = \frac{|v|^2}{2}$, we have:

$$\Psi_m\left(\frac{|v|^2}{2}\right) = c_m e^{-\rho} L_m^{(\frac{1}{2})}(\rho).$$

One can calculate that

$$\mathbf{A}(v) = c_m e^{-\rho}\left(2\rho \frac{d^2}{d\rho^2} L_m^{(\frac{1}{2})}(\rho) + (3-2\rho)\frac{d}{d\rho} L_m^{(\frac{1}{2})}(\rho)\right)$$

$$\mathbf{B}(v) = c_m e^{-\rho}\left(2\rho \frac{d^2}{d\rho^2} L_m^{(\frac{1}{2})}(\rho) + (3-4\rho)\frac{d}{d\rho} L_m^{(\frac{1}{2})}(\rho) + (2\rho-3) L_m^{(\frac{1}{2})}(\rho)\right).$$

By using the formulas (14_1), (7), (12) of Chapter IV in [11], that is, for $x \in \mathbb{R}$,

$$x\frac{d^2}{dx^2} L_m^{(\frac{1}{2})}(x) + (\frac{3}{2} - x)\frac{d}{dx} L_m^{(\frac{1}{2})}(x) + m L_m^{(\frac{1}{2})}(x) = 0;$$

$$(m+1) L_{m+1}^{(\frac{1}{2})}(x) - 2m L_m^{(\frac{1}{2})}(x) + (m + \frac{1}{2}) L_{m-1}^{(\frac{1}{2})}(x) = (\frac{3}{2} - x) L_m^{(\frac{1}{2})}(x);$$

$$x\frac{d}{dx} L_m^{(\frac{1}{2})}(x) = m L_m^{(\frac{1}{2})}(x) - (m + \frac{1}{2}) L_{m-1}^{(\frac{1}{2})}(x),$$

we have

$$\mathbf{A}(v) = -2m c_m e^{-\rho} L_m^{(\frac{1}{2})}(\rho) = -2m\sqrt{\mu}(v)\varphi_m(v),$$

$$\mathbf{B}(v) = -2(m+1) c_m e^{-\rho} L_{m+1}^{(\frac{1}{2})}(\rho) = -\sqrt{(2m+3)(2m+2)}\sqrt{\mu}(v)\varphi_{m+1}(v).$$

Substituting back to (2.2), we conclude that

$$\mathbf{\Gamma}(\varphi_n, \varphi_m) = -4m\delta_{0,n}\varphi_m(v) + \frac{4\sqrt{3(2m+3)(m+1)}}{3}\delta_{1,n}\varphi_{m+1}(v).$$

This ends the proof of the Proposition 2.1. □

Remark 2.2. Obviously, we can deduce from Proposition 2.1 that

$$\forall n \in \mathbb{N}, \quad \mathbf{\Gamma}(\varphi_0, \varphi_n) + \mathbf{\Gamma}(\varphi_n, \varphi_0) = -4n\varphi_n + 4\delta_{1,n}\varphi_1(v),$$

which satisfies that

$$\mathbf{\Gamma}(\varphi_0, \varphi_n) + \mathbf{\Gamma}(\varphi_n, \varphi_0) = -\lambda_n \varphi_n = -\mathcal{L}\varphi_n$$

with $\lambda_0 = \lambda_1 = 0$ and $\lambda_n = 4n$, $\forall n \geq 2$. This shows that the linear radially symmetric Landau operator $\mathcal{L}$ behaves as a fractional harmonic oscillator $\mathcal{H}$. For more details, see Proposition 2.1 in [5].

3. Explicit solution to the Cauchy problem

Now we solve explicitly the Cauchy problem associated with the non-cutoff radial symmetric spatially homogeneous Landau equation with Maxwellian molecules for the initial radial data $g^0 \in Q_r^\alpha(\mathbb{R}^3) \cap \mathcal{N}^\perp$ for any $\alpha < -\frac{3}{2}$.

We search a radial solution to the Cauchy problem (1.3) in the form

$$g(t) = \sum_{n=0}^{+\infty} g_n(t)\varphi_n \text{ with } g_n(t) = \langle g(t), \varphi_n \rangle$$

with initial data

$$g|_{t=0} = g^0 = \sum_{n=0}^{+\infty} \left\langle g^0, \varphi_n \right\rangle \varphi_n \, .$$

Remark that $g^0 \in Q_r^\alpha(\mathbb{R}^3) \cap \mathcal{N}^\perp$ with $\alpha < -\frac{3}{2}$ is equivalent to g^0 radial and

$$\|g^0\|^2_{Q_r^\alpha(\mathbb{R}^3)} = \sum_{n=2}^{+\infty} (2n + \frac{3}{2})^\alpha \left| \langle g^0, \varphi_n \rangle \right|^2 < +\infty,$$

see Appendix 5.

It follows from Proposition 2.1 that, for suitable radial symmetric function g, we have

$$\begin{aligned}\mathbf{\Gamma}(g, g) = & -\sum_{m=0}^{+\infty} g_0(t)\, g_m(t)\, 4m\varphi_m \\ & + \sum_{m=0}^{+\infty} g_1(t)\, g_m(t) \frac{4\sqrt{3(2m+3)(m+1)}}{3} \varphi_{m+1}.\end{aligned}$$

This implies that

$$\mathbf{\Gamma}(g, g) = \sum_{n=1}^{+\infty} \left[-4n g_0(t) g_n(t) + \frac{4\sqrt{3n(2n+1)}}{3} g_1(t) g_{n-1}(t) \right] \varphi_n.$$

For the radial symmetric function g, we also have

$$\mathcal{L}(g) = \sum_{n=0}^{+\infty} \lambda_n \, g_n(t)\, \varphi_n.$$

Formally, we take the inner product with φ_n on both sides of (1.3); we find that the functions $\{g_n(t)\}$ satisfy the following infinite system of differential equations

$$\begin{cases} \partial_t g_0(t) + \lambda_0 \, g_0(t) = 0; \\ \partial_t g_1(t) + \lambda_1 \, g_1(t) = 0; \\ \text{and} \quad \forall n \geq 2 \\ \partial_t g_n(t) + \lambda_n \, g_n(t) = -4n g_0(t) g_n(t) + \frac{4}{3}\sqrt{3n(2n+1)} g_1(t) g_{n-1}(t), \\ g_n(0) = \left\langle g^0, \varphi_n \right\rangle, \end{cases} \tag{3.1}$$

with initial data

$$g_n(0) = \left\langle g^0, \varphi_n \right\rangle, \quad \forall n \in \mathbb{N}.$$

Since $\lambda_0 = \lambda_1 = 0$ and for all $n \geq 2$, $\lambda_n = 4n$, then

$$g_0(t) \equiv \left\langle g^0, \varphi_0 \right\rangle = 0; \quad g_1(t) \equiv \left\langle g^0, \varphi_1 \right\rangle = 0.$$

The infinite system of differential equations (3.1) reduces to

$$\begin{cases} g_0(t) \equiv 0; \quad g_1(t) \equiv 0; \\ \partial_t g_n(t) + 4\, n\, g_n(t) = 0, \forall\, n \geq 2 \\ g_n(0) = \left\langle g^0, \varphi_n \right\rangle. \end{cases} \tag{3.2}$$

This is a linear ordinary differential equation system, which is consistent with the result in Section 3 of [14]. Direct calculation shows that

$$g_n(t) = \mathrm{e}^{-4nt} \left\langle g^0, \varphi_n \right\rangle, \ \forall\, n \geq 2.$$

The proof of Theorem 1.1 is reduced to prove the convergence of the following series

$$g(t) = \sum_{n=0}^{+\infty} g_n(t)\varphi_n \tag{3.3}$$

in the function space $Q_r^\alpha(\mathbb{R}^3)$ with $\alpha < -\frac{3}{2}$.

4. Proof of Theorem 1.1

We recall the definition of the weak solution to (1.3):

Definition 4.1. Let $g^0 \in \mathscr{S}'(\mathbb{R}^3)$, $g(t, v)$ is called a weak solution to the Cauchy problem (1.3) if it satisfies the following conditions:

$$g \in C^0([0, +\infty[; \mathscr{S}'(\mathbb{R}^3)), \quad g(0, v) = g^0(v).$$

For $T > 0$,

$$\mathcal{L}(g) \in L^2([0, T[; \mathscr{S}'(\mathbb{R}^3)), \quad \Gamma(g, g) \in L^2([0, T[; \mathscr{S}'(\mathbb{R}^3)),$$

$$\langle g(t), \phi(t)\rangle - \langle g^0, \phi(0)\rangle + \int_0^t \langle \mathcal{L}g(\tau), \phi(\tau)\rangle \mathrm{d}\tau$$
$$= \int_0^t \langle\, g(\tau), \partial_\tau \phi(\tau)\rangle\, \mathrm{d}\tau + \int_0^t \langle \Gamma(g(\tau), g(\tau)), \phi(\tau)\rangle \mathrm{d}\tau, \quad \forall t \geq 0,$$

for any $\phi(t) \in C^1\big([0, +\infty[; \mathscr{S}(\mathbb{R}^3)\big)$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We proceed to treat the proof by the following three steps.

Step 1. Existence of the solution to the Cauchy problem (1.3).

We construct an auxiliary uniform bounded function $\mathbb{S}_N g(t)$ with initial datum, which approximates $g^0 \in Q_r^\alpha(\mathbb{R}^3) \cap \mathcal{N}^\perp$ with $\alpha < -\frac{3}{2}$. Remark that $g^0 \in Q_r^\alpha(\mathbb{R}^3) \cap \mathcal{N}^\perp$ is equivalent to g^0 radial and

$$\|g^0\|^2_{Q_r^\alpha(\mathbb{R}^3)} = \sum_{n=2}^{+\infty} (2n + \frac{3}{2})^\alpha \left|\langle g^0, \varphi_n\rangle\right|^2 = \sum_{n=2}^{+\infty} (2n + \frac{3}{2})^\alpha \left|g_n^0\right|^2 < +\infty.$$

For all $2 < N \in \mathbb{N}$, we consider the Cauchy problem associated with the ODEs (3.2). Let us now fix some positive integer $N > 2$ and define the following function $\mathbb{S}_N g : [0, +\infty[\times\mathbb{R}^3 \to \mathscr{S}(\mathbb{R}^3)$ by

$$\mathbb{S}_N g(t) = \sum_{n=0}^{N} g_n(t)\varphi_n.$$

Then $\mathbb{S}_N g$ satisfies

$$\begin{cases} \partial_t \mathbb{S}_N g + \mathcal{L}\mathbb{S}_N g(t) = 0 \\ \mathbb{S}_N g(0) = \sum_{n=0}^{N} \langle g^0, \varphi_n\rangle\varphi_n. \end{cases} \tag{4.1}$$

Obviously, for any $N, P \in \mathbb{N}^+$, we have

$$\begin{aligned} &\|\mathcal{H}^{\frac{\alpha}{2}}\,(\mathbb{S}_{N+P}\, g(t) - \mathbb{S}_N g(t))\,\|_{L^2(\mathbb{R}^3)} \\ \le&\|\mathcal{H}^{\frac{\alpha}{2}}\left(\mathbb{S}_{N+P}\, g^0 - \mathbb{S}_N g^0\right)\|_{L^2(\mathbb{R}^3)} \to 0 \,\text{as}\, N \to +\infty, \\ &\int_0^t \|\mathcal{H}^{\frac{\alpha+1}{2}}\,(\mathbb{S}_{N+P}\, g(\tau) - \mathbb{S}_N g(\tau))\,\|_{L^2(\mathbb{R}^3)}\mathrm{d}\tau \\ \le&\|\mathcal{H}^{\frac{\alpha}{2}}\left(\mathbb{S}_{N+P}\, g^0 - \mathbb{S}_N g^0\right)\|_{L^2(\mathbb{R}^3)} \to 0 \,\text{as}\, N \to +\infty. \end{aligned}$$

Then $g \in L([0, +\infty[;\, Q^{\alpha}(\mathbb{R}^3))$, $\mathbb{P}g = 0$ and

$$\begin{aligned} &\mathbb{S}_N g(t) \to\ g(t) \,\text{in}\, Q^{\alpha}(\mathbb{R}^3) \,\text{as}\, N \to +\infty, \\ &\mathbb{S}_N g(t) \to\ g(t) \,\text{in}\, L^2([0, t];\, Q^{\alpha+1}(\mathbb{R}^3)) \,\text{as}\, N \to +\infty. \end{aligned}$$

It is obvious that

$$\mathcal{L}\mathbb{S}_N g(t) \to\ \mathcal{L}g(t) \,\text{in}\, Q^{\alpha-4}(\mathbb{R}^3) \,\text{as}\, N \to +\infty.$$

Equation (4.1) implies that

$$\mathbb{S}_N g(t) - \mathbb{S}_N g(0) \to \int_0^t \mathcal{L}g(\tau)\mathrm{d}\tau \,\text{in}\, Q^{\alpha-4}(\mathbb{R}^3) \,\text{as}\, N \to +\infty.$$

Then

$$g(t) \in C^1([0, +\infty[;\, Q^{\alpha-4}(\mathbb{R}^3)) \subset C^1([0, +\infty[;\, \mathscr{S}'(\mathbb{R}^3)).$$

Therefore, for any test function $\phi \in C^1([0, T], Q^{-1-\alpha}(\mathbb{R}^3))$, from (4.1), for any $0 < t \le T$, we have

$$\begin{aligned} &\langle \mathbb{S}_N g(t), \phi(t)\rangle - \langle g_N(0), \phi(0)\rangle + \int_0^t \langle \mathcal{L}\mathbb{S}_N g(\tau), \phi(\tau)\rangle \mathrm{d}\tau \\ &= \int_0^t \langle \mathbb{S}_N g(\tau), \partial_\tau\phi\rangle \mathrm{d}\tau. \end{aligned}$$

Passing to the limit as $N \to +\infty$, we get

$$\begin{aligned} &\langle g(t), \phi(t)\rangle - \langle g^0, \phi(0)\rangle + \int_0^t \langle \mathcal{L}g(\tau), \phi(\tau)\rangle \mathrm{d}\tau \\ &= \int_0^t \langle g(\tau), \partial_\tau\phi\rangle \mathrm{d}\tau + \int_0^t \langle \mathbf{\Gamma}(g(\tau), g(\tau)), \phi(\tau)\rangle \mathrm{d}\tau. \end{aligned}$$

This shows that $g \in L^\infty([0, +\infty[;\, Q^{\alpha}(\mathbb{R}^3)) \cap C^1([0, +\infty[;\, \mathscr{S}'(\mathbb{R}^3))$ is a weak solution to the Cauchy problem (1.3).

Step 2. Uniqueness of the solution to the Cauchy problem (1.3).

Assume that $\hat{g} \in L^\infty([0,+\infty[; Q^\alpha(\mathbb{R}^3)) \cap C^1([0,+\infty[; \mathscr{S}'(\mathbb{R}^3))$ is another radially symmetric solution to the Cauchy problem (1.3). Denote

$$h(t) = g(t) - \hat{g}(t).$$

Then $h \in L^\infty([0,+\infty[; Q^\alpha(\mathbb{R}^3)) \cap C^1([0,+\infty[; \mathscr{S}'(\mathbb{R}^3))$ and $h(0) = 0 \in \mathscr{S}'(\mathbb{R}^3)$, which means that

$$\langle h(0), \phi\rangle = 0, \qquad \forall \phi \in \mathscr{S}(\mathbb{R}^3).$$

For $T > 0$ and for any $0 < t \le T$ and any $\phi(t) \in C^1\big([0,T]; \mathscr{S}(\mathbb{R}^3)\big)$, one has

$$\langle h(t), \phi(t)\rangle + \int_0^t \langle h, \mathcal{L}\phi\rangle \mathrm{d}\tau = \int_0^t \langle h, \partial_\tau \phi\rangle \mathrm{d}\tau. \tag{4.2}$$

We define a smooth function

$$\phi(t) = \sum_{n=0}^N \langle h(t), \varphi_n\rangle \varphi_n \in C^1\big([0,T]; \mathscr{S}(\mathbb{R}^3)\big),$$

substituted $\mathcal{H}^\alpha \phi(t)$ into (4.2) as a test function, one can verify that

$$\|\mathbb{S}_N h\|_{Q^\alpha(\mathbb{R}^3)} \le 0.$$

Passing $N \to +\infty$, we have

$$\|h\|_{Q^\alpha(\mathbb{R}^3)} \le 0.$$

Thus $h = 0$ *in* $L^\infty([0,+\infty[; Q^\alpha(\mathbb{R}^3)) \cap C^1([0,+\infty[; \mathscr{S}'(\mathbb{R}^3))$.

Step 3. We prove now the regularity of the weak solution to the Cauchy problem (1.3) with the initial data $g^0 \in Q^\alpha(\mathbb{R}^3)$ with $\alpha < -\frac{3}{2}$.

It is obviously that

$$\|\mathrm{e}^{t\mathcal{L}} \mathcal{H}^{\frac{\alpha}{2}} \mathbb{S}_N g(t)\|_{L^2(\mathbb{R}^3)} \le \|\mathcal{H}^{\frac{\alpha}{2}} \mathbb{S}_N g^0\|_{L^2(\mathbb{R}^3)} \le \|\mathcal{H}^{\frac{\alpha}{2}} g^0\|_{L^2(\mathbb{R}^3)}.$$

We can deduce from the monotone convergence theorem that

$$\|\mathrm{e}^{t\mathcal{L}} \mathcal{H}^{\frac{\alpha}{2}} g(t)\|_{L^2(\mathbb{R}^3)} \le \|\mathcal{H}^{\frac{\alpha}{2}} g^0\|_{L^2(\mathbb{R}^3)}.$$

This shows the Gelfand–Shilov $S^{\frac{1}{2}}_{\frac{1}{2}}(\mathbb{R}^3)$ smoothing effect for the solution $g(t)$. This concludes the proof of Theorem 1.1. □

Proof of Remark 1.1. We show the Gelfand–Shilov smoothing effect with the initial datum (1.5); here we only need to prove $g^0 \in Q^\alpha_r(\mathbb{R}^3) \cap \mathcal{N}^\perp$ with $\alpha < -\frac{3}{2}$. Recalling the spectrum functions $\varphi_0(v)$ and $\varphi_1(v)$ at the beginning of this Section, we have

$$g^0 = \frac{1}{\sqrt{\mu}}\delta_{\mathbf{0}} - \sqrt{\mu} - \left(\frac{3}{2} - \frac{|v|^2}{2}\right)\sqrt{\mu} = \frac{1}{\sqrt{\mu}}\delta_{\mathbf{0}} - \varphi_0 - \sqrt{\frac{3}{2}}\varphi_1;$$

one can verify that

$$\langle g^0, \varphi_0\rangle = \langle \delta_{\mathbf{0}}, 1\rangle - \langle \varphi_0, \varphi_0\rangle = 0$$
$$\langle g^0, \varphi_1\rangle = \langle \delta_{\mathbf{0}}, \sqrt{\frac{2}{3}}\left(\frac{3}{2} - \frac{|v|^2}{2}\right)\rangle - \sqrt{\frac{3}{2}}\langle \varphi_1, \varphi_1\rangle = 0.$$

This shows that $g^0 \in \mathcal{N}^\perp$. Now we prove that $g^0 \in Q^\alpha_r(\mathbb{R}^3)$. Since $g^0 \in \mathcal{N}^\perp$, we can write g^0 in the form

$$g^0 = \sum_{k=2}^{+\infty} \langle g^0, \varphi_k\rangle \varphi_k,$$

where we can calculate that, for $k \ge 2$,

$$\langle g^0, \varphi_k\rangle = \langle \mu^{-\frac{1}{2}}\delta_{\mathbf{0}}, \varphi_k\rangle = \sqrt{\frac{2\Gamma(k+\frac{3}{2})}{\sqrt{\pi}k!}}.$$

By using the Stirling equivalent

$$\Gamma(x+1) \sim_{x\to+\infty} \sqrt{2\pi x}\left(\frac{x}{e}\right)^x,$$

we have that, $\forall k \geq 2$

$$\sqrt{\frac{2\Gamma(k+\frac{3}{2})}{\sqrt{\pi}k!}} \sim k^{\frac{1}{4}}.$$

Therefore, for any $\alpha < -\frac{3}{2}$,

$$\|g^0\|^2_{Q_r^\alpha(\mathbb{R}^3)} = \sum_{k=2}^{+\infty} k^\alpha |\langle g^0, \varphi_k\rangle|^2 \lesssim \sum_{k=2}^{+\infty} k^{\alpha+\frac{1}{2}} < +\infty.$$

This implies that $g^0 \in Q_r^\alpha(\mathbb{R}^3)$ with $\alpha < -\frac{3}{2}$; we end the proof of the Example. □

5. Appendix

Important known results but really needed for this paper are presented in this section. For the self-containedness of paper, we will present a proof of those properties.

Gelfand–Shilov spaces. The symmetric Gelfand–Shilov space $S_\nu^\nu(\mathbb{R}^3)$ can be characterized through a decomposition into the Hermite basis $\{H_\alpha\}_{\alpha\in\mathbb{N}^3}$ and the harmonic oscillator $\mathcal{H} = -\triangle + \frac{|v|^2}{4}$. For more details, see Theorem 2.1 in the book [3]

$$\begin{aligned} f \in S_\nu^\nu(\mathbb{R}^3) &\Leftrightarrow f \in C^\infty(\mathbb{R}^3), \exists\, \tau > 0, \|e^{\tau\mathcal{H}^{\frac{1}{2\nu}}} f\|_{L^2} < +\infty; \\ &\Leftrightarrow f \in L^2(\mathbb{R}^3), \exists\, \epsilon_0 > 0, \left\|\left(e^{\epsilon_0|\alpha|^{\frac{1}{2\nu}}} (f, H_\alpha)_{L^2}\right)_{\alpha\in\mathbb{N}^3}\right\|_{l^2} < +\infty; \\ &\Leftrightarrow \exists\, C > 0,\ A > 0,\ \|(-\triangle + \frac{|v|^2}{4})^{\frac{k}{2}} f\|_{L^2(\mathbb{R}^3)} \leq AC^k (k!)^\nu,\ k \in \mathbb{N}, \end{aligned}$$

where

$$H_\alpha(v) = H_{\alpha_1}(v_1)H_{\alpha_2}(v_2)H_{\alpha_3}(v_3),\ \alpha \in \mathbb{N}^3,$$

and for $x \in \mathbb{R}$,

$$H_n(x) = \frac{(-1)^n}{\sqrt{2^n n!\pi}} e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-x^2}) = \frac{1}{\sqrt{2^n n!\pi}}\left(x - \frac{d}{dx}\right)^n (e^{-\frac{x^2}{2}}).$$

For the harmonic oscillator $\mathcal{H} = -\triangle + \frac{|v|^2}{4}$ of 3-dimension and $s > 0$, we have

$$\mathcal{H}^{\frac{k}{2}} H_\alpha = (\lambda_\alpha)^{\frac{k}{2}} H_\alpha,\ \lambda_\alpha = \sum_{j=1}^3 (\alpha_j + \frac{1}{2}),\ k \in \mathbb{N},\ \alpha \in \mathbb{N}^3.$$

Shubin spaces. We refer the reader to the works [3,12] for the Shubin spaces. Let $\tau \in \mathbb{R}$. The Shubin spaces $Q^\tau(\mathbb{R}^3)$ can be also characterized through the decomposition into the Hermite basis:

$$\begin{aligned} f \in Q^\tau(\mathbb{R}^3) &\Leftrightarrow f \in \mathcal{S}'(\mathbb{R}^3),\ \left\|\mathcal{H}^{\frac{\tau}{2}} f\right\|_{L^2} < +\infty; \\ &\Leftrightarrow f \in \mathcal{S}'(\mathbb{R}^3),\ \left\|\left((|\alpha| + \frac{3}{2})^{\tau/2} (f, H_\alpha)_{L^2}\right)_{\alpha\in\mathbb{N}^3}\right\|_{l^2} < +\infty, \end{aligned}$$

where $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$,

$$H_\alpha(v) = H_{\alpha_1}(v_1)H_{\alpha_2}(v_2)H_{\alpha_3}(v_3),\ \alpha \in \mathbb{N}^3,$$

and for $x \in \mathbb{R}$, $n \in \mathbb{N}$,

$$H_n(x) = \frac{1}{(2\pi)^{\frac{1}{4}}} \frac{1}{\sqrt{n!}} \left(\frac{x}{2} - \frac{d}{dx}\right)^n (e^{-\frac{x^2}{4}}).$$

Thus, we have

$$Q^{-\tau}(\mathbb{R}^3) = \left(Q^{\tau}(\mathbb{R}^3)\right)', \tag{5.1}$$

where $\left(Q^{\tau}(\mathbb{R}^3)\right)'$ is the dual space of $Q^{\tau}(\mathbb{R}^3)$. We can also refer to the Appendix in [10].

Remark that, for $\tau > 0$,

$$Q^{\tau}(\mathbb{R}^3) \subsetneq H^{\tau}(\mathbb{R}^3),$$

where $H^{\tau}(\mathbb{R}^3)$ is the usual Sobolev space. In fact,

$$\mathcal{H}f \in L^2(\mathbb{R}^3) \;\Rightarrow\; \triangle f,\, |v|^2 f \in L^2(\mathbb{R}^3),$$

so that for the negative index, we have, by duality (5.1),

$$H^{-\tau}(\mathbb{R}^3) \subsetneq Q^{-\tau}(\mathbb{R}^3).$$

Moreover,

$$u \in Q_r^{\tau}(\mathbb{R}^3) \cap \mathcal{N}^{\perp} \Leftrightarrow u \in \mathcal{S}'(\mathbb{R}^3), \sum_{n=2}^{+\infty}(2n+\frac{3}{2})^{\tau}\langle u, \varphi_n\rangle^2 < +\infty.$$

Acknowledgements

The author deeply thanks Professor Léo Glangetas and Professor Chao-Jiang Xu for their valuable discussions all along this work. This research is supported by the Natural Science Foundation of China (No. 11701578) and the Fundamental Research Funds for the Central Universities of China, South-Central University for Nationalities (No. CZT18011).

References

[1] L. Glangetas, M. Najeme, Analytical regularizing effect for the radial homogeneous Boltzmann equation, Kinet. Relat. Models 6 (2013) 407–427.
[2] L. Glangetas, H.-G. Li, C.-J. Xu, Sharp regularity properties for the non-cutoff spatially homogeneous Boltzmann equation, Kinet. Relat. Models 9 (2016) 299–371.
[3] T. Gramchev, S. Pilipović, L. Rodino, Classes of Degenerate Elliptic Operators in Gelfand–Shilov Spaces. New Developments in Pseudo-Differential Operators, Birkhäuser, Basel, Switzerland, 2009, pp. 15–31.
[4] N. Lekrine, C.-J. Xu, Gevrey regularizing effect of the Cauchy problem for non-cutoff homogeneous Kac equation, Kinet. Relat. Models 2 (2009) 647–666.
[5] N. Lerner, Y. Morimoto, K. Pravda-Starov, C.-J. Xu, Phase space analysis and functional calculus for the linearized Landau and Boltzmann operators, Kinet. Relat. Models 6 (2013) 625–648.
[6] N. Lerner, Y. Morimoto, K. Pravda-Starov, C.-J. Xu, Gelfand–Shilov smoothing properties of the radially symmetric spatially homogeneous Boltzmann equation without angular cutoff, J. Differ. Equ. 256 (2014) 797–831.
[7] H.-G. Li, C.-J. Xu, Cauchy problem for the spatially homogeneous Landau equation with Shubin class initial datum and Gelfand–Shilov smoothing effect, arXiv:1708.06573.
[8] Y. Morimoto, S. Ukai, Gevrey smoothing effect of solutions for spatially homogeneous nonlinear Boltzmann equation without angular cutoff, J. Pseud.-Differ. Oper. Appl. 1 (2010) 139–159.
[9] Y. Morimoto, S. Ukai, C.-J. Xu, T. Yang, Regularity of solutions to the spatially homogeneous Boltzmann equation without angular cutoff, Discrete Contin. Dyn. Syst. 24 (2009) 187–212.
[10] Y. Morimoto, K. Pravda-Starov, C.-J. Xu, A remark on the ultra-analytic smoothing properties of the spatially homogeneous Boltzmann equation, Kinet. Relat. Models 6 (2013) 715–727.
[11] G. Sansone, Orthogonal Functions, Pure and Applied Mathematics, vol. IX, 1959.
[12] M. Shubin, Pseudodifferential Operators and Spectral Theory, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1987.
[13] C. Villani, On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations, Arch. Ration. Mech. Anal. 143 (1998) 273–307.
[14] C. Villani, On the spatially homogeneous Landau equation for Maxwellian molecules, Math. Models Methods Appl. Sci. 8 (1998) 957–983.
[15] C. Villani, A review of mathematical topics in collisional kinetic theory, in: S. Friedlander, D. Serre (Eds.), Handbook of Mathematical Fluid Dynamics, vol. 1, 2002, pp. 71–305.
[16] T.-F. Zhang, Z. Yin, Gevrey regularity of spatially homogeneous Boltzmann equation without cutoff, J. Differ. Equ. 253 (2012) 1172–1190.