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A topological nullstellensatz for tensor-triangulated categories

Un théorème des zéros topologique pour les catégories triangulées tensorielles

Abhishek Banerjee

Department of Mathematics, Indian Institute of Science, Bangalore-560012, India

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ABSTRACT

Let $\mathrm{Spec}(\mathbb{T})$ be the spectrum of a tensor-triangulated category $(\mathbb{T}, \otimes, \mathbf{1})$. We show that there is a homeomorphism between the spectral space of radical thick tensor ideals in $(\mathbb{T}, \otimes, \mathbf{1})$ and the collection of open subsets of $\mathrm{Spec}(\mathbb{T})$ in inverse topology. In fact, we prove a more general result in terms of supports on $(\mathbb{T}, \otimes, \mathbf{1})$ and work by combining methods from commutative algebra, topology and tensor triangular geometry.

RÉSUMÉ

Soit $\mathrm{Spec}(\mathbb{T})$ le spectre d'une catégorie triangulée tensorielle $(\mathbb{T}, \otimes, \mathbf{1})$ et notons par $\mathrm{Spec}(\mathbb{T})^{\mathrm{inv}}$ la topologie inverse sur $\mathrm{Spec}(\mathbb{T})$. Nous montrons qu'on dispose d'un homéomorphisme entre l'espace spectral des idéaux radicaux de $(\mathbb{T}, \otimes, \mathbf{1})$ et l'espace des sous-ensembles ouverts de $\mathrm{Spec}(\mathbb{T})^{\mathrm{inv}}$. En fait, nous obtenons un résulat plus général en termes des données de support sur $(\mathbb{T}, \otimes, \mathbf{1})$ en utilisant des idées provenant d'algèbre commutative, de topologie et de géométrie triangulée tensorielle.

1. Introduction

The spectrum $\mathrm{Spec}(\mathbb{T})$ of a tensor-triangulated category $(\mathbb{T}, \otimes, \mathbf{1})$ was introduced by Paul Balmer in [1]. As a set, the spectrum $\mathrm{Spec}(\mathbb{T})$ consists of the prime thick tensor ideals in $(\mathbb{T}, \otimes, \mathbf{1})$ (see Section 2 for definitions). Further, the spectrum of a tensor-triangulated category is a spectral space, i.e. it is homeomorphic to the Zariski spectrum of some commutative ring. The remarkable feature of this framework of Balmer [1] is that it unifies ideas from diverse branches of mathematics: from that of Thomason [26] in algebraic geometry to that of Devinatz, Hopkins, and Smith [11] in homotopy theory, and to that of Benson, Carlson and Rickard [10] in modular representation theory. Balmer's work was the beginning of tensor triangular geometry, which has emerged as a branch of inquiry in its own right (see, for instance, [1–7], [17,18], [21–24]). We refer the reader to [4] for an excellent survey of the landscape of this beautiful subject. Our purpose is to prove

E-mail address: abhishekbanerjee1313@gmail.com.

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a "topological nullstellensatz'-like result for a tensor-triangulated category $(\mathbb{T}, \otimes, \mathbf{1})$ by learning from some methods in commutative algebra, point-free topology, and tensor triangular geometry.

In this paper, we are motivated by looking at two nullstellensatz-like results, one in tensor triangular geometry, and the other in commutative algebra. The first was proved by Balmer [1] for the collection of radical thick tensor ideals in a tensor-triangulated category $(\mathbb{T}, \otimes, \mathbf{1})$. We recall here (see Hochster [14]) that any spectral space X carries a dual topology X^{inv}, with the complements of quasi-compact opens in X forming a basis of open sets for X^{inv}.

Theorem 1.1. *(See [1, Theorem 4.10].) Let $(\mathbb{T}, \otimes, \mathbf{1})$ be a tensor-triangulated category. Then, there is an order-preserving bijection between radical thick tensor ideals in $(\mathbb{T}, \otimes, \mathbf{1})$ and open subspaces in the inverse topology on the spectral space* $\mathrm{Spec}(\mathbb{T})$.

In [19], Kock and Pitsch provided a fresh perspective on Balmer's work in [1] and new conceptual proofs of classical results of Thomason [26] by using the approach of point-free topology, in which topological spaces are replaced by the lattice of their open subsets. In fact, such lattices satisfy certain special properties and are known as frames (see Johnstone [15] for details or Section 2 for a brief recollection). In particular, the collection of spectral spaces corresponds to the collection of coherent frames. As such, several of the results of Balmer [1] may be subsumed by the following theorem of Kock and Pitsch [19].

Theorem 1.2. *(See [19, Theorem 3.1.9].) Let $(\mathbb{T}, \otimes, \mathbf{1})$ be a tensor-triangulated category. Then, the radical thick tensor ideals of $\mathbb{T}$ form a coherent frame $Zar(\mathbb{T})$, provided there is only a set of them.*

The second result that motivates this paper is a "topological nullstellensatz" for radical ideals in a commutative ring recently established in [13]. It is a classical fact that closed subspaces of the spectrum $\mathrm{Spec}(R)$ of a commutative ring R are in one–one correspondence with radical ideals of R. By considering a "Zariski topology" on the collection of non-empty closed subsets of $\mathrm{Spec}(R)$; this bijection has recently been promoted by Finocchiaro, Fontana and Spirito [13] to a homeomorphism of topological spaces.

Theorem 1.3. *(See [13, Theorem 4.1].) Let R be a commutative ring. Then, there is a homeomorphism between the space of non-empty closed subspaces of* $\mathrm{Spec}(R)$ *and the inverse topology on the spectral space of proper radical ideals of R.*

Our aim in this paper is to promote the bijection in Theorem 1.1 to a homeomorphism between spectral spaces as in the topological nullstellensatz appearing in Theorem 1.3. We will use the method of frames to understand tensor triangular geometry as in Kock and Pitsch [19]. We work more generally with a support for the tensor-triangulated category $\mathbb{T}$ in the sense of [19]. A support $\Psi = (F, d)$ for $(\mathbb{T}, \otimes, \mathbf{1})$ consists of a frame F along with a rule that associates with each object $a \in \mathbb{T}$ an element $d(a) \in F$ satisfying certain conditions that are recalled in Section 2. In particular, Kock and Pitsch [19, Theorem 3.2.3] showed that the Zariski frame $Zar(\mathbb{T})$ is an initial object in the category of supports on $(\mathbb{T}, \otimes, \mathbf{1})$, just as the Zariski frame of a commutative ring R (the frame of radical ideals) is the initial support on R (see Joyal [16]).

Accordingly, given a support $\Psi = (F, d)$ for the tensor-triangulated category $(\mathbb{T}, \otimes, \mathbf{1})$, there is a morphism $u : Zar(\mathbb{T}) \longrightarrow F$ of frames which factors as a regular epimorphism $Zar(\mathbb{T}) \longrightarrow F^{\Psi}$ followed by a monomorphism $F^{\Psi} \longrightarrow F$ (see Section 2). On the other hand, let $\mathcal{K}$ be a thick tensor ideal of $(\mathbb{T}, \otimes, \mathbf{1})$ and let $\bigvee\limits_{b \in \mathcal{K}} d(b)$ be the join in the frame F of all elements $d(b)$ for $b \in \mathcal{K}$. We will say that the thick tensor ideal $\mathcal{K}$ is Ψ-closed if it satisfies

$$d(a) \leq \bigvee_{b \in \mathcal{K}} d(b) \qquad \Rightarrow \qquad a \in \mathcal{K} \tag{1.1}$$

Then, we begin by showing that the collection of Ψ-closed ideals in $(\mathbb{T}, \otimes, \mathbf{1})$ forms a frame $Zar^{\Psi}(\mathbb{T})$. We also show that there is an isomorphism of frames between $Zar^{\Psi}(\mathbb{T})$ and the frame F^{Ψ}, a fact that generalizes Theorem 1.1.

From (1.1), it is clear that the Ψ-closed ideals are precisely the fixed points of the following closure operator (see Definition 3.2) on the collection $Id(\mathbb{T})$ of thick tensor ideals of $\mathbb{T}$.

$$c_{\Psi} : Id(\mathbb{T}) \longrightarrow Id(\mathbb{T}) \qquad \mathcal{K} \mapsto \left\{ a \in \mathbb{T} \mid d(a) \leq \bigvee_{b \in \mathcal{K}} d(b) \right\} \tag{1.2}$$

For any object $a \in \mathbb{T}$, let $\mathbb{T}(a)$ denote the smallest thick tensor ideal containing a. When the support $\Psi = (F, d)$ is such that $c_{\Psi} : Id(\mathbb{T}) \longrightarrow Id(\mathbb{T})$ is actually a closure operator of finite type, we show that the $\{c_{\Psi}(\mathbb{T}(a))\}_{a \in \mathbb{T}}$ are the finite elements of the frame $Zar^{\Psi}(\mathbb{T})$ (and vice versa). This enables us to show that the frame $Zar^{\Psi}(\mathbb{T})$ of Ψ-closed ideals is actually a coherent frame. Our main result is the following, which promotes the bijection in Theorem 1.1 to a homeomorphism between spectral spaces.

Theorem 1.4. *Let $\Psi = (F, d)$ be a support on a tensor-triangulated category $(\mathbb{T}, \otimes, \mathbf{1})$ such that $c_{\Psi} : Id(\mathbb{T}) \longrightarrow Id(\mathbb{T})$ is a closure operator of finite type. Then, the following spaces are spectral and there is a homeomorphism between them:*

(a) the set $Zar^{\Psi}(\mathbb{T})$ of Ψ-closed thick tensor ideals of $(\mathbb{T}, \otimes, \mathbf{1})$ equipped with the topology generated by the open sets $\{U(a) \cap Zar^{\Psi}(\mathbb{T})\}_{a\in\mathbb{T}}$ where

$$U(a) = \{\mathcal{K} \in Id(\mathbb{T}) \mid a \notin \mathcal{K}\}$$

(b) the set F^{Ψ} equipped with the topology generated by the open sets $\{W^{\Psi}(a)\}_{a\in\mathbb{T}}$ where

$$W^{\Psi}(a) = \{y \in F^{\Psi} \mid y \not\geq d(a)\}$$

As such, the result of Theorem 1.4 may be seen as a "topological nullstellensatz" for tensor-triangulated categories for a given support $\Psi = (F, d)$. Finally, we apply the result of Theorem 1.4 to the derived category of perfect complexes over a scheme. We recall that the topological nullstellensatz of Finocchiaro, Fontana, and Spirito [13] gave us a homeomorphism for the space of closed subspaces of $\mathrm{Spec}(R)$, where R is a commutative ring. Therefore, the following result gives us a counterpart of the topological nullstellensatz in Theorem 1.3 for the case of non-affine (topologically noetherian) schemes.

Corollary 1.5. *Let Z be a topologically Noetherian scheme and let $D^{\mathrm{perf}}(Z)$ be the derived category of perfect complexes over Z equipped with the usual derived tensor product. Then, there is a homeomorphism between the spectral space of radical thick tensor ideals in $D^{\mathrm{perf}}(Z)$ and the collection $\mathcal{U}(Z^{\mathrm{inv}})$ of open subsets of Z^{inv} equipped with the topology generated by the opens $\{V \in \mathcal{U}(Z^{\mathrm{inv}}) \mid V \not\supseteq U\}$ for all $U \in \mathcal{U}(Z^{\mathrm{inv}})$.*

2. Support data and Ψ-closed thick tensor ideals

Throughout, we suppose that $(\mathbb{T}, \otimes, \mathbf{1})$ is a tensor-triangulated category, i.e. $\mathbb{T}$ is a triangulated category carrying a symmetric monoidal product $\otimes : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbb{T}$ that is exact in each variable. We will always assume that the category $\mathbb{T}$ is essentially small.

Definition 2.1. (See [1].) A thick tensor ideal $\mathcal{K}$ of $(\mathbb{T}, \otimes, \mathbf{1})$ is a full subcategory such that $0 \in \mathcal{K}$ and $\mathcal{K}$ satisfies the following conditions:

(a) the subcategory $\mathcal{K}$ is thick, i.e. for objects $a, b, c \in \mathbb{T}$ with $c \cong a \oplus b$, the object $c \in \mathcal{K}$ if and only if both $a, b \in \mathcal{K}$;

(b) the subcategory $\mathcal{K}$ is triangulated, i.e. if $a \longrightarrow b \longrightarrow c \longrightarrow \Sigma(a)$ is any distinguished triangle in $\mathbb{T}$ and any two out of a, b, c lie in $\mathcal{K}$, so does the third;

(c) the subcategory $\mathcal{K}$ is a tensor ideal, i.e. if $a \in \mathcal{K}$ then $a \otimes b \in \mathcal{K}$ for any $b \in \mathbb{T}$.

A proper thick tensor ideal $\mathcal{P}$ is said to be prime if $a \otimes b \in \mathcal{P}$ implies that either $a \in \mathcal{P}$ or $b \in \mathcal{P}$. The collection of prime ideals of $(\mathbb{T}, \otimes, \mathbf{1})$ is given by $\mathrm{Spec}(\mathbb{T})$.

In [1], Balmer began the study of tensor triangular geometry by showing that $\mathrm{Spec}(\mathbb{T})$ is a spectral space in the sense of Hochster [14], i.e. it is homeomorphic to the Zariski spectrum of a commutative ring. A basis of closed sets for the topology on $\mathrm{Spec}(\mathbb{T})$ is given by the collection

$$\mathrm{supp}(a) := \{\mathcal{P} \in \mathrm{Spec}(\mathbb{T}) \mid a \notin \mathcal{P}\} \tag{2.1}$$

as a varies over all the objects in $\mathbb{T}$. A thick tensor ideal $\mathcal{K}$ in $\mathbb{T}$ is said to be radical if it satisfies (see [1, § 4])

$$\mathcal{K} = Rad(\mathcal{K}) := \{a \in \mathbb{T} \mid \exists\, n \geq 1 \text{ such that } a^{\otimes n} \in \mathcal{K}\} \tag{2.2}$$

Since $\mathrm{Spec}(\mathbb{T})$ is a spectral space, we can form (see, for instance, [25, Tag 08YF]) the inverse topology $\mathrm{Spec}(\mathbb{T})^{\mathrm{inv}}$ whose basis of open sets is given by the complements of quasi-compact opens in $\mathrm{Spec}(\mathbb{T})$. Then, Balmer's result [1, Theorem 4.10] shows that there is an order-preserving correspondence between radical thick tensor ideals in $(\mathbb{T}, \otimes, 1)$ and open sets in $\mathrm{Spec}(\mathbb{T})^{\mathrm{inv}}$.

Following Kock and Pitsch [19], we will now describe how these results may be understood in frame-theoretic terms. We recall that a lattice is a partially ordered set $(A, \leq)$ such that every finite subset $S \subseteq A$ has a least upper bound (called the join $\bigvee_{s\in S} s$) and a greatest lower bound (called the meet $\bigwedge_{s\in S} s$). The join of the empty set is denoted by 0 and the meet of the empty set is denoted by 1. A frame A is a complete lattice such that finite meets distribute over arbitrary joins (see [15, Chapter II]):

$$a \wedge \left(\bigvee_{s\in S} s\right) = \bigvee_{s\in S} (a \wedge s) \qquad \forall\, a \in A,\ S \subseteq A \tag{2.3}$$

Given any topological space, it is evident that its open subsets form a frame, with joins being given by union and finite meets being given by intersection.

At the same time, the collection of radical thick tensor ideals in $(\mathbb{T}, \otimes, \mathbf{1})$ may be treated as a frame: the meet of any two radical ideals $\mathcal{K}_1$ and $\mathcal{K}_2$ is given by intersection, while the join of any collection $\{\mathcal{K}_i\}_{i\in I}$ of radical ideals is $Rad(\sum_{i\in I}\mathcal{K}_i)$, the smallest radical ideal containing all the ideals $\{\mathcal{K}_i\}_{i\in I}$. Following Kock and Pitsch [19], we will refer to the frame of radical thick tensor ideals in $(\mathbb{T}, \otimes, \mathbf{1})$ as the Zariski frame $Zar(\mathbb{T})$. By virtue of [1, Theorem 4.10], we know that $Zar(\mathbb{T})$ is same as the frame of open sets corresponding to the topological space $\mathrm{Spec}(\mathbb{T})^{\mathrm{inv}}$.

The following definition is due to Balmer [1, Definition 3.1], although we state below its reformulation by Kock and Pitsch [19, Definition 3.2.1] in terms of frames.

Definition 2.2. Let $(\mathbb{T}, \otimes, \mathbf{1})$ be a tensor-triangulated category. A support Ψ on $(\mathbb{T}, \otimes, \mathbf{1})$ is a pair (F, d) where F is a frame and $d : obj(\mathbb{T}) \longrightarrow F$ is a map satisfying the following conditions:

(a) $d(0) = 0$ and $d(\mathbf{1}) = 1$;
(b) $d(a \oplus b) = d(a) \vee d(b)$ for any $a, b \in \mathbb{T}$;
(c) $d(a \otimes b) = d(a) \wedge d(b)$ for any $a, b \in \mathbb{T}$;
(d) $d(\Sigma(a)) = d(a)$ for any $a \in \mathbb{T}$, where Σ denotes the translation functor on the triangulated category $\mathbb{T}$;
(e) $d(b) \leq d(a) \vee d(c)$ for any distinguished triangle $a \longrightarrow b \longrightarrow c \longrightarrow \Sigma(a)$ in $\mathbb{T}$.

A morphism $f : (F, d) \longrightarrow (F', d')$ of supports on $(\mathbb{T}, \otimes, \mathbf{1})$ is a frame map $f : F \longrightarrow F'$ compatible with d and d'.

We note that an isomorphism $a \cong a'$ in $\mathbb{T}$ gives rise to a distinguished triangle $a \xrightarrow{\cong} a' \longrightarrow 0 \longrightarrow \Sigma(a)$ and it follows from the rules in Definition 2.2 that $d(a) = d(a')$. As such, the category $\mathbb{T}$ being essentially small, we can always talk about the join $\bigvee_{a\in\mathcal{S}} d(a)$ for any collection $\mathcal{S}$ of objects in $\mathbb{T}$ without causing any set-theoretic complications.

For any object $a \in \mathbb{T}$, let $Rad(a)$ denote the smallest radical thick tensor ideal containing a. Then, the pair $\Psi_0 := (Zar(\mathbb{T}), Rad)$ is initial in the category of supports on $(\mathbb{T}, \otimes, \mathbf{1})$ (see [19, Theorem 3.2.3]). This is a reformulation of Balmer's result [1, Theorem 3.2] in terms of frames.

Definition 2.3. Let $\Psi = (F, d)$ be a support for the tensor-triangulated category $(\mathbb{T}, \otimes, \mathbf{1})$. We will say that a thick tensor ideal $\mathcal{K}$ is Ψ-closed if it satisfies the condition:

$$d(a) \leq \bigvee_{b\in\mathcal{K}} d(b) \qquad \Rightarrow \qquad a \in \mathcal{K} \tag{2.4}$$

for any object $a \in \mathbb{T}$. The collection of all Ψ-closed ideals will be denoted by $Zar^{\Psi}(\mathbb{T})$.

In particular, when the support is chosen to be $\Psi_0 = (Zar(\mathbb{T}), Rad)$, it is clear that the collection of Ψ_0-closed ideals is identical to the collection of radical thick tensor ideals in $(\mathbb{T}, \otimes, \mathbf{1})$.

Let $\Psi = (F, d)$ be a support for the tensor-triangulated category $(\mathbb{T}, \otimes, \mathbf{1})$. Then, for any $x \in F$, it is clear from Definition 2.2 that the collection of objects

$$g(x) := \{a \in \mathbb{T} \mid d(a) \leq x\} \subseteq \mathbb{T} \tag{2.5}$$

is a radical thick tensor ideal in $(\mathbb{T}, \otimes, \mathbf{1})$. We also notice that g is an order-preserving map, i.e. if $x \leq y$ in F, then the ideals $g(x) \subseteq g(y)$ in $Zar(\mathbb{T})$.

Any partially ordered set $(A, \leq)$ may be treated as a category whose objects are the elements of A, with a single morphism from x to y if $x \leq y$. As such, the order-preserving map g corresponds to a functor $g : F \longrightarrow Zar(\mathbb{T})$.

Since $\Psi_0 = (Zar(\mathbb{T}), Rad)$ is the initial object in the category of supports on $\mathbb{T}$, the support (F, d) gives rise to a morphism of frames $u : Zar(\mathbb{T}) \longrightarrow F$. This morphism may also be seen as a functor u from $Zar(\mathbb{T})$ to F.

Lemma 2.4. *The functor $g : F \longrightarrow Zar(\mathbb{T})$ is right adjoint to the functor $u : Zar(\mathbb{T}) \longrightarrow F$.*

Proof. Since the functor $u : Zar(\mathbb{T}) \longrightarrow F$ comes from a morphism of frames, we know that it must have a right adjoint (see [15, § II.1]). We should note here that this right adjoint is only a homomorphism of meet-semilattices and not necessarily a morphism of frames. We will check that $g : F \longrightarrow Zar(\mathbb{T})$ is the right adjoint to u. Translated in terms of partially ordered sets and order-preserving maps, we must verify that:

$$u(\mathcal{K}) \leq x \qquad \Leftrightarrow \qquad \mathcal{K} \subseteq g(x) \tag{2.6}$$

for any radical thick tensor ideal $\mathcal{K}$ and any $x \in F$.

Now, for any object $a \in \mathbb{T}$, the explicit description of u (see the proof of [19, Theorem 3.2.3]) provides that $u(Rad(a)) = d(a)$. Since u is a morphism of frames, it preserves arbitrary joins. This leads to

$$u(\mathcal{K}) = \bigvee_{a\in\mathcal{K}} u(Rad(a)) = \bigvee_{a\in\mathcal{K}} d(a) \tag{2.7}$$

for any radical thick tensor ideal $\mathcal{K}$. Suppose first that $u(\mathcal{K}) \leq x$ for some $x \in F$. From (2.7), it follows that $d(a) \leq x$ for each $a \in \mathcal{K}$. Applying the definition of g from (2.5), we see that $\mathcal{K} \subseteq g(x)$.

Conversely, suppose that $\mathcal{K} \subseteq g(x)$ for some $x \in F$ and some radical thick tensor ideal $\mathcal{K}$. Again, this gives $d(a) \leq x$ for each $a \in \mathcal{K}$. Now, the join $\bigvee_{a\in\mathcal{K}} d(a)$ being the least upper bound of the collection $\{d(a)\}_{a\in\mathcal{K}}$, it follows from (2.7) that $u(\mathcal{K}) = \bigvee_{a\in\mathcal{K}} d(a) \leq x$. $\square$

We now recall that a category $\mathcal{C}$ is said to be 'algebraic' if it is equipped with a functor $U : \mathcal{C} \longrightarrow Set$ that is 'monadic', i.e. having a left adjoint satisfying certain properties (see [15, § I.3]). We know that the category Frm of frames is algebraic and this in particular implies that every morphism $f : F_1 \longrightarrow F_2$ in Frm may be factored uniquely as $f = i \circ q$, where q is a regular epimorphism and i is a monomorphism of frames (see [15, § II.2.1]).

Given the support $\Psi = (F, d)$ on $(\mathbb{T}, \otimes, \mathbf{1})$, we now consider a new frame F^Ψ as follows: we factor the morphism $u : Zar(\mathbb{T}) \longrightarrow F$ of frames as $Zar(\mathbb{T}) \xrightarrow{u'} F^\Psi \xrightarrow{i'} F$ where u' (resp. i') is a regular epimorphism (resp. a monomorphism) in Frm.

Theorem 2.5. *Let $\Psi = (F, d)$ be a support for the tensor-triangulated category $(\mathbb{T}, \otimes, \mathbf{1})$. Then, the collection $Zar^\Psi(\mathbb{T})$ of Ψ-closed ideals is a frame. Further, there is an isomorphism of frames between $Zar^\Psi(\mathbb{T})$ and the quotient frame F^Ψ.*

Proof. From Lemma 2.4, we know that $g : F \longrightarrow Zar(\mathbb{T})$ is right adjoint to the functor $u : Zar(\mathbb{T}) \longrightarrow F$. We set $j := g \circ u : Zar(\mathbb{T}) \longrightarrow Zar(\mathbb{T})$.

By definition, $Zar(\mathbb{T})$ is the frame of radical thick tensor ideals in $(\mathbb{T}, \otimes, \mathbf{1})$. Since $j : Zar(\mathbb{T}) \longrightarrow Zar(\mathbb{T})$ is defined by composing the morphism u of frames with its right adjoint, it follows from [15, § II.2] that the collection of radical thick tensor ideals fixed by j forms a frame that is isomorphic to the quotient frame F^Ψ.

Finally, it follows from the expressions in (2.5) and (2.7) that for any radical thick tensor ideal $\mathcal{K}$ we have

$$j(\mathcal{K}) := \{a \in \mathbb{T} \mid d(a) \leq \bigvee_{b\in\mathcal{K}} d(b)\} \tag{2.8}$$

From Definition 2.3, it is now clear that the collection of radical thick tensor ideals fixed by j is exactly the collection $Zar^\Psi(\mathbb{T})$ of Ψ-closed ideals in $(\mathbb{T}, \otimes, \mathbf{1})$. $\square$

Remark 2.6. It should be remarked that while the finite meet of elements in the fixed point frame $Zar^\Psi(\mathbb{T})$ is identical to their meet in $Zar(\mathbb{T})$, the join of a family in $Zar^\Psi(\mathbb{T})$ is not the same as their join in $Zar(\mathbb{T})$ (see the proof of [15, Lemma II.2.2]). In fact, the join of a family $\{\mathcal{K}_i\}_{i\in I}$ in the fixed point frame $Zar^\Psi(\mathbb{T})$, which we shall denote by $\overset{\Psi}{\underset{i\in I}{\vee}} \mathcal{K}_i$ is given by

$$\overset{\Psi}{\bigvee_{i\in I}} \mathcal{K}_i := j\left(Rad\left(\sum_{i\in I} \mathcal{K}_i\right)\right) = \{a \in \mathbb{T} \mid d(a) \leq \bigvee_{b\in Rad(\sum_{i\in I}\mathcal{K}_i)} d(b)\} = \{a \in \mathbb{T} \mid d(a) \leq \bigvee_{b\in\sum_{i\in I}\mathcal{K}_i} d(b)\} \tag{2.9}$$

The last equality in (2.9) follows from the fact that the map d appearing in the support (F, d) satisfies $d(b^{\otimes n}) = d(b)$ for any object $b \in \mathbb{T}$ and any $n \geq 1$.

3. Coherent frames and Ψ-closed ideals

For a commutative ring R, it was proved in [13] that the collection of radical ideals carries the structure of a spectral space. We have shown in [8] that the methods of [13] can be applied more generally to abelian categories. Further, we have studied in [9] the spectral spaces of submodules of a triangulated category that is a module over $(\mathbb{T}, \otimes, \mathbf{1})$. The latter is in keeping with the general philosophy that notions in abelian categories should have counterparts in triangulated categories (see, for instance, Krause [20, § 1]).

We continue with a support $\Psi = (F, d)$ for the tensor-triangulated category $(\mathbb{T}, \otimes, \mathbf{1})$. Additionally, we set

$$Zar^\Psi_\bullet(\mathbb{T}) := Zar^\Psi(\mathbb{T}) \backslash \{\mathbb{T}\} \tag{3.1}$$

to be the set of proper Ψ-closed ideals.

We denote by Top the category of topological spaces and continuous maps. Assigning to a topological space its frame of open sets determines a functor $Top \longrightarrow Frm^{op}$. This functor has a right adjoint, known as the functor of points (see Johnstone [15, Chapter II] for details). A "point" of a frame A is taken to be a frame map $\phi : A \longrightarrow \{0, 1\}$, which is motivated by the fact that $\{0, 1\}$ is the frame of open sets corresponding to the one point space $\{*\}$. The collection $pt(A)$ of points of A forms a topological space in which the open sets are of the form $\{\phi : A \longrightarrow \{0, 1\} \mid \phi(x) = 1\}$ for some $x \in A$.

Given a frame A, the space $pt(A)$ of points of A is always a sober topological space (i.e. every irreducible closed subset has a unique generic point). In fact, we know (see [15, § II.1.7]) that the adjunction between topological spaces and frames restricts to a contravariant equivalence between sober spaces and "frames that have enough points," also known as spatial frames.

In particular, we know that every spectral space is sober. We now recall the notion of coherent frames, which are the frame-theoretic counterpart of spectral spaces.

Definition 3.1. (See [15, § II.3].) Let A be a frame. An element $x \in A$ is said to be finite if for any subset $S \subseteq A$ such that $x \leq \bigvee_{s \in S} s$, there exists a finite subset $S' \subseteq S$ such that $x \leq \bigvee_{s \in S'} s$.

A frame A is said to be coherent if it satisfies the following two conditions:

(i) every element of A may be expressed as a join of finite elements;

(ii) the finite elements form a sublattice of A. Equivalently, the top element 1 is finite and the meet of any two finite elements is finite.

Given a set S of objects of $\mathbb{T}$, we will denote by $\mathbb{T}(S)$ the thick tensor ideal generated by S, i.e. the smallest thick tensor ideal in $(\mathbb{T}, \otimes, \mathbf{1})$ containing all the elements of S. When $S = \{s\}$ consists of a single element, we denote the thick tensor ideal generated by $\{s\}$ simply by $\mathbb{T}(s)$. Using Definition 2.1 we also note that for any finite collection $S = \{s_1, ..., s_k\}$ of objects of $\mathbb{T}$, we have $\mathbb{T}(s_1, ..., s_k) = \mathbb{T}(s_1 \oplus s_2 \oplus ... \oplus s_k)$.

We will now give conditions for the frame $Zar^{\Psi}(\mathbb{T})$ to be coherent. For this, we will have to consider "closure operators" on thick tensor ideals defined in a manner similar to closure operators in the case of modules in commutative algebra (see [13, § 3] and also [12]). We have shown in [9] how closure operators on the collection $Id(\mathbb{T})$ of thick tensor ideals can be used to construct spectral spaces.

Definition 3.2. (See [9, Definition 2.4].) Let $(\mathbb{T}, \otimes, \mathbf{1})$ be a tensor-triangulated category and let $Id(\mathbb{T})$ be the collection of its thick tensor ideals. An operator $c : Id(\mathbb{T}) \longrightarrow Id(\mathbb{T})$ will be said to be:

(a) extensive if $\mathcal{K} \subseteq c(\mathcal{K})$ for each $\mathcal{K} \in Id(\mathbb{T})$;
(b) order-preserving if $\mathcal{K} \subseteq \mathcal{K}'$ implies that $c(\mathcal{K}) \subseteq c(\mathcal{K}')$;
(c) idempotent if $c(\mathcal{K}) = c(c(\mathcal{K}))$ for each $\mathcal{K} \in Id(\mathbb{T})$;
(d) finite type if $c(\mathcal{K}) = \bigcup_{a \in \mathcal{K}} c(\mathbb{T}(a))$.

We will refer to an operator satisfying (a), (b) and (c) as a closure operator on $Id(\mathbb{T})$.

Lemma 3.3. *Let $c : Id(\mathbb{T}) \longrightarrow Id(\mathbb{T})$ be a closure operator of finite type. Then, for any collection $\{\mathcal{K}_i\}_{i \in I}$ of thick tensor ideals, we have*

$$c\left(\sum_{i \in I} \mathcal{K}_i\right) = \bigcup_{J \in \mathcal{P}^f(I)} c\left(\sum_{j \in J} \mathcal{K}_j\right) \tag{3.2}$$

where $\mathcal{P}^f(I)$ denotes the collection of finite subsets of I.

Proof. Given any $a \in \sum_{i \in I} \mathcal{K}_i$, it follows (see [19, Lemma 3.1.2]) that there is a finite subset $J \subseteq I$ such that $a \in \sum_{j \in J} \mathcal{K}_j$. Then, $\mathbb{T}(a) \subseteq \sum_{j \in J} \mathcal{K}_j$ and $c(\mathbb{T}(a)) \subseteq c(\sum_{j \in J} \mathcal{K}_j)$. Since c is of finite type, we know that $c(\sum_{i \in I} \mathcal{K}_i) = \bigcup_{a \in \sum_{i \in I} \mathcal{K}_i} c(\mathbb{T}(a))$. Hence, $c(\sum_{i \in I} \mathcal{K}_i) \subseteq \bigcup_{J \in \mathcal{P}^f(I)} c\left(\sum_{j \in J} \mathcal{K}_j\right)$. Because c is extensive, the reverse inclusion is obvious. This proves the result. $\square$

Lemma 3.4. *Let S be a set of objects of $\mathbb{T}$ and let $\mathbb{T}(S)$ be the thick tensor ideal generated by the elements of S. Then, for any support $\Psi = (F, d)$, we have:*

$$\bigvee_{a \in \mathbb{T}(S)} d(a) = \bigvee_{s \in S} d(s) \tag{3.3}$$

Proof. Let A be a set of objects from $\mathbb{T}$. Following [19], let $Gen(A)$ be the collection of objects of any of the following forms:

(i) an interated suspension $\sum^n(a)$ or iterated desuspension $\sum^{-n}(a)$ of some $a \in A$;
(ii) a finite direct sum of objects of A or a direct summand of an object of A;
(iii) the form $a \otimes b$ with $a \in A$ and $b \in \mathbb{T}$;
(iv) an extension of two objects in A.

Considering the conditions appearing in the definition of a support (F,d) and the above ways in which elements of $Gen(A)$ arise from those of A, it is clear that $d(b) \leq \bigvee\limits_{a\in A} d(a)$ for any $b \in Gen(A)$. This implies that:

$$\bigvee_{b\in Gen(A)} d(b) = \bigvee_{a\in A} d(a) \tag{3.4}$$

On the other hand, we know from [19, § 3.1] that the union $Gen^{\omega}(S) = \bigcup_{k\in\mathbb{N}} Gen^k(S)$ gives the thick tensor ideal $\mathbb{T}(S)$ generated by S. Combining with (3.4), we have the result. □

Given the support $\Psi = (F,d)$ on $(\mathbb{T}, \otimes, \mathbf{1})$, we now proceed to define an operator on $Id(\mathbb{T})$ as follows:

$$c_\Psi : Id(\mathbb{T}) \longrightarrow Id(\mathbb{T}) \qquad \mathcal{K} \mapsto \left\{ a \in \mathbb{T} \mid d(a) \leq \bigvee_{b\in\mathcal{K}} d(b) \right\} \tag{3.5}$$

From (3.5), we note that for any $\mathcal{K} \in Id(\mathbb{T})$, we must have $\bigvee\limits_{b\in\mathcal{K}} d(b) = \bigvee\limits_{b\in c_\Psi(\mathcal{K})} d(b)$. It follows that c_Ψ is extensive, order-preserving and idempotent, i.e. the support $\Psi = (F,d)$ defines a closure operator c_Ψ on $Id(\mathbb{T})$. It is also evident that, if $\mathcal{K}$ is a radical thick tensor ideal, then $c_\Psi(\mathcal{K}) = j(\mathcal{K})$ in the notation of Section 2.

Proposition 3.5. *The following statements are equivalent:*

(a) the operator $c_\Psi : Id(\mathbb{T}) \longrightarrow Id(\mathbb{T})$ is a closure operator of finite type;
(b) for each $a \in \mathbb{T}$, the element $c_\Psi(\mathbb{T}(a))$ of the frame $Zar^\Psi(\mathbb{T})$ is finite.

Proof. (a) ⇒ (b): We consider some $a \in \mathbb{T}$ and a family of elements $\{\mathcal{K}_i\}_{i\in I}$ from $Zar^\Psi(\mathbb{T})$ such that:

$$c_\Psi(\mathbb{T}(a)) \subseteq \overset{\Psi}{\bigvee_{i\in I}} \mathcal{K}_i = \{b' \in \mathbb{T} \mid d(b') \leq \bigvee_{b\in \sum_{i\in I}\mathcal{K}_i} d(b)\} = c_\Psi\left(\sum_{i\in I} \mathcal{K}_i\right) \tag{3.6}$$

Since c_Ψ is of finite type, it follows from Lemma 3.3 that $c_\Psi\left(\sum_{i\in I}\mathcal{K}_i\right) = \bigcup\limits_{J\in\mathcal{P}^f(I)} c_\Psi\left(\sum_{j\in J}\mathcal{K}_j\right)$. Hence, there is a finite subset $J \subseteq I$ such that $a \in c_\Psi(\mathbb{T}(a))$ lies in $c_\Psi(\sum_{j\in J}\mathcal{K}_j) = \overset{\Psi}{\bigvee\limits_{j\in J}} \mathcal{K}_j$. Then, the ideal $\mathbb{T}(a)$ generated by a lies in $\overset{\Psi}{\bigvee\limits_{j\in J}} \mathcal{K}_j$ and so does its closure $c_\Psi(\mathbb{T}(a))$. This shows that each $c_\Psi(\mathbb{T}(a))$ is a finite element of the frame $Zar^\Psi(\mathbb{T})$.

(b) ⇒ (a): We consider an element $\mathcal{K} \in Id(\mathbb{T})$ and some $a \in c_\Psi(\mathcal{K})$. Then, the finite element $c_\Psi(\mathbb{T}(a)) \subseteq c_\Psi(\mathcal{K})$. We notice that

$$c_\Psi(\mathcal{K}) = c_\Psi\left(\sum_{b\in\mathcal{K}} c_\Psi(\mathbb{T}(b))\right) = \overset{\Psi}{\bigvee_{b\in\mathcal{K}}} c_\Psi(\mathbb{T}(b)) \tag{3.7}$$

From the finiteness of $c_\Psi(\mathbb{T}(a))$, it follows that there exists a finite set $\{b_1, ..., b_n\} \subseteq \mathcal{K}$ such that

$$c_\Psi(\mathbb{T}(a)) \subseteq \overset{\Psi}{\bigvee_{1\leq k\leq n}} c_\Psi(\mathbb{T}(b_k)) = c_\Psi\left(\sum_{1\leq k\leq n} c_\Psi(\mathbb{T}(b_k))\right) = \{a' \in \mathbb{T} \mid d(a') \leq d(b_1 \oplus ... \oplus b_n)\} = c_\Psi(\mathbb{T}(b_1 \oplus ... \oplus b_n))$$

Then, $a \in c_\Psi(\mathbb{T}(a)) \subseteq c_\Psi(\mathbb{T}(b_1 \oplus ... \oplus b_n))$ and we see that c_Ψ is of finite type. □

Proposition 3.6. *Suppose that the operator $c_\Psi : Id(\mathbb{T}) \longrightarrow Id(\mathbb{T})$ is a closure operator of finite type. Then,*

(a) the collection $\{c_\Psi(\mathbb{T}(a))\}_{a\in\mathbb{T}}$ is the collection of finite elements of the frame $Zar^\Psi(\mathbb{T})$;
(b) the frame $Zar^\Psi(\mathbb{T})$ is coherent.

Proof. (a) Suppose that $\mathcal{K}_0 \in Id(\mathbb{T})$ is a finite element of $Zar^\Psi(\mathbb{T})$. It is clear that:

$$\mathcal{K}_0 = \sum_{a\in\mathcal{K}_0} \mathbb{T}(a) \subseteq \sum_{a\in\mathcal{K}_0} c_\Psi(\mathbb{T}(a)) \subseteq c_\Psi\left(\sum_{a\in\mathcal{K}_0} c_\Psi(\mathbb{T}(a))\right) = \overset{\Psi}{\bigvee_{a\in\mathcal{K}_0}} c_\Psi(\mathbb{T}(a)) \tag{3.8}$$

Since $\mathcal{K}_0 \in Zar^{\Psi}(\mathbb{T})$ is a finite element, we have a finite collection of elements $\{a_1, ..., a_n\} \subseteq \mathcal{K}_0$ such that

$$\mathcal{K}_0 \subseteq \bigvee_{1\leq k\leq n}^{\Psi} c_\Psi(\mathbb{T}(a_k)) = c_\Psi\left(\sum_{1\leq k\leq n} c_\Psi(\mathbb{T}(a_k))\right) \tag{3.9}$$

Applying Lemma 3.4, the right-hand side of (3.9) becomes

$$\begin{aligned} c_\Psi\left(\sum_{1\leq k\leq n} c_\Psi(\mathbb{T}(a_k))\right) &= \{a \in \mathbb{T} \mid d(a) \leq \bigvee_{1\leq k\leq n} \bigvee_{b\in c_\Psi(\mathbb{T}(a_k))} d(b)\} \\ &= \{a \in \mathbb{T} \mid d(a) \leq \bigvee_{1\leq k\leq n} \bigvee_{b\in \mathbb{T}(a_k)} d(b)\} \\ &= \{a \in \mathbb{T} \mid d(a) \leq \bigvee_{1\leq k\leq n} d(a_k) = d(a_1 \oplus ... \oplus a_k)\} = c_\Psi(\mathbb{T}(a_1 \oplus ... \oplus a_k)) \end{aligned} \tag{3.10}$$

This gives $\mathcal{K}_0 \subseteq c_\Psi(\mathbb{T}(a_1 \oplus ... \oplus a_n))$. On the other hand, $\mathcal{K}_0 \in Zar^{\Psi}(\mathbb{T})$ satisfies $c_\Psi(\mathbb{T}(a_1 \oplus ... \oplus a_n)) \subseteq c_\Psi(\mathcal{K}_0) = \mathcal{K}_0$. Hence, the finite element $\mathcal{K}_0 \in Zar^{\Psi}(\mathbb{T})$ is of the form $\mathcal{K}_0 = c_\Psi(\mathbb{T}(a_1 \oplus ... \oplus a_n))$. Since c_Ψ is of finite type, the converse follows from Proposition 3.5.

(b) We see that any $\mathcal{K} \in Zar^{\Psi}(\mathbb{T})$ may be expressed as a join of finite elements as follows:

$$\mathcal{K} = c_\Psi(c_\Psi(\mathcal{K})) = c_\Psi\left(\sum_{a\in\mathcal{K}} c_\Psi(\mathbb{T}(a))\right) = \bigvee_{a\in\mathcal{K}}^{\Psi} c_\Psi(\mathbb{T}(a)) \tag{3.11}$$

We also notice that the top element $\mathbb{T} = c_\Psi(\mathbb{T}(\mathbf{1})) \in Zar^{\Psi}(\mathbb{T})$ is finite. Finally, we consider two finite elements $c_\Psi(\mathbb{T}(a_1))$ and $c_\Psi(\mathbb{T}(a_2))$ in the frame $Zar^{\Psi}(\mathbb{T})$. Their meet is given by

$$c_\Psi(\mathbb{T}(a_1)) \cap c_\Psi(\mathbb{T}(a_2)) = \{a \in \mathbb{T} \mid d(a) \leq \bigvee_{b\in c_\Psi(\mathbb{T}(a_1))} d(b) \text{ and } d(a) \leq \bigvee_{b\in c_\Psi(\mathbb{T}(a_2))} d(b)\} \tag{3.12}$$

From Lemma 3.4, we know that

$$\bigvee_{b\in c_\Psi(\mathbb{T}(a_1))} d(b) = \bigvee_{b\in\mathbb{T}(a_1)} d(b) = d(a_1) \qquad \bigvee_{b\in c_\Psi(\mathbb{T}(a_2))} d(b) = \bigvee_{b\in\mathbb{T}(a_2)} d(b) = d(a_2) \tag{3.13}$$

Since $d(a_1 \otimes a_2) = d(a_1) \wedge d(a_2)$ is the greatest lower bound of $\{d(a_1), d(a_2)\}$ in F, it follows from (3.12) that

$$c_\Psi(\mathbb{T}(a_1)) \cap c_\Psi(\mathbb{T}(a_2)) = \{a \in \mathbb{T} \mid d(a) \leq d(a_1 \otimes a_2)\} = c_\Psi(\mathbb{T}(a_1 \otimes a_2)) \tag{3.14}$$

Hence, the meet of two finite elements in $Zar^{\Psi}(\mathbb{T})$ is still finite. From Definition 3.1, it now follows that $Zar^{\Psi}(\mathbb{T})$ is a coherent frame. □

4. The lower interval topology and spectral spaces

Given a partially ordered set $(A, \leq)$ and an element $x \in A$, we set $\uparrow(x) := \{y \in A \mid y \geq x\}$ and refer to $\uparrow(x)$ as the principal upper set generated by x. A 'lower interval' in A is the complement of a principal upper set. We put

$$L(x) := A \setminus \uparrow(x) = \{y \in A \mid y \not\geq x\} \tag{4.1}$$

Then, the lower interval topology on A is defined to be the topology generated by taking the lower intervals $\{L(x)\}_{x\in A}$ as a subbasis of open sets (see [15, § II.1.8]). This topological space will be denoted by $LI(A)$.

Suppose in particular that A is a coherent frame. We will now show that the space $LI(A)$ is actually a spectral space. If the top element in the frame A is denoted by 1, we will also show that $LI(A\setminus\{1\})$ is a spectral space.

Proposition 4.1. *Let A be a coherent frame and let A^ω be the collection of finite elements of A. Then,*

(a) the space $LI(A)$, i.e. the set A equipped with the lower interval topology, is a spectral space. The collection $\mathcal{L} = \{L(x) \mid x \in A^\omega\}$ forms a subbasis of quasi-compact open subspaces of $LI(A)$;

(b) let $1 \in A$ be the top element in the frame A. Then, the space $LI(A\setminus\{1\})$, i.e. the set $A\setminus\{1\}$, equipped with the lower interval topology, is a spectral space.

Proof. (a) We will first show that $LI(A)$ is a T_0 space. We consider elements $x_1, x_2 \in A$ with $x_1 \neq x_2$. Then, either $x_1 \not\geq x_2$ or $x_2 \not\geq x_1$. It follows that at least one of the open sets $L(x_1)$ and $L(x_2)$ contains exactly one of the two points x_1 and x_2.

For any collection of elements $\{x_i\}_{i\in I}$ from A, it is evident from the definitions that $\bigcup_{i\in I} L(x_i) = L(\bigvee_{i\in I} x_i)$. Since A is a coherent frame, we know that every element may be expressed as the join of elements in A^ω. As such, it is clear that the topology $LI(A)$ can be generated by taking the collection $\mathcal{L} = \{L(x) \mid x \in A^\omega\}$ as a subbasis of open sets.

We now define a new topology on the set A by taking the following collection as a subbasis of opens:

$$\mathcal{L}' = \{\uparrow (y) \mid y \in A^{\omega}\} \cup \{L(x) \mid x \in A^{\omega}\} \tag{4.2}$$

This topological space will be denoted by A'. In particular, the subbasis $\mathcal{L} = \{L(x) \mid x \in A^{\omega}\}$ for the space $LI(A)$ is a collection of clopen (closed and open) sets of the space A'. We have already seen that $LI(A)$ is a T_0 space. In order to show that $LI(A)$ is spectral, it therefore suffices to check that A' is quasi-compact (see Hochster [14, Proposition 7]).

As such, suppose that we have a covering of A' by a collection of subbasis elements from $\mathcal{L}'$:

$$A' = \left(\bigcup_{j\in J} \uparrow (y_j)\right) \cup \left(\bigcup_{i\in I} L(x_i)\right) = \left(\bigcup_{j\in J} \uparrow (y_j)\right) \cup L(x) \tag{4.3}$$

Here $x = \bigvee_{i\in I} x_i$. Since $x \notin L(x)$, we can find some $j_0 \in J$ such that $x \in \uparrow (y_{j_0})$, i.e. $x \geq y_{j_0}$. But since y_{j_0} is a finite element, we can choose a finite subcollection $\{x_1, ..., x_n\} \subseteq \{x_i\}_{i\in I}$ such that $y_{j_0} \leq \bigvee\limits_{k=1}^{n} x_k$. Then, $L(y_{j_0}) \subseteq \bigcup\limits_{k=1}^{n} L(x_k)$, and we have obtained a finite subcovering $A' = \uparrow (y_{j_0}) \cup \bigcup\limits_{k=1}^{n} L(x_k)$. From the Alexander subbase theorem (see, for instance, [25, Tag 08ZP]), it follows that A' is quasi-compact. This proves that $LI(A)$ is spectral. Finally, for any $x' \in A^{\omega}$, the set $L(x')$ is a closed subset of the quasi-compact space A' and hence quasi-compact as a subspace of A'. Since the topology $LI(A)$ is coarser than the topology A', it follows that $L(x')$ is quasi-compact as a subspace of $LI(A)$.

(b) We put $A_\bullet := A \backslash \{1\}$ and $\uparrow_\bullet (x) := \uparrow (x) \backslash \{1\}$ for any $x \in A$. From the definition in (4.1), it is clear that the top element 1 does not lie in any of the $L(x)$, $x \in A^{\omega}$. In this case, we can consider the topology $A'_\bullet$ on the set $A_\bullet$ by taking as subbasis the collection:

$$\mathcal{L}'_\bullet = \{\uparrow_\bullet (y) \mid y \in A^{\omega}\} \cup \{L(x) \mid x \in A^{\omega}\} \tag{4.4}$$

As in part(a), we may consider a covering of $A'_\bullet$ by a collection of subbasis elements from $\mathcal{L}'_\bullet$:

$$A'_\bullet = \left(\bigcup_{j\in J} \uparrow_\bullet (y_j)\right) \cup \left(\bigcup_{i\in I} L(x_i)\right) = \left(\bigcup_{j\in J} \uparrow_\bullet (y_j)\right) \cup L\left(\bigvee_{i\in I} x_i\right) \tag{4.5}$$

If $\bigvee_{i\in I} x_i = x \neq 1$, the result follows by repeating the argument in (a). If $x = 1$, we use the fact that $1 \in A$ is a finite element of the coherent frame A. Then, there is a finite subcollection $\{x_1, ..., x_n\} \subseteq \{x_i\}_{i\in I}$ such that $1 \leq \bigvee\limits_{k=1}^{n} x_k$. Then, $A'_\bullet = L(1) = \bigcup\limits_{k=1}^{n} L(x_k)$, and we have a finite subcovering. □

5. A topological nullstellensatz for $(\mathbb{T}, \otimes, 1)$

For any $a \in \mathbb{T}$, we now set

$$U(a) := \{\mathcal{K} \in Id(\mathbb{T}) \mid a \notin \mathcal{K}\} \tag{5.1}$$

In this section, we always suppose that c_Ψ is a closure operator of finite type. Then, by Proposition 3.6, we know that $Zar^{\Psi}(\mathbb{T})$ is a coherent frame and it follows from Proposition 4.1 that $LI(Zar^{\Psi}(\mathbb{T}))$ is a spectral space. It also follows from Proposition 4.1 that $LI(Zar^{\Psi}_\bullet(\mathbb{T}))$ is a spectral space.

Lemma 5.1. *(a) The topology on $LI(Zar^{\Psi}(\mathbb{T}))$, i.e. the set $Zar^{\Psi}(\mathbb{T})$ equipped with the lower interval topology, is generated by the open sets $\{U(a) \cap Zar^{\Psi}(\mathbb{T})\}_{a\in\mathbb{T}}$.*

(b) The topology on $LI(Zar^{\Psi}_\bullet(\mathbb{T}))$, the set $Zar^{\Psi}_\bullet(\mathbb{T}) = Zar^{\Psi}(\mathbb{T})\backslash\{\mathbb{T}\}$ equipped with the lower interval topology, is generated by the open sets $\{U(a) \cap Zar^{\Psi}_\bullet(\mathbb{T})\}_{a\in\mathbb{T}}$.

Proof. We first prove (a). From the definition in (4.1), we know that the lower interval topology $LI(Zar^{\Psi}(\mathbb{T}))$ is generated by the open sets (for all $\mathcal{K} \in Zar^{\Psi}(\mathbb{T})$)

$$L(\mathcal{K}) = \{\mathcal{K}' \in Zar^{\Psi}(\mathbb{T}) \mid \mathcal{K}' \not\supseteq \mathcal{K}\} = \bigcup_{a\in\mathcal{K}} \{\mathcal{K}' \in Zar^{\Psi}(\mathbb{T}) \mid a \notin \mathcal{K}'\} = \bigcup_{a\in\mathcal{K}} (U(a) \cap Zar^{\Psi}(\mathbb{T})) \tag{5.2}$$

The result now follows from (5.2) and the fact that the set $U(a) \cap Zar^{\Psi}(\mathbb{T}) = \{\mathcal{K}' \in Zar^{\Psi}(\mathbb{T}) \mid \mathcal{K}' \not\supseteq \mathbb{T}(a)\} = \{\mathcal{K}' \in Zar^{\Psi}(\mathbb{T}) \mid \mathcal{K}' \not\supseteq c_\Psi(\mathbb{T}(a))\} = L(c_\Psi(\mathbb{T}(a)))$. The result of (b) follows similarly. □

Let $\Psi = (F, d)$ be a support for the tensor-triangulated category $(\mathbb{T}, \otimes, \mathbf{1})$. As mentioned in Section 2, the frame map $u : Zar(\mathbb{T}) \longrightarrow F$ factors as a regular epimorphism $u' : Zat(\mathbb{T}) \longrightarrow F^{\Psi}$ followed by a monomorphism $i' : F^{\Psi} \longrightarrow F$ of frames.

Lemma 5.2. *The topology on the space $LI(F^{\Psi})$, i.e. the set F^{Ψ} equipped with the lower interval topology, is generated by the open sets*

$$W^{\Psi}(a) := \{y \in F^{\Psi} \mid y \ngeq d(a)\} \qquad \forall\, a \in \mathbb{T} \tag{5.3}$$

Proof. Considering $F^{\Psi} = Im(u)$ as a subframe of F, the space $LI(F^{\Psi})$ has a topology generated by the open sets $L^{\Psi}(x) = \{y \in F^{\Psi} \mid y \ngeq x\}$ for all $x \in F^{\Psi}$. We consider some radical thick tensor ideal $\mathcal{K}$ such that $u(\mathcal{K}) = \bigvee\limits_{a\in\mathcal{K}} d(a) = x$. It follows that

$$L^{\Psi}(x) = \{y \in F^{\Psi} \mid y \ngeq \bigvee_{a\in\mathcal{K}} d(a)\} = \bigcup_{a\in\mathcal{K}} \{y \in F^{\Psi} \mid y \ngeq d(a)\} = \bigcup_{a\in\mathcal{K}} W^{\Psi}(a) \tag{5.4}$$

Since $d(a) = u(Rad(a))$, we also have $W^{\Psi}(a) = \{y \in F^{\Psi} \mid y \ngeq d(a)\} = L^{\Psi}(d(a))$. Combining with (5.4), we obtain the result. $\square$

From Theorem 2.5, we know that there is an isomorphism between the frames $Zar^{\Psi}(\mathbb{T})$ and the quotient frame F^{Ψ}. We have supposed that c_{Ψ} is a closure operator of finite type and it follows from Proposition 3.6 that $Zar^{\Psi}(\mathbb{T})$ is a coherent frame. Hence, F^{Ψ} is coherent. We are now ready to promote the correspondence in Theorem 2.5 to a homeomorphism of spectral spaces.

Theorem 5.3. *Let $\Psi = (F, d)$ be a support for the tensor-triangulated category $(\mathbb{T}, \otimes, \mathbf{1})$ such that c_{Ψ} is a closure operator of finite type. Then,*

(a) the following spaces are homeomorphic:

(i) the spectral space $LI(Zar^{\Psi}(\mathbb{T}))$, i.e. the set $Zar^{\Psi}(\mathbb{T})$ of Ψ-closed ideals with topology generated by the open sets $\{U(a) \cap Zar^{\Psi}(\mathbb{T})\}_{a\in\mathbb{T}}$

(ii) the space $LI(F^{\Psi})$, i.e. the set F^{Ψ} equipped with the topology generated by the open sets $\{W^{\Psi}(a)\}_{a\in\mathbb{T}}$.

(b) setting $F^{\Psi}_{\bullet} := F^{\Psi}\backslash\{1\}$, the following spaces are homeomorphic:

(i) the spectral space $LI(Zar^{\Psi}_{\bullet}(\mathbb{T}))$ of proper Ψ-closed ideals with topology generated by the open sets $\{U(a) \cap Zar^{\Psi}_{\bullet}(\mathbb{T})\}_{a\in\mathbb{T}}$

(ii) the space $LI(F^{\Psi}_{\bullet})$, i.e. the set $F^{\Psi}_{\bullet}$ equipped with the topology generated by the open sets $\{W^{\Psi}(a)\}_{a\in\mathbb{T}}$.

Proof. Since $Zar^{\Psi}(\mathbb{T})$ is a coherent frame, we know from Proposition 4.1 that $LI(Zar^{\Psi}(\mathbb{T}))$ is a spectral space. Applying the LI construction to the isomorphic frames $Zar^{\Psi}(\mathbb{T})$ and F^{Ψ} leads to a homeomorphism between the spectral space $LI(Zar^{\Psi}(\mathbb{T}))$ and $LI(F^{\Psi})$. Using Lemma 5.1 and Lemma 5.2 respectively for the explicit descriptions of the topologies on $LI(Zar^{\Psi}(\mathbb{T}))$ and $LI(F^{\Psi})$, we have (a).

The isomorphism of frames between $Zar^{\Psi}(\mathbb{T})$ and the quotient frame F^{Ψ} restricts to an isomorphism of partially ordered sets between $Zar^{\Psi}_{\bullet}(\mathbb{T})$ and $F^{\Psi}_{\bullet}$. Applying Proposition 4.1 to the coherent frame $Zar^{\Psi}(\mathbb{T})$, we know that $LI(Zar^{\Psi}_{\bullet}(\mathbb{T}))$ is a spectral space. The result of (b) now follows by applying the LI construction to the isomorphic partially ordered sets $Zar^{\Psi}_{\bullet}(\mathbb{T})$ and $F^{\Psi}_{\bullet}$ and considering the explicit descriptions of the topologies $LI(Zar^{\Psi}_{\bullet}(\mathbb{T}))$ and $LI(F^{\Psi}_{\bullet})$. $\square$

The following result is an immediate consequence of Theorem 5.3.

Corollary 5.4. *Let $\Psi = (F, d)$ be a support for the tensor-triangulated category $(\mathbb{T}, \otimes, \mathbf{1})$ such that c_{Ψ} is a closure operator of finite type. Then, both $LI(F^{\Psi})$ and $LI(F^{\Psi}_{\bullet})$ are spectral spaces.*

We now describe explicitly the case of the initial support $\Psi_0 = (Zar(\mathbb{T}), Rad)$.

Corollary 5.5. *Let $(\mathbb{T}, \otimes, \mathbf{1})$ be a tensor-triangulated category and let $\mathrm{Spec}(\mathbb{T})^{\mathrm{inv}}$ be the spectrum of $\mathbb{T}$ equipped with the inverse topology. Then, the following spaces are homeomorphic:*

(1) the spectral space of radical thick tensor ideals in $(\mathbb{T}, \otimes, \mathbf{1})$ with topology generated by the open sets

$$\{\mathcal{K} \in Id(\mathbb{T}) \mid \mathcal{K} \text{ is radical and } a \notin \mathcal{K}\} \qquad \forall\, a \in \mathbb{T} \tag{5.5}$$

(2) the space $\mathcal{U}(\mathrm{Spec}(\mathbb{T})^{\mathrm{inv}})$ of open sets of $\mathrm{Spec}(\mathbb{T})^{\mathrm{inv}}$ with topology generated by taking the collection

$$\{V \in \mathcal{U}(\mathrm{Spec}(\mathbb{T})^{\mathrm{inv}}) \mid V \nsupseteq U\} \qquad \forall\, U \in \mathcal{U}(\mathrm{Spec}(\mathbb{T})^{\mathrm{inv}}) \tag{5.6}$$

to be open sets.

Proof. It is clear that the operation

$$\mathcal{K} \mapsto Rad(\mathcal{K}) := \{a \in \mathbb{T} \mid a^{\otimes n} \in \mathcal{K} \text{ for some } n \geq 1\} \qquad \forall\, \mathcal{K} \in Id(\mathbb{T}) \tag{5.7}$$

is a closure operator of finite type. Hence, the set of radical thick tensor ideals forms a spectral space with the lower interval topology generated by the collection in (5.5). As mentioned in Section 2, we know from [1, Theorem 4.10] that there is an order-preserving correspondence between radical thick tensor ideals and open subsets of $\mathrm{Spec}(\mathbb{T})^{\mathrm{inv}}$. Now, it is clear that the lower interval topology on the frame of open subsets of $\mathrm{Spec}(\mathbb{T})^{\mathrm{inv}}$ is generated by (5.6) and this proves the result. □

In Corollary 5.5, we have thus promoted the bijection in [1, Theorem 4.10] to a homeomorphism of spectral spaces. Finally, we conclude by presenting the counterpart of the topological nullstellensatz of Finocchiaro, Fonatana and Spirito [13] in the case of schemes that are not necessarily affine.

Corollary 5.6. *Let Z be a topologically noetherian scheme and let $D^{\mathrm{perf}}(Z)$ be the derived category of perfect complexes over Z equipped with the usual derived tensor product. Then, there is a homeomorphism between the spectral space of radical thick tensor ideals in $D^{\mathrm{perf}}(Z)$ and the collection $\mathcal{U}(Z^{\mathrm{inv}})$ of open subsets of Z^{inv} equipped with the topology generated by the opens*

$$\{V \in \mathcal{U}(Z^{\mathrm{inv}}) \mid V \not\supseteq U\} \qquad \forall\, U \in \mathcal{U}(Z^{\mathrm{inv}}) \tag{5.8}$$

Proof. From [1, Corollary 5.6] (see also [26, Theorem 3.15]), we know that there is a homeomorphism $Z \simeq \mathrm{Spec}(D^{\mathrm{perf}}(Z))$. This makes Z into a spectral space, and we can consider the inverse topology Z^{inv}. The result now follows from Theorem 5.5. □

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