



Functional analysis

# On some eigenvalue inequalities for Schatten–von Neumann operators



*Sur quelques inégalités pour les valeurs propres d'opérateurs de Schatten–von Neumann*

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## ABSTRACT

Michael Gil' recently obtained some bounds for eigenvalues in [J. Funct. Anal. 267 (2014) 3500–3506] and [Commun. Contemp. Math. 18 (2016) 1550022], which improve some classical results related to this aspect. We revisit these results by providing genuinely different arguments (e.g., using Aluthge transform, majorization). New results are derived along our discussions.

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## R É S U M É

Dans les articles [J. Funct. Anal. 267 (2014) 3500–3506] et [Commun. Contemp. Math. 18 (2016) 1550022], Michael Gil' a récemment obtenu des bornes pour les valeurs propres d'opérateurs de Schatten–von Neumann qui améliorent des énoncés classiques dans ce contexte. Nous reprenons ces résultats et en donnons des preuves véritablement différentes (par exemple, en utilisant la transformation d'Aluthge, la majoration). Au fil des arguments, nous obtenons de nouveaux résultats.

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## 1. Introduction

We consider a separable Hilbert space  $\mathcal{H}$ . For a linear operator  $A$  on  $\mathcal{H}$ ,  $A^*$  is the adjoint operator,  $|A| := \sqrt{A^*A}$  is the absolute value of  $A$  (e.g., [10, p. 1]). If  $A$  is compact, then every nonzero element in the spectrum of  $A$  is an eigenvalue, and we denote by  $\lambda_k(A)$  ( $k = 1, 2, \dots$ ) the eigenvalues of  $A$  (if there exist any) with their multiplicities enumerated in the non-increasing order of their absolute values:  $|\lambda_k(A)| \geq |\lambda_{k+1}(A)|$ ;  $s_k(A)$  ( $k = 1, 2, \dots$ ) the singular values of  $A$  with their multiplicities, also enumerated in the non-increasing order. By  $SN_p$  ( $p \geq 1$ ) we mean the Schatten–von Neumann ideal of compact operators  $A$  with the finite norm

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$$\|A\|_p := (\text{trace } |A|^p)^{1/p} = \left( \sum_{k=1}^{\infty} s_k^p(A) \right)^{1/p}.$$

Most studied cases are the trace class operators  $SN_1$ , the Hilbert–Schmidt operators  $SN_2$  and bounded operators (viz, when  $p = \infty$ ). In addition,  $[A, A^*] = AA^* - A^*A$  is the self-commutator and

$$[A, A^*]_p = [A^p, (A^*)^p] = A^p(A^*)^p - (A^*)^p A^p, \quad p = 1, 2, \dots$$

The following classical result of comparing eigenvalues and singular values of a compact operator is pretty known (e.g., [6, Corollary II.3.1])

$$\sum_{k=1}^m |\lambda_k(A)|^2 \leq \sum_{k=1}^m s_k^2(A), \quad m = 1, 2, \dots \quad (1)$$

By Weyl's majorant theorem ([11], [10, p. 7]), (1) could be generalized to

$$\sum_{k=1}^m |\lambda_k(A)|^p \leq \sum_{k=1}^m s_k^p(A), \quad p \geq 1, m = 1, 2, \dots \quad (2)$$

If  $A \in SN_p$ , then (2) immediately yields

$$\sum_{k=1}^{\infty} |\lambda_k(A)|^p \leq \|A\|_p^p, \quad p \geq 1. \quad (3)$$

In [4] Gil' obtained a partial improvement of (3) as follows.

**Theorem 1.1.** [4, Theorem 1.1] For any  $A \in SN_{2p}$  ( $p = 1, 2, \dots$ ), one has:

$$\sum_{k=1}^{\infty} |\lambda_k(A)|^{2p} \leq \left[ \|A\|_{2p}^{4p} - \frac{1}{4} \|[A, A^*]_p\|_1^2 \right]^{1/2}. \quad (4)$$

In another paper [5], by making use of the result from [4], Gil' proved a strengthening of (1)

**Theorem 1.2.** [5, Theorem 1.1] For any compact operator  $A$ , one has

$$\left( \sum_{k=1}^m |\lambda_k(A)|^2 \right)^2 \leq 2 \sum_{1 \leq k < j \leq m} s_k^2(A) s_j^2(A) + \sum_{k=1}^m s_k^2(A^2), \quad m = 2, 3, \dots \quad (5)$$

This note intends to give new proofs of the previous two theorems. Our ideas of proof are quite different from the original ones given by Gil', and may facilitate new methods for treating these kinds of inequalities. New results and comments are included along our discussions.

## 2. Proof of Theorem 1.1 and more

Let  $A$  be an operator on  $\mathcal{H}$ , and let  $A = U|A| = |A^*|U$  be the polar decomposition of  $A$ . The Aluthge transform [1] is defined by

$$\Delta(A) = |A|^{1/2} U |A|^{1/2}.$$

Though the isometry  $U$  in the polar decomposition may not be unique,  $\Delta(A)$  is invariant with respect to the choice of  $U$ . It is clear from the definition that the Aluthge transform preserves the spectrum. Moreover, if  $A \in SN_p$ , then

$$\|\Delta(A)\|_p \leq \|A\|_p.$$

This follows immediately from the positivity of the  $2 \times 2$  block operator matrix

$$\begin{pmatrix} |A| & \Delta(A) \\ \Delta^*(A) & |A| \end{pmatrix}$$

defined on  $\mathcal{H} \oplus \mathcal{H}$ .

The key step in Gil’s proof of Theorem 1.1 is the following inequality [4, (2.6)]: Assuming  $A$  an operator on  $\mathcal{H}$  with  $\dim \mathcal{H} = n$ ,

$$\sum_{k=1}^n |\lambda_k(A)|^2 \leq \left[ \|A\|_2^4 - \frac{1}{4} \|[A, A^*]\|_1^2 \right]^{1/2}. \tag{6}$$

With (6), one replaces  $A$  with  $A^p$  ( $p = 1, 2, \dots$ ) and note that  $\|A^p\|_2 \leq \|A\|_{2p}^p$ . Then by a standard limit argument one considers  $A \in SN_p$ , so that (4) follows. We remark that inequality (6) has also appeared in [8, (18)] in the context of measuring non-normality for matrices.

We shall prove the following result, which leads to (6).

**Theorem 2.1.** *Let  $A \in SN_2$ . Then*

$$\|\Delta(A)\|_2^2 \leq \left[ \|A\|_2^4 - \frac{1}{4} \|[A, A^*]\|_1^2 \right]^{1/2}. \tag{7}$$

**Proof.** First of all, we prove a slightly more general result

**Claim.** *If  $M, N \in SN_1$  are selfadjoint positive definite, then*

$$\|M - N\|_1^2 \leq (\text{trace}(M + N))^2 - 4(\text{trace } M^{1/2}N^{1/2})^2. \tag{8}$$

**Proof of Claim.** Compute

$$\begin{aligned} & \|M - N\|_1^2 \\ &= \left( \frac{1}{2} \left\| (M^{1/2} + N^{1/2})(M^{1/2} - N^{1/2}) + (M^{1/2} - N^{1/2})(M^{1/2} + N^{1/2}) \right\|_1 \right)^2 \\ &\leq \left( \frac{1}{2} \left\| (M^{1/2} + N^{1/2})(M^{1/2} - N^{1/2}) \right\|_1 + \frac{1}{2} \left\| (M^{1/2} - N^{1/2})(M^{1/2} + N^{1/2}) \right\|_1 \right)^2 \\ &= \left\| (M^{1/2} + N^{1/2})(M^{1/2} - N^{1/2}) \right\|_1^2 \\ &\leq \|M^{1/2} + N^{1/2}\|_1 \|M^{1/2} - N^{1/2}\|_1 \\ &= \|M^{1/2} + N^{1/2}\|_1 \|M^{1/2} - N^{1/2}\|_1 \\ &= \text{trace}(M^{1/2} + N^{1/2})^2 \text{trace}(M^{1/2} - N^{1/2})^2 \\ &= (\text{trace}(M + N))^2 - 4(\text{trace } M^{1/2}N^{1/2})^2, \end{aligned}$$

in which the second inequality is by the Cauchy–Schwarz inequality. This proves the Claim.  $\square$

Note that

$$\begin{aligned} \|\Delta(A)\|_2^2 &= \| |A|^{1/2}U|A|^{1/2} \|_2^2 = \text{trace } |A|^{1/2}U^*|A|U|A|^{1/2} \\ &= \text{trace } U|A|U^*|A| = \text{trace } |A^*||A|. \end{aligned}$$

Substituting  $M = AA^*, N = A^*A$  into (8) gives (7).  $\square$

If we replace  $A$  with  $\Delta(A)$  in (6), then we have

$$\sum_{k=1}^n |\lambda_k(A)|^2 \leq \left[ \|\Delta(A)\|_2^4 - \frac{1}{4} \|\Delta(A), \Delta^*(A)\|_1^2 \right]^{1/2}. \tag{9}$$

Though it is known that  $\|\Delta(A)\|_2 \leq \|A\|_2$ , we could not claim that the bound in (9) is always better than that in (6). Indeed, numerical simulations suggest that

$$\|[A, A^*]\|_p \geq \|\Delta(A), \Delta^*(A)\|_p.$$

We note that even the case  $p = 2$  remains open (see [7, Conjecture 3.3]).

The following result is a complement of Theorem 1.1.

**Theorem 2.2.** Let  $A \in SN_p$  ( $p = 1, 2, \dots$ ). Then

$$\sum_{k=1}^{\infty} |\lambda_k(A)|^p \leq \left[ \|A\|_p^{2p} - \frac{1}{4} \| |(A^p)^* | - |A^p| \|_1^2 \right]^{1/2}.$$

**Proof.** It suffices to show  $p = 1$  case and this boils down to the following

$$\|\Delta(A)\|_1 \leq \left[ \|A\|_1^2 - \frac{1}{4} \| |A^*| - |A| \|_1^2 \right]^{1/2}. \tag{10}$$

A simple calculation gives

$$\|\Delta(A)\|_1 = \| |A^*|^{1/2} |A|^{1/2} \|_1,$$

so (10) is seen to be equivalent to

$$\| |A^*| - |A| \|_1^2 \leq 4\|A\|_1^2 - 4\| |A^*|^{1/2} |A|^{1/2} \|_1^2. \tag{11}$$

We need a strengthening of (8).

**Claim.** If  $M, N \in SN_1$  are selfadjoint positive definite, then

$$\|M - N\|_1^2 \leq (\text{trace}(M + N))^2 - 4\|M^{1/2}N^{1/2}\|_1^2. \tag{12}$$

The proof of (12) is similar to the proof of (8). We sketch the difference here.

**Proof of Claim.** Without loss of generality, we assume that the underlying Hilbert space is of finite dimension; the general case is a straightforward extension. Consider the polar decomposition of  $M^{1/2}N^{1/2}$ ,

$$M^{1/2}N^{1/2} = V|M^{1/2}N^{1/2}|,$$

where  $V$  is unitary. Hence,  $\|M^{1/2}N^{1/2}\|_1 = \text{trace } M^{1/2}N^{1/2}V^*$ . Compute

$$\begin{aligned} & \|M - N\|_1^2 \\ &= \left( \frac{1}{2} \left\| (M^{1/2} + N^{1/2}V^*)(M^{1/2} - VN^{1/2}) + (M^{1/2} - N^{1/2}V^*)(M^{1/2} + VN^{1/2}) \right\|_1 \right)^2 \\ &\leq \|M^{1/2} + N^{1/2}V^*\|_2^2 \|M^{1/2} - VN^{1/2}\|_2^2 \\ &= (\text{trace}(M + N))^2 - 4\|M^{1/2}N^{1/2}\|_1^2. \end{aligned}$$

This proves the Claim. Substituting  $M = |A^*|, N = |A|$  into (12) gives (11).  $\square$

**Corollary 2.3.** Let  $A \in SN_2$ . Then

$$\sum_{k=1}^{\infty} |\text{Im}\lambda_k(A)|^2 \leq \|\Im A\|_2^2 - \frac{1}{2}\|A\|_2^2 + \frac{1}{2} \left[ \|A\|_2^4 - \frac{1}{4} \| |(A^2)^* | - |A^2| \|_1^2 \right]^{1/2}, \tag{13}$$

where  $\Im A = (A - A^*)/2i$ .

**Proof.** We use the equality ([3, Lemma 6.5.2])

$$\|A\|_2^2 - \sum_{k=1}^{\infty} |\lambda_k(A)|^2 = 2\|\Im A\|_2^2 - 2\sum_{k=1}^{\infty} |\text{Im}\lambda_k(A)|^2.$$

Due to Theorem 2.2

$$\sum_{k=1}^{\infty} |\lambda_k(A)|^2 \leq \left[ \|A\|_2^4 - \frac{1}{4} \| |(A^2)^* | - |A^2| \|_1^2 \right]^{1/2}.$$

Therefore,

$$\|A\|_2^2 - \left[ \|A\|_2^4 - \frac{1}{4} \| |(A^2)^* | - |A^2| \|_1^2 \right]^{1/2} \leq 2\|\Im A\|_2^2 - 2\sum_{k=1}^{\infty} |\text{Im}\lambda_k(A)|^2.$$

So (13) follows.  $\square$

Corollary 2.3 complements [4, Theorem 1.2]. Evidently, (13) is another refinement of the classical inequality (see [6, Theorem II.6.1]):

$$\sum_{k=1}^{\infty} |\operatorname{Im} \lambda_k(A)|^2 \leq \|\Im A\|_2^2.$$

### 3. Proof of Theorem 1.2

In this section, we assume  $A$  to be a compact operator on  $\mathcal{H}$ . Specifying  $p = 1$  in (2) gives

$$\sum_{k=1}^m |\lambda_k(A)| \leq \sum_{k=1}^m s_k(A), \quad m = 1, 2, \dots \tag{14}$$

Now we replace  $A$  with  $A^2$  in (14) to get

$$\sum_{k=1}^m |\lambda_k(A)|^2 \leq \sum_{k=1}^m s_k(A^2), \quad m = 1, 2, \dots$$

Taking square on both sides yields

$$\left( \sum_{k=1}^m |\lambda_k(A)|^2 \right)^2 \leq 2 \sum_{1 \leq k < j \leq m} s_k(A^2) s_j(A^2) + \sum_{k=1}^m s_k^2(A^2), \quad m = 2, 3, \dots$$

It therefore remains to show

$$\sum_{1 \leq k < j \leq m} s_k(A^2) s_j(A^2) \leq \sum_{1 \leq k < j \leq m} s_k^2(A) s_j^2(A), \quad m = 2, 3, \dots \tag{15}$$

in order to complete the proof of Theorem 1.2.

Denote by  $E_k(t_1, \dots, t_m)$ ,  $k = 1, \dots, m$ , the  $k$ th elementary symmetric function of the  $m$  variables  $t_1, \dots, t_m$ . We present the following proposition.

**Proposition 3.1.** *Let  $A$  be a compact operator and let  $x = (s_1(A^p), \dots, s_m(A^p))$ ,  $y = (s_1^p(A), \dots, s_m^p(A))$ ,  $m, p = 1, 2, \dots$ . Then*

$$E_k(x) \leq E_k(y), \quad 1 \leq k \leq m. \tag{16}$$

**Proof.** A special case of Horn’s majorant theorem ([2], [10, p. 7]) says

$$\prod_{k=1}^m s_k(A^p) \leq \prod_{k=1}^m s_k^p(A), \quad m, p = 1, 2, \dots$$

Denote by  $\log x = (\log s_1(A^p), \dots, \log s_m(A^p))$ ,  $y = (\log s_1^p(A), \dots, \log s_m^p(A))$ . The previous inequality is the same as saying that  $\log x$  is weakly majorized by  $\log y$  (e.g., [12, p. 56]). Since  $E_k(e^{t_1}, \dots, e^{t_m})$  is convex, increasing with respect to  $t_i$ ,  $i = 1, 2, \dots, m$ ; see [9, Theorem 9], (16) follows by a standard result in majorization theory (e.g., [12, p. 66]).  $\square$

Inequality (15) is just a special case of (16) when  $p = k = 2$ .

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