



Harmonic analysis

On the Calderón–Zygmund structure of Petermichl's kernel <sup>☆</sup>*Sur la structure Calderón–Zygmund du noyau de Petermichl*

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## ABSTRACT

We show that Petermichl's dyadic operator  $\mathcal{P}$  (Petermichl (2000) [8]) is a Calderón–Zygmund-type operator on an adequate metric normal space of homogeneous type. We also compare the maximal operators associated with truncations of the kernel and to the summability of the Haar series.

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## R É S U M É

Nous démontrons que l'opérateur dyadique de Petermichl  $\mathcal{P}$  est un opérateur de type Calderón–Zygmund sur un espace normal métrique de type homogène. Nous comparons les opérateurs maximaux associés aux troncatures du noyau et à la sommabilité de la série de Haar.

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## 1. Introduction

In [8], Stefanie Petermichl proves a remarkable identity that provides the Hilbert kernel  $\frac{1}{x-y}$  in  $\mathbb{R}$  through a mean value of dilations and translations of a basic kernel defined in terms of dyadic families on  $\mathbb{R}$ . The basic kernel for a fixed dyadic system  $\mathcal{D}$  is described in terms of Haar wavelets. Assume that  $\mathcal{D}$  is the standard dyadic family on  $\mathbb{R}$ , i.e.  $\mathcal{D} = \cup_{j \in \mathbb{Z}} \mathcal{D}^j$  with  $\mathcal{D}^j = \{I_k^j : k \in \mathbb{Z}\}$  and  $I_k^j = [\frac{k}{2^j}, \frac{k+1}{2^j})$ . Let  $\mathcal{H}$  be the standard Haar system built on the dyadic intervals in  $\mathcal{D}$ . There is a natural bijection between  $\mathcal{H}$  and  $\mathcal{D}$ . We shall use  $\mathcal{D}$  as the index set and we shall write  $h_I$  to denote the function  $h_I(x) = |I|^{-1/2} (\mathcal{X}_{I^-}(x) - \mathcal{X}_{I^+}(x))$  where  $I^-$  and  $I^+$  are the respective left and right halves of  $I$ ,  $\mathcal{X}_E$  is, as usual, the indicator function of  $E$  and  $|E|$  denotes the Lebesgue measure of the measurable set  $E$ . With the above notation, the basic Petermichl's operator on  $L^2(\mathbb{R})$  is given by

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$$\mathcal{P}f(x) = \sum_{I \in \mathcal{D}} \langle f, h_I \rangle (h_{I^-}(x) - h_{I^+}(x)), \tag{1.1}$$

where, as usual,  $\langle f, h_I \rangle = \int_{\mathbb{R}} f(y)h_I(y) dy$ . Hence, at least formally, the operator  $\mathcal{P}$  is defined by the nonconvolution non-symmetric kernel

$$\begin{aligned} P(x, y) &= \sum_{h \in \mathcal{D}} h_I(y)(h_{I^-}(x) - h_{I^+}(x)) \\ &= P^+(x, y) + P^-(x, y); \end{aligned}$$

with

$$P^+(x, y) = \sum_{I \in \mathcal{D}^+} h_I(y)(h_{I^-}(x) - h_{I^+}(x)) \tag{1.2}$$

and  $\mathcal{D}^+ = \{I_k^j \in \mathcal{D} : k \geq 0\}$ .

Let us observe that, for  $x \geq 0, y \geq 0$  and  $x \neq y$ , the series  $\sum_{I \in \mathcal{D}^+} h_I(y)[h_{I^-}(x) - h_{I^+}(x)]$  is absolute convergent. In fact,

$$\begin{aligned} \sum_{I \in \mathcal{D}^+} |h_I(y)| |h_{I^-}(x) - h_{I^+}(x)| &= \sum_{I \in \mathcal{D}^+, I \supseteq I(x,y)} \frac{1}{\sqrt{|I|}} |h_{I^-}(x) - h_{I^+}(x)| \\ &\leq \sum_{I \in \mathcal{D}^+, I \supseteq I(x,y)} \frac{2\sqrt{2}}{|I|} = \frac{4\sqrt{2}}{|I(x, y)|} \end{aligned}$$

where  $I(x, y)$  is the smallest dyadic interval in  $\mathbb{R}$  containing  $x$  and  $y$ .

In §2, we show that  $\mathcal{P}^+$  (and  $\mathcal{P}^-$ ), the operator induced by the kernel  $P^+$  (resp.  $P^-$ ), is of Calderón–Zygmund type in the normal space of homogeneous type  $\mathbb{R}^+$  (resp.  $\mathbb{R}^-$ ) with the dyadic ultrametric  $\delta(x, y) = \inf\{|I| : x, y \in I \text{ and } I \in \mathcal{D}\}$  and Lebesgue measure. In §3, we compare the maximal operators induced by the geometric truncations of the kernel with the maximal operator of the partial sums of the Haar series.

### 2. Petermichl’s operator as a Calderón–Zygmund operator

Following [7], a linear and continuous operator  $T : \mathcal{D}(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)$ , with  $\mathcal{D}$  and  $\mathcal{D}'$  the test functions and the distributions on  $\mathbb{R}^n$ , is a Calderón–Zygmund operator if there exists  $K \in L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta)$  where  $\Delta$  is the diagonal of  $\mathbb{R}^n \times \mathbb{R}^n$  such that

(1) there exists  $C_0 > 0$  with

$$|K(x, y)| \leq \frac{C_0}{|x - y|^n}, \quad x \neq y;$$

(2) there exist  $C_1$  and  $\gamma > 0$  such that

$$(2.a) \quad |K(x', y) - K(x, y)| \leq C_1 \frac{|x' - x|^\gamma}{|x - y|^{n+\gamma}} \text{ when } 2|x' - x| \leq |x - y|;$$

$$(2.b) \quad |K(x, y') - K(x, y)| \leq C_1 \frac{|y' - y|^\gamma}{|x - y|^{n+\gamma}} \text{ when } 2|y' - y| \leq |x - y|;$$

(3)  $T$  extends to  $L^2(\mathbb{R}^n)$  as a continuous linear operator;

(4) for  $\varphi$  and  $\psi \in \mathcal{D}(\mathbb{R}^n)$  with  $\text{supp } \varphi \cap \text{supp } \psi = \emptyset$ , we have:

$$\langle T\varphi, \psi \rangle = \iint_{\mathbb{R}^n \times \mathbb{R}^n} K(x, y)\varphi(x)\psi(y) dx dy.$$

With a little effort, the notions of Calderón–Zygmund operator and Calderón–Zygmund kernel  $K$  – i.e. satisfying (1) and (2) – can be extended to normal metric spaces of homogeneous type. Even when the formulation can be stated in quasi-metric spaces for our application, it shall be enough in the following context. Let  $(X, d)$  be a metric space. If there exists a Borel measure  $\mu$  on  $X$  such that for some constants  $0 < \alpha \leq \beta < \infty$  such that the inequalities  $\alpha r \leq \mu(B(x, r)) \leq \beta r$  hold for every  $r > 0$  and every  $x \in X$ , we shall say that  $(X, d, \mu)$  is a normal space. As usual  $B(x, r) = \{y \in X : d(x, y) < r\}$ . In particular,  $(X, d, \mu)$  is a space of homogeneous type in the sense of [4], [6], [5], [2], and many problems of harmonic analysis find there a natural place to be solved.

In this setting in [6] a fractional-order inductive limit topology is given to the space of compactly supported Lipschitz  $\gamma$  functions ( $0 < \gamma < 1$ ). We shall still write  $\mathcal{D} = \mathcal{D}(X, d)$  to denote this test functions space. And  $\mathcal{D}' = \mathcal{D}'(X, d)$  its dual, the space of distributions. So, the extension of the definition of Calderón–Zygmund operators to this setting becomes natural.

**Definition 1.** Let  $(X, d, \mu)$  be a normal metric measure space such that continuous functions are dense in  $L^1(X, \mu)$ . We say that a linear and continuous operator  $T : \mathcal{D} \rightarrow \mathcal{D}'$  is Calderón–Zygmund on  $(X, d, \mu)$  if there exists  $K \in L^1_{\text{loc}}(X \times X \setminus \Delta)$ , where  $\Delta$  is the diagonal in  $X \times X$ , such that

(i) there exists  $C_0 > 0$  with

$$|K(x, y)| \leq \frac{C_0}{d(x, y)}, \quad x \neq y;$$

(ii) there exist  $C_1 > 0$  and  $\gamma > 0$  such that

(ii.a)  $|K(x', y) - K(x, y)| \leq C_1 \frac{d(x', x)^\gamma}{d(x, y)^{1+\gamma}}$  when  $2d(x', x) \leq d(x, y)$ ;

(ii.b)  $|K(x, y') - K(x, y)| \leq C_1 \frac{d(y, y')^\gamma}{d(x, y)^{1+\gamma}}$  when  $2d(y', y) \leq d(x, y)$ ;

(iii)  $T$  extends to  $L^2(X, \mu)$  as a continuous linear operator;

(iv) for  $\varphi$  and  $\psi \in \mathcal{D}$  with  $d(\text{supp } \varphi, \text{supp } \psi) > 0$  we have:

$$(T\varphi, \psi) = \iint_{X \times X} K(x, y) \varphi(x) \psi(y) d(\mu \times \mu)(x, y).$$

Our first result shows that  $\mathcal{P}^+$  and  $\mathcal{P}^-$  are Calderón–Zygmund operators. In what follows, we shall keep using  $P$  for  $\mathcal{P}^+$  and  $\mathcal{P}$  for  $\mathcal{P}^-$ .

**Theorem 2.** *There exists a metric  $\delta$  on  $\mathbb{R}^+ = \{x : x \geq 0\}$  such that  $(\mathbb{R}^+, \delta, |\cdot|)$  is a normal space where  $\delta$ -continuous functions are dense in  $L^1(\mathbb{R}^+, dx)$  and  $P$  can be written, for  $x \neq y$  both in  $\mathbb{R}^+$ , as*

$$P(x, y) = \frac{\Omega(x, y)}{\delta(x, y)}, \tag{2.1}$$

where  $\Omega$  is bounded,  $\delta$ -smooth and homogeneous of degree zero, i.e.  $\Omega(2^j x, 2^j y) = \Omega(x, y)$ . Moreover,  $\mathcal{P}$  is a Calderón–Zygmund operator on  $(\mathbb{R}^+, \delta, |\cdot|)$ .

**Proof.** For  $x \neq y$ , two points in  $\mathbb{R}^+$ , define  $\delta(x, y) = \inf\{|I| : x, y \in I \in \mathcal{D}\}$ . Define also  $\delta(x, x) = 0$  for every  $x \in \mathbb{R}^+$ . It is easy to see that  $\delta$  is an ultra-metric on  $\mathbb{R}^+$ . This means that the triangle inequality improves to  $\delta(x, z) \leq \sup\{\delta(x, y), \delta(y, z)\}$  for every  $x, y$  and  $z \in \mathbb{R}^+$ . Notice that  $|x - y| \leq \delta(x, y)$ , but they are certainly not equivalent. Also, for  $x \in \mathbb{R}^+$  and  $r > 0$  given, taking  $m \in \mathbb{Z}$  such that  $2^{-m} < r \leq 2^{-m+1}$ , we see that  $B_\delta(x, r) = \{y \in \mathbb{R}^+ : \delta(x, y) < r\} = \{y \in \mathbb{R}^+ : \delta(x, y) \leq 2^{-m}\} = I^m_{k(x)}$ , where  $k(x)$  is the only index  $k \in \mathbb{N} \cup \{0\}$  such that  $x \in I^m_k$ . Hence, the Lebesgue measure of  $B_\delta(x, r)$  is that of the interval  $I^m_{k(x)}$ . Precisely,  $|B_\delta(x, r)| = 2^{-m}$ . So that  $\frac{r}{2} \leq |B_\delta(x, r)| < r$ , for every  $x \in \mathbb{R}^+$  and every  $r > 0$ . In terms of our above definitions  $(\mathbb{R}^+, \delta, |\cdot|)$  is a normal metric space. The integrability properties of powers of  $\delta$  resemble completely those of the powers of  $x$ . In fact, for fixed  $x \in \mathbb{R}^+$ , the function of  $y \in \mathbb{R}^+$  given by  $1/\delta^\alpha(x, y)$  is integrable inside a  $\delta$ -ball when  $\alpha < 1$ . It is integrable outside a  $\delta$ -ball when  $\alpha > 1$ . In particular,  $1/\delta(x, y)$  is neither locally nor globally integrable on  $\mathbb{R}^+$ .

Notice now that real-valued simple functions built on the dyadic intervals are continuous as functions defined on  $(\mathbb{R}^+, \delta)$ . In fact, for  $I \in \mathcal{D}$ , we have that  $|\mathcal{X}_I(x) - \mathcal{X}_I(y)|$  equals zero for  $x$  and  $y$  in  $I$  or for  $x$  and  $y$  outside  $I$ . Assume that  $x \in I$  and  $y \notin I$ , then  $\delta(x, y) \geq 2|I|$ , so that  $|\mathcal{X}_I(x) - \mathcal{X}_I(y)| \leq \delta(x, y)(2|I|)^{-1}$  for every  $x$  and  $y \in \mathbb{R}^+$ . In other words, for  $I \in \mathcal{D}$ ,  $\mathcal{X}_I$  is Lipschitz with respect to  $\delta$  with constant  $(2|I|)^{-1}$ . Hence,  $\delta$ -continuous functions are dense in  $L^1(\mathbb{R}^+, dx)$ .

The operator  $\mathcal{P}$  is actually defined as an operator in  $L^2(\mathbb{R}^+, dx)$ . For  $f \in L^2(\mathbb{R}^+, dx)$ ,

$$\begin{aligned} \mathcal{P}f(x) &= \sum_{I \in \mathcal{D}^+} \langle f, h_I \rangle (h_{I^-}(x) - h_{I^+}(x)) \\ &= \sum_{I \in \mathcal{D}^+} \langle f, h_I \rangle h_{I^-}(x) - \sum_{I \in \mathcal{D}^+} \langle f, h_I \rangle h_{I^+}(x). \end{aligned}$$

Hence  $\|\mathcal{P}f\|_2^2 \leq 2 \sum_{I \in \mathcal{D}^+} |\langle f, h_I \rangle|^2 = 2 \|f\|_2^2$ , which proves (iii) in Definition 1. In particular, if  $\varphi$  is a simple function built on the dyadic intervals, we see that  $\mathcal{P}\varphi \in L^2(\mathbb{R}^+, dx)$ , so that when  $\psi$  is another simple function such that  $\delta(\text{supp } \varphi, \text{supp } \psi) > 0$ , the two variables function  $F(x, y) = \varphi(x)\psi(y)$  is simple in  $\mathbb{R}^+ \times \mathbb{R}^+$  and for some  $\varepsilon > 0$ ,  $\text{supp } F \cap \{\delta < \varepsilon\} = \emptyset$ , we have that, since only a finite subset of  $\mathcal{D}^+$  is actually involved,

$$\iint_{\mathbb{R}^+ \times \mathbb{R}^+} \left( \sum_{I \in \mathcal{D}^+} h_I(y)[h_{I^-}(x) - h_{I^+}(x)] \right) \varphi(y)\psi(x) dy dx$$

$$\begin{aligned}
 &= \int_{x \in \mathbb{R}^+} \left( \int_{y \in \mathbb{R}^+} P(x, y) \varphi(y) \right) \psi(x) \, dx \\
 &= \int_{x \in \mathbb{R}^+} \mathcal{P}\varphi(x) \psi(x) \, dx \\
 &= \langle \mathcal{P}\varphi, \psi \rangle.
 \end{aligned}$$

Hence,  $P(x, y) = \sum_{I \in \mathcal{D}^+} h_I(y)[h_{I^-}(x) - h_{I^+}(x)]$  is the kernel for  $\mathcal{P}$ . Let us now show that  $P(x, y) = \frac{\Omega(x, y)}{\delta(x, y)}$  for  $x \neq y$ . For  $J \in \mathcal{D}^+$ , define

$$\Omega_J(x, y) = \Theta_J^1(y) \Theta_J^2(x)$$

where

$$\begin{aligned}
 \Theta_J^1(y) &= \mathcal{X}_{J^-}(y) - \mathcal{X}_{J^+}(y) \\
 \Theta_J^2(x) &= (\mathcal{X}_{J^-}(x) + \mathcal{X}_{J^+}(x)) - (\mathcal{X}_{J^{--}}(x) + \mathcal{X}_{J^{++}}(x)).
 \end{aligned}$$

Let us denote with  $I(x, y)$  the smallest interval containing  $x$  and  $y$ , then we have

$$P(x, y) = \sum_{I \in \mathcal{D}^+} h_I(y)[h_{I^-}(x) - h_{I^+}(x)] = \sqrt{2} \sum_{I \in \mathcal{D}^+, I \supseteq I(x, y)} \frac{1}{|I|} \Omega_I(x, y).$$

Since  $|I(x, y)| = \delta(x, y)$  and in the last series, we are adding on all the dyadic ancestors of  $I(x, y)$ , including  $I(x, y)$  itself,

$$P(x, y) = \frac{\sqrt{2}}{\delta(x, y)} \sum_{m=0}^{\infty} \frac{1}{2^m} \Omega_{I^{(m)}(x, y)}(x, y) = \frac{\Omega(x, y)}{\delta(x, y)}$$

with  $I^{(m)}(x, y)$  the  $m$ -th ancestor of  $I(x, y)$  and

$$\Omega(x, y) = \sqrt{2} \sum_{m=0}^{\infty} 2^{-m} \Omega_{I^{(m)}(x, y)}(x, y).$$

Hence (i) in Definition 1 holds with  $C_0 = 2^{5/2}$ .

Let us check (ii.a). Let  $x, y$  and  $x' \in \mathbb{R}^+$  be such that  $\delta(x, x') \leq \frac{1}{2} \delta(x, y)$ . Let  $I(x, y)$  be the smallest dyadic interval containing  $x$  and  $y$ . Then  $|I(x, y)| = \delta(x, y)$ . In a similar way  $|I(x, x')| = \delta(x, x')$  and  $|I(x', y)| = \delta(x', y)$ . Since those three intervals are all dyadic and since  $|I(x, x')| \leq \frac{1}{2} |I(x, y)|$ , we necessarily must have that  $x'$  belongs to the same half of  $I(x, y)$  as  $x$  does. Hence  $I(x', y) = I(x, y)$  and certainly also are the same all the ancestors  $I^{(m)}(x', y) = I^{(m)}(x, y)$ . Now,

$$\begin{aligned}
 \frac{1}{\sqrt{2}} |P(x', y) - P(x, y)| &= \left| \frac{\Omega(x', y)}{\delta(x', y)} - \frac{\Omega(x, y)}{\delta(x, y)} \right| \\
 &\leq \frac{|\Omega(x', y) - \Omega(x, y)|}{\delta(x, y)} + |\Omega(x', y)| \left| \frac{1}{\delta(x', y)} - \frac{1}{\delta(x, y)} \right| \\
 &= I + II.
 \end{aligned}$$

In order to estimate  $I$ , let us first explore the  $\delta$ -regularity of each  $\Omega_J$ . Let us prove that

- (a) for fixed  $y \in \mathbb{R}^+$  we have that  $|\Omega_J(x', y) - \Omega_J(x, y)| \leq \frac{8}{|J|} \delta(x, x')$ ; and
- (b) for fixed  $x \in \mathbb{R}^+$ ,  $|\Omega_J(x, y') - \Omega_J(x, y)| \leq \frac{2}{|J|} \delta(y, y')$ .

Let us check (a). The regularity in the second variable is similar. Since the indicator function of a dyadic interval  $I$  is  $\delta$ -Lipschitz with constant  $\frac{1}{2|I|}$ , we have

$$\begin{aligned}
 |\Omega_J(x', y) - \Omega_J(x, y)| &= \left| \Theta_J^1(y) (\Theta_J^2(x') - \Theta_J^2(x)) \right| \\
 &= \left| \Theta_J^2(x') - \Theta_J^2(x) \right| \\
 &\leq |\mathcal{X}_{J^-}(x') - \mathcal{X}_{J^-}(x)| + |\mathcal{X}_{J^+}(x') - \mathcal{X}_{J^+}(x)| + \\
 &\quad + |\mathcal{X}_{J^{--}}(x') - \mathcal{X}_{J^{--}}(x)| + |\mathcal{X}_{J^{++}}(x') - \mathcal{X}_{J^{++}}(x)|
 \end{aligned}$$

$$\leq 4 \frac{4}{2^{|J|}} \delta(x, x').$$

Since the series defining  $\Omega$  is absolutely convergent, from the above remarks, we have

$$\begin{aligned} I &\leq \frac{1}{\delta(x, y)} \sum_{m=0}^{\infty} 2^{-m} |\Omega_{I^{(m)}(x', y)}(x', y) - \Omega_{I^{(m)}(x, y)}(x, y)| \\ &= \frac{1}{\delta(x, y)} \sum_{m=0}^{\infty} 2^{-m} |\Omega_{I^{(m)}(x, y)}(x', y) - \Omega_{I^{(m)}(x, y)}(x, y)| \\ &\leq \frac{8}{\delta(x, y)} \sum_{m=0}^{\infty} 2^{-m} \frac{\delta(x, x')}{|I^{(m)}(x, y)|} \\ &= 16 \frac{\delta(x, x')}{\delta^2(x, y)}. \end{aligned}$$

Let us estimate  $II$ . Since  $|\Omega|$  is bounded above by 2 and  $\delta$  is a metric on  $\mathbb{R}^+$ , we have

$$II \leq 2 \frac{|\delta(x, y) - \delta(x', y)|}{\delta(x, y)\delta(x', y)} \leq 2 \frac{\delta(x, x')}{\delta(x, y)\delta(x', y)}$$

as we already observed, under the current conditions,  $\delta(x', y) = \delta(x, y)$ . And we get the desired type estimate  $II \leq 2 \frac{\delta(x, x')}{\delta(x, y)}$ .

Hence  $|P(x', y) - P(x, y)| \leq \sqrt{2} \frac{14}{3} \frac{\delta(x, x')}{\delta^2(x, y)}$  when  $\delta(x, x') \leq \frac{1}{2} \delta(x, y)$ .

The analogous procedure, using (b) and a similar geometric consideration for  $x, y, y'$  with  $\delta(y, y') \leq \frac{1}{2} \delta(x, y)$  gives

$$|P(x, y') - P(x, y)| \leq \sqrt{2} 12 \frac{\delta(y, y')}{\delta^2(x, y)}. \quad \square$$

### 3. Comparison of the maximal operators

As usual, for Calderón–Zygmund operators, the truncations of the kernel and the associated maximal operator play a central role in the analysis of the boundedness properties of the operator. For  $0 < \varepsilon < R < \infty$ , set

$$P_{\varepsilon, R}(x, y) = \mathcal{X}_{\{\varepsilon \leq \delta(x, y) < R\}} P(x, y) = \mathcal{X}_{\{\varepsilon \leq \delta(x, y) < R\}} \frac{\Omega(x, y)}{\delta(x, y)}.$$

Sometimes, for example when  $P$  acts on  $L^p(\mathbb{R}^+, dx)$  with  $p > 1$ , only the local truncation about the diagonal is actually needed. For  $\varepsilon > 0$ ,  $P_{\varepsilon, \infty}(x, y) = \mathcal{X}_{\{\delta(x, y) \geq \varepsilon\}}(x, y) P(x, y)$ . Since the original form of Petermichl’s kernel is given in terms of the Haar–Fourier analysis, a scale truncation is still possible and natural. For  $l < m$  both in  $\mathbb{Z}$ , we consider also the scale truncation of  $P$  between  $2^l$  and  $2^m$ . In other words,

$$P^{l, m}(x, y) = \sum_{\{I \in \mathcal{D}^+ : 2^l \leq |I| < 2^m\}} h_I(y)[h_{I^-}(x) - h_{I^+}(x)].$$

Since  $\delta$  takes only dyadic values,  $P_{\varepsilon, R}$  can also be written as  $P_{2^\lambda, 2^\mu}$  for  $\lambda$  and  $\mu \in \mathbb{Z}$ . For simplicity, we shall write  $P_{\lambda, \mu}$  to denote  $P_{2^\lambda, 2^\mu}$ . Hence, in our notation the distinction between the two truncations is only positional:  $P^{l, m}$  is scale truncation;  $P_{l, m}$  is metric truncation. Let us compare these two kernels and the operators induced by them. The calligraphic versions  $\mathcal{P}^{l, m}$  and  $\mathcal{P}_{l, m}$  denote the operators induced by  $P^{l, m}$  and  $P_{l, m}$ , respectively.

In the next statement, we use two notations for the ancestry of a dyadic interval. Given  $I \in \mathcal{D}^+$ ,  $I^{(n)}$  denotes, as before, the  $n$ -th ancestor of  $I$ . Instead,  $\widehat{I}^j$  denotes the only, if any, ancestor of  $I$  in the level  $\mathcal{D}^j$  of the dyadic interval. For instance, if  $I = [\frac{3}{2}, 2)$ , then  $I^{(1)} = [1, 2)$ ,  $I^{(2)} = [0, 2)$ ,  $\widehat{I}^0 = [1, 2)$ ,  $\widehat{I}^3 = [0, 8)$ .

**Lemma 3.** *Let  $l$  and  $m$  in  $\mathbb{Z}$  with  $l < m$ . Then*

(1)  $P^{l, m}(x, y) = P_{l, m}(x, y) + Q_{l, m}(x, y)$ , where

$$Q_{l, m}(x, y) = \begin{cases} 0, & \text{for } \delta(x, y) \geq 2^m; \\ \sqrt{2} \sum_{j=l}^{m-1} 2^{-j} \Omega_{\widehat{I}^j(x, y)}(x, y), & \text{for } 0 < \delta(x, y) < 2^l; \\ -\frac{\sqrt{2}}{\delta(x, y)} \sum_{n=\log_2 \frac{2}{\delta(x, y)}}^{\infty} 2^{-n} \Omega_{I^{(n)}(x, y)}(x, y), & \text{when } 2^l \leq \delta(x, y) < 2^m; \end{cases}$$

(2)  $P^{l,m}$  belongs to  $L^1(\mathbb{R}^+, dx)$  in each variable when the other variable remains fixed. Moreover

$$\int_{y \in \mathbb{R}^+} P^{l,m}(x, y) dx = \int_{y \in \mathbb{R}^+} P^{l,m}(x, y) dy = 0;$$

(3)  $|Q_{l,m}(x, y)| \leq 2\sqrt{2} \left( 2^{-l} \mathcal{X}_{\{\delta(x,y) < 2^l\}}(x, y) + 2^{-m} \mathcal{X}_{\{\delta(x,y) < 2^m\}} \right);$

(4) the inequality  $\left| \int_{y \in \mathbb{R}^+} Q_{l,m}(x, y) dy \right| \leq 2\sqrt{2}$  holds for every  $l, m$  in  $\mathbb{Z}$  and every  $x \in \mathbb{R}^+$ ;

(5) the sequence  $\int_{y \in \mathbb{R}^+} Q_{l,0}(x, y) dy$  converges uniformly in  $x \in \mathbb{R}^+$  for  $l$  tending to  $-\infty$ .

**Proof.** Let us rewrite together the two truncations of  $P$  for the same values of  $l$  and  $m$  with  $l < m$ ,

$$P^{l,m}(x, y) = \sum_{I \in \mathcal{D}^+, 2^l \leq |I| < 2^m} h_I(y)[h_{I^-}(x) - h_{I^+}(x)];$$

$$P_{l,m}(x, y) = \mathcal{X}_{\{2^l \leq \delta(x,y) < 2^m\}}(x, y) \frac{\Omega(x, y)}{\delta(x, y)}$$

with  $\Omega(x, y) = \sqrt{2} \sum_{n=0}^{\infty} 2^{-n} \Omega_{I^{(n)}(x,y)}(x, y)$ . Let us compute  $P^{l,m}(x, y)$  for the three bands around the diagonal  $\Delta$  of  $\mathbb{R}^+ \times \mathbb{R}^+$  determined by  $2^l$  and  $2^m$ . First, assume that  $0 < \delta(x, y) < 2^l$ . Then

$$P^{l,m}(x, y) = \sqrt{2} \sum_{\substack{I \in \mathcal{D}^+ \\ 2^l \leq |I| < 2^m}} \frac{1}{|I|} \Omega_I(x, y).$$

Since  $\text{supp } \Omega_I \subset I \times I$ , once  $(x, y)$  is given, with  $\delta(x, y) < 2^l$ , the sum above is performed only on those dyadic intervals  $I$  for which  $2^l \leq |I| < 2^m$  that contain  $I(x, y)$ ; the smallest dyadic interval containing both  $x$  and  $y$ . Hence,

$$P^{l,m}(x, y) = \sqrt{2} \sum_{j=l}^{m-1} \frac{1}{2^j} \Omega_{I^j(x,y)}(x, y) = Q_{l,m}(x, y) = Q_{l,m}(x, y) + P_{l,m}(x, y)$$

in the  $\delta$ -strip  $\{(x, y) : \mathbb{R}^+ \times \mathbb{R}^+ : \delta(x, y) < 2^l\}$ . Second, assume that  $\delta(x, y) \geq 2^m$ . Then no dyadic interval  $I$  containing both  $x$  and  $y$  has a measure less than  $2^m$ , so that  $P^{l,m}$  vanishes when  $\delta(x, y) \geq 2^m$  and again  $P^{l,m} = Q_{l,m} + P_{l,m}$ . The third and last case to be considered is when  $2^l \leq \delta(x, y) < 2^m$ . Again the non-vanishing condition for  $\Omega_I(x, y)$  requires  $I \supseteq I(x, y)$ , hence

$$P^{l,m}(x, y) = \sqrt{2} \sum_{\substack{I \in \mathcal{D} \\ |I| < 2^m \\ I \supseteq I(x,y)}} \frac{1}{|I|} \Omega_I(x, y).$$

Since  $I \supseteq I(x, y)$  then, in the above sum,  $I$  has to be an ancestor of  $I(x, y)$ . Hence  $|I| = 2^n |I(x, y)| = 2^n \delta(x, y)$  for some  $n = 0, 1, 2, \dots$ . The upper restriction on the measure of  $I$ ,  $|I| < 2^m$ , provides an upper bound for  $n$ . In fact, since  $2^m > |I| = 2^n \delta(x, y)$ ,  $n \leq (\log_2 2^m \delta^{-1}(x, y)) - 1$ . Notice that  $2^m \delta^{-1}(x, y)$  is an integral power of 2, so that  $\log_2 2^m \delta^{-1}(x, y) \in \mathbb{Z}$ . Hence,

$$\begin{aligned} P^{l,m} &= \frac{\sqrt{2}}{\delta(x, y)} \sum_{n=0}^{\log_2 \frac{2^m}{\delta(x,y)} - 1} \frac{1}{2^n} \Omega_{I^{(n)}(x,y)}(x, y) \\ &= \frac{\sqrt{2}}{\delta(x, y)} \left( \Omega(x, y) - \sum_{n=\log_2 \frac{2^m}{\delta(x,y)}}^{\infty} \frac{1}{2^n} \Omega_{I^{(n)}(x,y)}(x, y) \right) \\ &= P_{l,m}(x, y) + Q_{l,m}(x, y), \end{aligned}$$

and (1) is proved.

In order to prove (2), notice that for  $x$  fixed  $P^{l,m}(x, \cdot)$  is a finite linear combination of Haar functions in the variable  $y$ . Hence,  $P^{l,m}(x, \cdot)$  is an  $L^1(\mathbb{R}^+, dx)$  function and its integral in  $y$  vanishes, since each Haar function has mean value zero. An analogous argument hold for  $y$  fixed and  $P^{l,m}(\cdot, y)$ .

Let us get the bound in (3). We only have to check it in the bands  $\{\delta(x, y) < 2^l\}$  and  $\{2^l \leq \delta(x, y) < 2^m\}$ . Let us first take  $\delta(x, y) < 2^l$ . Then,

$$|Q_{l,m}(x, y)| = \sqrt{2} \left| \sum_{j=l}^{m-1} 2^{-j} \Omega_{\widehat{\gamma}^j(x,y)}(x, y) \right| \leq \sqrt{2} \sum_{j=l}^m 2^{-j} \leq 2\sqrt{2}2^{-l},$$

as desired. Assume now that  $2^l \leq \delta(x, y) < 2^m$ . Then,

$$|Q_{l,m}(x, y)| \leq \sqrt{2} \frac{1}{\delta(x, y)} \sum_{n=\log_2 \frac{2^m}{\delta(x,y)}}^{\infty} 2^{-n} = 2\sqrt{2} \frac{1}{\delta(x, y)} \frac{\delta(x, y)}{2^m} = 2\sqrt{2}2^{-m}.$$

For the proof of (4), notice that from (3), we have that, for fixed  $x$  and fixed  $l$  and  $m$ , as a function of  $y$ ,  $Q_{l,m}(x, y)$ , and hence  $P_{l,m}(x, y)$  are integrable. Then,

$$\left| \int_{y \in \mathbb{R}^+} Q_{l,m}(x, y) dy \right| \leq 2\sqrt{2} \int_{y \in \mathbb{R}^+} \left\{ 2^{-l} \chi_{\{\delta(x,y) < 2^l\}}(x, y) + 2^{-m} \chi_{\{\delta(x,y) < 2^m\}}(x, y) \right\} dy = 2\sqrt{2}.$$

Let us prove (5). From the expression in (1) for  $Q_{l,0}$ , we have

$$\begin{aligned} \int_{y \in \mathbb{R}^+} Q_{l,0}(x, y) dy &= \sqrt{2} \int_{B_\delta(x, 2^l)} \left( \sum_{j=l}^{-1} 2^{-j} \Omega_{\widehat{\gamma}^j(x,y)}(x, y) \right) dy \\ &\quad - \sqrt{2} \int_{B_\delta(x, 1) \setminus B_\delta(x, 2^l)} \frac{1}{\delta(x, y)} \left( \sum_{n=\log_2 \frac{1}{\delta(x,y)}}^{\infty} \frac{1}{2^n} \Omega_{I^{(n)}(x,y)}(x, y) \right) dy \\ &= \sqrt{2} \sum_{j=l}^{-1} 2^{-j} \int_{B_\delta(x, 2^j)} \Omega_{\widehat{\gamma}^j(x,y)}(x, y) dy - \sqrt{2} \sum_{i=l}^{-1} 2^{-i} \int_{\{y:\delta(x,y)=2^i\}} \left( \sum_{n=-i}^{\infty} \frac{1}{2^n} \Omega_{I^{(n)}(x,y)}(x, y) \right) dy \\ &= \sqrt{2} \left( \sum_{j=l}^{-1} 2^{-j} 2^j \widehat{\sigma}_{l,j}(x) - \frac{1}{2} \sum_{i=l}^{-1} 2^{-i} \sum_{n=-i}^{\infty} 2^{-n} 2^i \sigma_{n,i}(x) \right), \end{aligned}$$

where  $\widehat{\sigma}_{l,j}(x) = \int_{B_\delta(x, 2^j)} \Omega_{\widehat{\gamma}^j(x,y)}(x, y) dy$  and  $\sigma_{n,i}(x) = \int_{\{\delta(x,y)=2^i\}} \Omega_{I^{(n)}(x,y)}(x, y) dy$  and  $f_E f$  denotes the mean value of  $f$  on  $E$ , so that

$$\int_{y \in \mathbb{R}^+} Q_{l,0}(x, y) dy = \sqrt{2} \sum_{i=0}^{-l-1} 2^{-i} \widehat{\sigma}_{l,i+l}(x) - \frac{\sqrt{2}}{2} \left( \sum_{n=1}^{-l} 2^{-n} \sum_{i=-n}^{-1} \sigma_{n,i}(x) + \sum_{n=-l+1}^{\infty} 2^{-n} \sum_{i=l}^{-1} \sigma_{n,i}(x) \right).$$

Since, in the definitions of  $\widehat{\sigma}$  and  $\sigma$ , we are taking mean values of functions with  $L^\infty$ -norm equal to 1, we certainly have that  $|\widehat{\sigma}| \leq 1$  and  $|\sigma| \leq 1$ . Hence,  $\left| \sum_{i=-n}^{-1} \sigma_{n,i}(x) \right| \leq n$ , and  $\left| \sum_{i=l}^{-1} \sigma_{n,i}(x) \right| \leq |l| = -l$ . So the first term in the expression for the integral is dominated by the geometric series  $\sum_{i \geq 0} 2^{-i}$ , the second term is dominated by the convergent series  $\sum_{n=1}^{\infty} n 2^{-n}$ , and the third term is bounded by  $|l| \sum_{n=-l+1}^{\infty} 2^{-n}$ , which tends to zero as  $|l|$  tends to infinity.  $\square$

Let us notice that (4) and (5) in the above lemma hold also integrating in the variable  $x$ . Let

$$M_{dy} f(x) = \sup_{x \in I \in \mathcal{D}^+} \frac{1}{|I|} \int_I |f(y)| dy$$

be the dyadic maximal operator. Set

$$\begin{aligned} \mathcal{P}^* f(x) &= \sup_{\substack{l < m \\ l, m \in \mathbb{Z}}} \left| \int_{\mathbb{R}^+} P^{l,m}(x, y) f(y) dy \right| \\ \mathcal{P}_* f(x) &= \sup_{\substack{l < m \\ l, m \in \mathbb{Z}}} |P_{l,m}(x, y)| \end{aligned}$$

**Theorem 4.** *The inequalities*

$$\mathcal{P}_* f(x) \leq 4\sqrt{2}M_{dy} f(x) + \mathcal{P}^* f(x) \quad (3.1)$$

and

$$\mathcal{P}^* f(x) \leq 4\sqrt{2}M_{dy} f(x) + \mathcal{P}_* f(x) \quad (3.2)$$

hold for every locally integrable function  $f$  defined on  $\mathbb{R}^+$ .

**Proof.** Inequalities (3.1) and (3.2) follow from (1) and (3) in Lemma 3.  $\square$

The above theorem, together with properties (2), (4) and (5) in Lemma 3, and the results in [1] and [3], give classical boundedness properties in Lebesgue spaces of  $\mathcal{P}_*$ , so that, from (3.2) and the boundedness properties of  $M_{dy}$  we obtain the corresponding bounds for  $\mathcal{P}^*$ .

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