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A Serrin-type symmetry result on model manifolds: An extension of the Weinberger argument

Un résultat de symétrie de type Serrin pour les variétés modèles : une extension de l'argument de Weinberger

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A R T I C L E I N F O

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A B S T R A C T

We consider the classical "Serrin's symmetry result" for the overdetermined boundary value problem related to the equation $\Delta u = -1$ in a model manifold of non-negative Ricci curvature. Using an extension of the Weinberger classical argument we prove a Euclidean symmetry result under a suitable "compatibility" assumption between the solution and the geometry of the model.

R É S U M É

Nous considérons le résultat classique de « symétrie de Serrin » pour les problèmes à valeurs à la frontière surdéterminés, pour l'équation $\Delta u = -1$ sur une variété modèle de courbure de Ricci positive ou nulle. Utilisant une extension de l'argument également classique de Weinberger, nous montrons un résultat de symétrie euclidienne sous une hypothèse de « compatibilité » entre la solution et la géométrie du modèle.

1. Preliminaries and statement of the result

A classical result obtained by Serrin in [16] is the following one.

Theorem 1. *Let Ω be a bounded domain in the Euclidean space $\mathbb{R}^m$ whose boundary is of class C^2. Suppose that Ω supports a solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ to the overdetermined problem*

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$$\begin{cases} \Delta u = -1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = \text{constant} & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where ν denotes the exterior unit normal to $\partial\Omega$. Then Ω is a ball and u has this specific form

$$u(r) = \frac{1}{2m}(b^2 - r^2), \tag{2}$$

where b is the radius of the ball and r denotes distance from its centre.

This result is known as the "Serrin's symmetry result" or the "Serrin's rigidity result". The technique used by Serrin to prove this result is a refinement of the famous *reflection principle* due to Alexandrov in [2], and is the so-called "moving planes method" together with the Maximum Principle and a new version of Hopf's boundary point Lemma. In particular, Alexandrov introduced this method to prove that a closed (i.e. compact without boundary) hypersurface embedded in the Euclidean space $\mathbb{R}^m$ with constant mean curvature must be a sphere. Moreover, in [11], Kumaresan and Prajapat used the same method of the moving planes to prove the analogous of the "Serrin's symmetry result" in the case of bounded domains of the hyperbolic space $\mathbb{H}^m$ and of the hemisphere $\mathbb{S}^m_+$.

We mention that the technique of Serrin inspired the study of various properties and symmetry results for positive solutions to elliptic partial differential equations in bounded and unbounded domains of the Euclidean space (see the seminal paper by Gidas, Ni and Nirenberg [10]).

In this article, we focus on the more analytic approach by Weinberger [18], which is based on the Maximum Principle, the integration by parts, the Cauchy–Schwarz inequality, and the Bochner–Weitzenboch formula. We try to extend his proof to the so-called model manifolds with non-negative Ricci curvature.

We mention that the approach of Weinberger inspired several works in the context of elliptic partial differential equations (see, e.g., [3,5–9,12,13,17] and references therein).

In general, as we will see, the importance and the convenience of the model manifolds lie in the fact that their geometry and some natural differential operators (such as the Laplacian) have a particularly simple and explicit description.

First of all, we recall the definition of the m-dimensional model manifold.

Definition 2. A Riemannian manifold $(\mathbb{M}^m_\sigma, g_{\mathbb{M}^m_\sigma})$ is called a model manifold if:

$$\mathbb{M}^m_\sigma := \frac{[0,R) \times \mathbb{S}^{m-1}}{\sim} \quad \text{and} \quad g_{\mathbb{M}^m_\sigma} := \mathrm{d}r \otimes \mathrm{d}r + \sigma^2(r) g_{\mathbb{S}^{m-1}};$$

where $R \in (0, +\infty]$, $\sim$ is the relation that identifies all the points of $\{0\} \times \mathbb{S}^{m-1}$ and $\sigma : [0,R) \to [0,+\infty)$ is a smooth function such that:

- $\sigma(r) > 0$, for all $r > 0$;
- $\sigma^{(2k)}(0) = 0$, for all $k = 0, 1, 2, \ldots$;
- $\sigma'(0) = 1$.

Moreover, the unique point corresponding to $r = 0$ is called the pole of the model and denoted by $o \in \mathbb{M}^m_\sigma$; σ is called the warping function.

Important examples of model manifolds are the so called space-forms: $\mathbb{R}^m$, $\mathbb{H}^m$ and $\mathbb{S}^m$. Explicitly,

- the Euclidean space $\mathbb{R}^m$ is isometric to the model manifold $\mathbb{M}^m_\sigma$ with $\sigma(r) = r : [0,+\infty) \to [0,+\infty)$;
- the hyperbolic space $\mathbb{H}^m$ is isometric to the model manifold $\mathbb{M}^m_\sigma$ with $\sigma(r) = \sinh(r) : [0,+\infty) \to [0,+\infty)$;
- the standard sphere $\mathbb{S}^m \setminus \{N\}$ is isometric to the model manifold $\mathbb{M}^m_\sigma$ with $\sigma(r) = \sin(r) : [0,\pi) \to [0,+\infty)$.

We also recall that in $\mathbb{M}^m_\sigma$ the Ricci curvature has the following explicit expression (see e.g. [14]). Given $x \in \mathbb{M}^m_\sigma$ and $X \in \nabla r(x)^\perp$ in $T_x\mathbb{M}^m_\sigma$ a unit vector, we have

$$\mathrm{Ric}_{\mathbb{M}^m_\sigma}(X,X) = (m-2)\frac{1-(\sigma')^2}{\sigma^2} - \frac{\sigma''}{\sigma},$$

and

$$\mathrm{Ric}_{\mathbb{M}^m_\sigma}(\nabla r, \nabla r) = -(m-1)\frac{\sigma''}{\sigma}.$$

With these preliminaries, the main Theorem of this article is the following one.

Theorem 3. *Let $\Omega \subset \mathbb{M}^m_\sigma$ be a smooth domain with $o \in \Omega$. Assume that $\Omega \Subset B_{\tilde{R}}(o)$ where the ray $\tilde{R} > 0$ is such that the following conditions on σ are satisfied on the interval $[0, \tilde{R})$:*

(*a*) $\mathrm{Ric}_{\mathbb{M}^m_\sigma} \geq 0$, *i.e.* $\sigma'' \leq 0$ *and* $(m-2)\left(1-(\sigma')^2\right) - \sigma\,\sigma'' \geq 0$;
(*b*) $\sigma' > 0$.

If Ω supports a solution u to (1) and u satisfies the following "compatibility" condition

$$\int_\Omega \frac{(\sigma''\sigma^{m-1})'}{\sigma^{m-1}} u^2 \geq 0, \tag{3}$$

then we have that Ω is a Euclidean ball of radius ρ centred in the pole o of the model and u has the specific form:

$$u(r) = \frac{1}{2m}(\rho^2 - r^2) \tag{4}$$

where $r(x) = dist(x, o)$.

Remark 4. We analyse the hypothesis of the Theorem.

- Condition (*b*) appears in other articles on the subject, see for instance [4] by Ciraolo and Vezzoni.
- The "compatibility" condition (3) describes a property of the solution in relation to the geometry of the model. It is automatically satisfied by any solution to (1) in the case of the Euclidean space and it can not be reduced to a simple condition on the model, like

  $$(\sigma''\sigma^{m-1})' \geq 0.$$

  Indeed, in this case, the three conditions are compatible only with the flat case: consider $f(r) := \sigma''(r)\sigma^{m-1}(r)$. Then $f(0) = 0$ and if $f'(r) \geq 0$, i.e. $f(r)$ is non-decreasing, so $f(r) \geq 0$ for $r > 0$. But $\sigma''(r) \leq 0$ according to (*a*), so we have that $\sigma''(r) = 0$. In this case, the result is well known and is presented in Weinberger's article.
- Moreover, in [1], Alessandrini and Magnanini consider a symmetry result for a overdetermined problem, and they assume a "compatibility" condition as an integral on the boundary of the domain involving the solution and its gradient.

Remark 5. Observe that, by the Strong Maximum Principle, a solution u to (1) is positive in Ω. Moreover since $\partial_\nu u = constant \neq 0$ on Ω we obtain that $|\nabla u| \neq 0$ on $\partial\Omega$ and the smooth hypersurface $\partial\Omega = \{u = 0\}$ has exterior normal given by

$$\nu = -\frac{\nabla u}{|\nabla u|}\,|_{\partial\Omega}\,.$$

This implies that

$$\partial_\nu u = -\,|\nabla u| \text{ on } \partial\Omega.$$

2. Explicit computations towards the proof of Theorem 3

The Laplace–Beltrami operator Δ of $\mathbb{M}^m_\sigma$ acts on C^2-functions $u : \mathbb{M}^m_\sigma \to \mathbb{R}$ as follows:

$$\begin{aligned}\Delta u &= \partial_r^2 u + (m-1)\frac{\sigma'}{\sigma}\partial_r u + \frac{1}{\sigma^2}\bar{\Delta} u \\ &= \frac{\partial_r(\sigma^{m-1}\partial_r u)}{\sigma^{m-1}} + \frac{1}{\sigma^2}\bar{\Delta} u,\end{aligned} \tag{5}$$

where $\bar{\Delta}$ denotes the Laplacian on the standard sphere $(\mathbb{S}^{m-1}, g_{\mathbb{S}^{m-1}})$. Using this expression, we obtain the following lemma.

Lemma 6. *The following general formula holds:*

$$\Delta(\sigma\,\partial_r u) = \sigma\,\partial_r \Delta u + 2\sigma'\,\Delta u + (2-m)\sigma''\,\partial_r u. \tag{6}$$

Remark 7. In particular, if $\sigma(r) = r$ and, hence, $\mathbb{M}^m_\sigma = \mathbb{R}^m$, we obtain

$$\Delta(r\,\partial_r u) = r\,\partial_r \Delta u + 2\Delta u, \tag{7}$$

which is the traditional formula used by Weinberger to prove Serrin's result.

Proof. We compute

$$\begin{aligned}\sigma\,\partial_r(\Delta u) &= \sigma\left\{\partial_r^3 u + (m-1)\frac{\sigma''\sigma-(\sigma')^2}{\sigma^2}\partial_r u + (m-1)\frac{\sigma'}{\sigma}\partial_r^2 u - 2\frac{\sigma'}{\sigma^3}\bar{\Delta}u + \frac{1}{\sigma^2}\partial_r(\bar{\Delta}u)\right\} \\ &= \sigma\,\partial_r^3 u + (m-2)\sigma''\partial_r u + \sigma''\partial_r u - 2(m-1)\frac{(\sigma')^2}{\sigma}\partial_r u + (m-1)\frac{(\sigma')^2}{\sigma}\partial_r u + \\ &\quad + (m+1)\sigma'\partial_r^2 u - 2\sigma'\partial_r^2 u - 2\frac{\sigma'}{\sigma^2}\bar{\Delta}u + \frac{1}{\sigma}\partial_r(\bar{\Delta}u) + \frac{1}{\sigma^2}\bar{\Delta}(\sigma\,\partial_r u) - \frac{1}{\sigma^2}\bar{\Delta}(\sigma\,\partial_r u) \\ &= \Delta(\sigma\,\partial_r u) + (m-2)\sigma''\partial_r u - 2\sigma'\Delta u + \frac{1}{\sigma}\partial_r(\bar{\Delta}u) - \frac{1}{\sigma^2}\bar{\Delta}(\sigma\,\partial_r u),\end{aligned}$$

i.e.

$$\Delta(\sigma\,\partial_r u) = \sigma\,\partial_r(\Delta u) + (2-m)\sigma''\partial_r u + 2\sigma'\Delta u. \quad \square$$

Now we focus on the solution u to (1) (from now on, we put the constant in (1) equal to c) and we show the following lemma.

Lemma 8. *Let Ω and u satisfy (1). Then:*

$$(m+2)\int_\Omega u\,\sigma' = mc^2\int_\Omega \sigma' + \frac{(m-2)}{2}\int_\Omega \frac{(\sigma''\sigma^{m-1})'}{\sigma^{m-1}}u^2. \tag{8}$$

Remark 9. In particular, if $\sigma(r) = r$ and, hence, $\mathbb{M}^m_\sigma = \mathbb{R}^m$, we obtain

$$(m+2)\int_\Omega u = mc^2|\Omega|, \tag{9}$$

as in the original Weinberger argument, where $|\Omega|$ is the volume of the domain Ω.

Proof. First of all we observe that, in this setting, formula (6) becomes

$$\Delta(\sigma\,\partial_r u) = -2\sigma' + (2-m)\sigma''\,\partial_r u.$$

So by Green's Theorem

$$\begin{aligned}\int_\Omega -2\sigma' u + (2-m)\sigma''\,\partial_r u\, u + \sigma\,\partial_r u &= \int_\Omega \Delta(\sigma\,\partial_r u)u - \sigma\,\partial_r u\,\Delta u \\ &= \int_{\partial\Omega} \partial_\nu(\sigma\,\partial_r u)u - \sigma\,\partial_r u\,\partial_\nu u \\ &= -\int_{\partial\Omega} \sigma\,(\partial_\nu u)^2\partial_\nu r \\ &= -c^2\int_{\partial\Omega}\sigma\,\partial_\nu r \\ &= -c^2\int_\Omega \sigma\,\Delta r + g_{\mathbb{M}^m_\sigma}(\nabla r, \nabla\sigma) \\ &= -c^2\int_\Omega \sigma\,(m-1)\frac{\sigma'}{\sigma} + \sigma' \\ &= -c^2 m\int_\Omega \sigma',\end{aligned}$$

where we have used the fact that $u = 0$ on $\partial\Omega$ and that $\partial_\nu u = c$ on $\partial\Omega$.

Now note that

$$\begin{aligned}\int_{\Omega} \sigma\, \partial_r u &= \int_{\Omega} g_{\mathbb{M}^m_\sigma}(\nabla u, \nabla(\int_0^r \sigma(s)\,\mathrm{d}s)) \\ &= -\int_{\Omega} u\,\Delta(\int_0^r \sigma(s)\,\mathrm{d}s) \\ &= -m\int_{\Omega} u\sigma'.\end{aligned}$$

Using this and the previous computation we have

$$(m+2)\int_{\Omega} u\,\sigma' = mc^2\int_{\Omega} \sigma' + (2-m)\int_{\Omega} \sigma''\, u\partial_r u. \tag{10}$$

Finally we observe that

$$\begin{aligned}\int_{\Omega} \sigma''\, u\partial_r u &= \int_{\Omega} g_{\mathbb{M}^m_\sigma}(\nabla\sigma', \nabla(\frac{1}{2}u^2)) \\ &= -\frac{1}{2}\int_{\Omega} \Delta\sigma'\, u^2 \\ &= -\frac{1}{2}\int_{\Omega} \frac{(\sigma''\sigma^{m-1})'}{\sigma^{m-1}} u^2,\end{aligned} \tag{11}$$

where the second and the third equations are obtained using the condition $u=0$ on $\partial\Omega$ and the expression (5), respectively. □

3. Proof of Theorem 3

Now we are ready to prove the main result of this paper.

Proof of Theorem 3. Let u and Ω as in the statement of Theorem 3; by the Bochner–Weitzenboch formula and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\Delta\left(m|\nabla u|^2 + 2u\right) &= 2m|\mathrm{Hess}(u)|^2 + 2m\mathrm{Ric}_{\mathbb{M}^m_\sigma}(\nabla u, \nabla u) + 2\Delta u \\ &\geq 2(m|\mathrm{Hess}(u)|^2 + \Delta u) \\ &= 2\left(m|\mathrm{Hess}(u)|^2 - (\Delta u)^2\right) \\ &\geq 0 \text{ on } \Omega,\end{aligned} \tag{12}$$

and the equality holds if and only if

$$\mathrm{Hess}(u) = \frac{\Delta u}{m} g_{\mathbb{M}^m_\sigma}$$

and

$$\mathrm{Ric}_{\mathbb{M}^m_\sigma}(\nabla u, \nabla u) = 0.$$

Since, according to Remark 5,

$$\left(m|\nabla u|^2 + 2u\right) = mc^2 \text{ on } \partial\Omega, \tag{13}$$

we conclude from the Strong Maximum Principle that either

$$\left(m|\nabla u|^2 + 2u\right) < mc^2 \text{ on } \Omega \tag{14}$$

or

$$\left(m|\nabla u|^2 + 2u\right) \equiv mc^2 \text{ on } \Omega. \tag{15}$$

By contradiction, assume that condition (14) is satisfied. According to (b), we can multiply both members of (14) by σ' and integrate in order to obtain

$$m\int_\Omega |\nabla u|^2\sigma' + 2\int_\Omega u\,\sigma' < mc^2\int_\Omega \sigma'. \tag{16}$$

Now we use the identity (8) to deal with the second term, i.e.

$$2\int_\Omega u\,\sigma' = mc^2\int_\Omega \sigma' + \frac{(m-2)}{2}\int_\Omega \frac{(\sigma''\sigma^{m-1})'}{\sigma^{m-1}}u^2 - m\int_\Omega u\sigma'. \tag{17}$$

Note that, by the divergence theorem,

$$m\int_\Omega \sigma'\operatorname{div}(u\nabla u) = -m\int_\Omega \sigma'' u\,\partial_r u. \tag{18}$$

Moreover,

$$m\int_\Omega \sigma'\operatorname{div}(u\nabla u) = m\int_\Omega \sigma'|\nabla u|^2 - m\int_\Omega \sigma' u.$$

So

$$m\int_\Omega \sigma'|\nabla u|^2 = m\int_\Omega \sigma' u - m\int_\Omega \sigma'' u\,\partial_r u. \tag{19}$$

Substituting (17) and (19) in (16), we obtain:

$$-m\int_\Omega \sigma'' u\,\partial_r u + m\int_\Omega \sigma' u + mc^2\int_\Omega \sigma' + \frac{(m-2)}{2}\int_\Omega \frac{(\sigma''\sigma^{m-1})'}{\sigma^{m-1}}u^2 - m\int_\Omega u\sigma' < mc^2\int_\Omega \sigma'.$$

Lastly, we use the identity (11) to deduce

$$\frac{m}{2}\int_\Omega \frac{(\sigma''\sigma^{m-1})'}{\sigma^{m-1}}u^2 + \frac{(m-2)}{2}\int_\Omega \frac{(\sigma''\sigma^{m-1})'}{\sigma^{m-1}}u^2 < 0,$$

i.e.

$$-(m-1)\int_\Omega \frac{(\sigma''\sigma^{m-1})'}{\sigma^{m-1}}u^2 > 0; \tag{20}$$

and this contradicts the “compatibility” condition (3).

Therefore, (15) holds true and $m|\nabla u|^2 + 2u$ must be constant in Ω. Since its Laplacian then vanishes, we conclude from (12) that equality must hold in the Cauchy–Schwarz inequality, i.e. we have proved that u is a solution to (recall that $\Delta u = -1$ in Ω)

$$\operatorname{Hess}(u) = -\frac{1}{m} g_{\mathbb{M}^m_\sigma} \text{ in } \Omega. \tag{21}$$

Now, let $\rho := dist(o, \partial\Omega)$ and take $B_\rho(o) \subset \Omega$. Since $\partial\Omega$ is compact, there exists $p \in \partial\Omega$ such that $p \in \partial\Omega \cap \partial B_\rho(o)$. In particular, since $u = 0$ on $\partial\Omega$, we have that

$$u(p) = 0.$$

If we prove that u is a radial function in $B_\rho(o)$ then

$$u = 0 \text{ on } \partial B_\rho(o).$$

On the other hand, by the Strong Maximum Principle,

$$u > 0 \text{ in } \Omega.$$

Therefore we can conclude that $\partial B_\rho(o) \cap \Omega = \emptyset$ and, hence, $\Omega = B_\rho(o)$.

So the keypoint is to prove that $u : B_\rho(o) \to \mathbb{R}$, solution to (21), is a radial function in $B_\rho(o)$. To this end, take $x \in B_\rho(o)$. Since $\mathbb{M}^m_\sigma$ is geodesically complete there exist a minimizing and normalized geodesic $\gamma \subset B_\rho(o)$ from o to x. Let $y(t) := u \circ \gamma(t)$ and note that, along γ, equation (21) implies:

$$\begin{aligned} y''(t) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}(u \circ \gamma)(t) \\ &= \frac{d}{\mathrm{d}t} g_{\mathbb{M}^m_\sigma}(\nabla u(\gamma(t)), \dot{\gamma}(t)) \\ &= g_{\mathbb{M}^m_\sigma}(D_{\dot{\gamma}} \nabla u(\gamma(t)), \dot{\gamma}(t)) + g_{\mathbb{M}^m_\sigma}(\nabla u(\gamma(t)), D_{\dot{\gamma}} \dot{\gamma}(t)) \\ &= g_{\mathbb{M}^m_\sigma}((D_{\dot{\gamma}(t)} \nabla u)(\gamma(t)), \dot{\gamma}(t)) \\ &= \mathrm{Hess}(u) \mid_{\gamma(t)} (\dot{\gamma}(t), \dot{\gamma}(t)) \\ &= -\frac{1}{m}. \end{aligned}$$

The solutions to $y''(t) = -\dfrac{1}{m}$ are given by

$$y(t) = -\frac{1}{2m} t^2 + \alpha t + \beta$$

where $\alpha, \beta \in \mathbb{R}$. Now taking $t = r(x)$ we get

$$u(x) = u \circ \gamma(r(x)) = y(r(x)) = -\frac{1}{2m} r(x)^2 + \alpha r(x) + \beta \tag{22}$$

which is radial. To determine the two constant in (22), we recall that u satisfies the following,

$$\begin{cases} u(\rho) = 0 \\ u(r) > 0 \quad \text{for } 0 < r < \rho \end{cases}$$

i.e., using the explicit formula of u, we obtain

$$\begin{cases} -\dfrac{1}{2m}\rho^2 + \alpha\rho + \beta = 0 \\ -\dfrac{1}{2m}\left(\dfrac{\rho}{2}\right)^2 + \alpha\dfrac{\rho}{2} + \beta > 0 \quad \text{for } r = \dfrac{\rho}{2} \end{cases}$$

substituting the expression $\beta = \dfrac{1}{2m}\rho^2 - \alpha\rho$ in the second equation, we get

$$\alpha < \frac{3}{4m}\rho.$$

But, since u must be a C^2-function, we have that $\alpha = 0$; indeed, if we consider the Euclidean case where $r(x) = d(x, 0) = |x|$, the gradient of u becomes

$$\nabla u(x) = -\frac{1}{m} x + \alpha \frac{x}{|x|} \tag{23}$$

which is not a C^1 function in the origin (i.e. the pole of the Euclidean space) unless $\alpha = 0$. In a generic model, the expression (23) holds in a system of normal coordinates in the pole. So the same conclusion holds and $\beta = \dfrac{1}{2m}\rho^2$; with this constants the function u becomes

$$u(r) = -\frac{1}{2m} r^2 + \frac{1}{2m}\rho^2$$

which is exactly the expression (4); observe that, since u is radial, $\partial_\nu u = u'(r)$ and the condition $\partial_\nu u = \textit{constant}$ in $\partial\Omega = \partial B_\rho(o)$ is automatically satisfied.

Moreover, we recall that if $f : \mathbb{M}^m_\sigma \to \mathbb{R}$ is a smooth radial function, then its Hessian takes the following expression

$$\mathrm{Hess}(f) = f'' \mathrm{d}r \otimes \mathrm{d}r + f' \sigma \sigma' g_{\mathbb{S}^{m-1}}. \tag{24}$$

Using this expression with the function u we get

$$\mathrm{Hess}(u) = -\frac{1}{m}\mathrm{d}r \otimes \mathrm{d}r - \frac{1}{m} r\sigma\sigma' g_{\mathbb{S}^{m-1}}, \tag{25}$$

and using this latter in (21) we obtain

$$-\frac{1}{m}\mathrm{d}r \otimes \mathrm{d}r - \frac{1}{m} r\sigma\sigma' g_{\mathbb{S}^{m-1}} = -\frac{1}{m}(\mathrm{d}r \otimes \mathrm{d}r + \sigma^2 g_{\mathbb{S}^{m-1}})$$

i.e.

$$-\frac{1}{m} r\sigma\sigma' g_{\mathbb{S}^{m-1}} = -\frac{1}{m}\sigma^2 g_{\mathbb{S}^{m-1}}.$$

It follows that $\sigma(r) = r$, so, in the ball $B_\rho(o)$, not only the solution to (21) is radial, but also the metric $g_{\mathbb{M}^m_\sigma}$ is the Euclidean metric. This implies that the ball $B_\rho(o)$ is a Euclidean ball, and the claim follows. □

Remark 10 *(An alternative end of the proof).* From the equality sign in the Bochner inequality (12) we get

$$\mathrm{Ric}_{\mathbb{M}^m_\sigma}(\nabla u, \nabla u) = 0. \tag{26}$$

From the explicit expression of u (formula (4)), we see that the only critical point is in $r = 0$, i.e. in the pole o of the model. So the condition on the Ricci curvature becomes

$$\mathrm{Ric}_{\mathbb{M}^m_\sigma}(\nabla r, \nabla r) = 0 \text{ in } B_\rho(o) \setminus \{o\}. \tag{27}$$

From the explicit expression of $\mathrm{Ric}_{\mathbb{M}^m_\sigma}(\nabla r, \nabla r)$, we get $\sigma'' = 0$ in $(0, \rho)$, and we conclude that $\sigma(r) = r$, i.e. $B_\rho(o)$ is a Euclidean ball.

Remark 11. In [15], by Ros, we can find a similar spirit where, using the Reilly's formula, he obtained a generalization of Alexandrov theorem for compact hypersurfaces with constant higher order mean curvatures; in this article, equation (21) is used to prove a Euclidean symmetry result on a generic compact Riemannian manifold of non-negative Ricci curvature with smooth boundary with mean curvature positive everywhere.

Remark 12. In this remark, we provide an example that shows that if the "compatibility" condition (3) is not satisfied, then we can not have Euclidean symmetry. According to the result of Kumaresan and Prajapat [11] we know that if we take a domain $\Omega \subset \mathbb{S}^m$ such that $\bar{\Omega} \subset \mathbb{S}^m_+$, and there exist a solution u to the Serrin's symmetry problem (1), then Ω must be a geodesic ball and u must be radially symmetric. We know that the hemisphere is isometric to the model $\mathbb{M}^m_\sigma$ with $\sigma(r) = \sin(r)\,|_{[0,\pi/2]}$; so in this example, conditions (a) and (b) of Theorem 3 are clearly satisfied and the "compatibility" condition (3) becomes:

$$\int_\Omega \frac{(\sigma''\sigma^{m-1})'}{\sigma^{m-1}} u^2 = \int_\Omega -m\cos(r)u^2(r),$$

which is negative due to the monotonicity of the integral and to the fact that the function $r \mapsto \cos(r)u^2(r)$ is positive in Ω.

In conclusion, the "compatibility" condition is not satisfied, and the symmetry result is not Euclidean, since the ball Ω is a geodesic ball, i.e. the metric in this ball is the metric of the sphere.

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