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An integrable connection on the configuration space of a Riemann surface of positive genus

Une connexion intégrable sur l'espace de configuration d'une surface de Riemann de genre positif

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ABSTRACT

Let X be a Riemann surface of positive genus. Denote by $X^{(n)}$ the configuration space of n distinct points on X. We use the Betti–de Rham comparison isomorphism on $H^1(X^{(n)})$ to define an integrable connection on the trivial vector bundle on $X^{(n)}$ with fiber the universal algebra of the Lie algebra associated with the descending central series of π_1 of $X^{(n)}$. The construction is inspired by the Knizhnik–Zamolodchikov system in genus zero and its integrability follows from Riemann period relations.

RÉSUMÉ

Soit X une surface de Riemann de genre positif. Nous notons $X^{(n)}$ l'espace des configurations de n points distincts sur X. Nous utilisons l'isomorphisme de comparaison de Betti–de Rham sur $H^1(X^{(n)})$ pour définir une connexion intégrable sur le fibré vectoriel trivial sur $X^{(n)}$, dont la fibre est l'algèbre universelle de l'algèbre de Lie associée à la série centrale descendante du π_1 de $X^{(n)}$. La construction s'inspire du système de Knizhnik–Zamolodchikov en genre zéro ; l'intégrabilité résulte des relations de périodes de Riemann.

Fix $n \geq 1$. Let $\mathfrak{g}_0$ be the graded complex Lie algebra associated with the descending central series[1] of the classical pure braid group PB_n, i.e. the fundamental group of

$$\mathbb{C}^{(n)} := \{(z_1, \ldots, z_n) : z_i \in \mathbb{C},\ z_i \neq z_j \text{ for } i \neq j\}.$$

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[1] Let G be any group and $G_1 := G \supset \cdots \supset G_k \supset G_{k+1} := [G_k, G] \supset \cdots$ be its descending central series. By the graded complex Lie algebra associated with the descending central series of G, we mean the positively graded Lie algebra with degree k component $G_k \big/ G_{k+1} \otimes \mathbb{C}$ and Lie bracket induced by the commutator operator in G. See [4].

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It is generated by degree 1 elements $\{s_{ij} : 1 \le i, j \le n,\ i \ne j\}$, subject to the relations

$$\begin{aligned} s_{ij} &= s_{ji} \\ [s_{ij}, s_{kl}] &= 0 \qquad (i, j, k, l \text{ distinct}) \\ [s_{ij} + s_{ik}, s_{jk}] &= 0. \end{aligned} \tag{1}$$

The element $s_{ij} \in H_1(\mathbb{C}^{(n)}, \mathbb{C})$ (= degree 1 part of $\mathfrak{g}_0$) is the homology class of the j-th strand going positively around the i-th, while all other strands stay constant.

Let $\widehat{U\mathfrak{g}_0}$ be the completion of the universal algebra of $\mathfrak{g}_0$. Let $\mathcal{O}(\mathbb{C}^{(n)})$ (resp. $\Omega^{\cdot}(\mathbb{C}^{(n)})$) be the space of analytic functions (resp. complex of holomorphic differentials) on $\mathbb{C}^{(n)}$. The relations (1) assure that the Knizhnik–Zamolodchikov connection

$$\nabla_{KZ} :\ \widehat{U\mathfrak{g}_0} \otimes \mathcal{O}(\mathbb{C}^{(n)}) \longrightarrow\ \widehat{U\mathfrak{g}_0} \otimes \Omega^1(\mathbb{C}^{(n)})$$

defined by

$$\nabla_{KZ} f = df - \overbrace{\left(\sum_{i<j} \frac{1}{2\pi i} s_{ij} \otimes \frac{d(z_i - z_j)}{z_i - z_j}\right)}^{\lambda_0} f$$

is integrable. This connection and its more general variants are of great importance in conformal field theory, representation theory, and number theory.

The connection ∇_{KZ} is related to the comparison isomorphism

$$comp_{\mathbb{C}^{(n)}} : H^1(\mathbb{C}^{(n)}, \mathbb{C}) \longrightarrow H^1_{\mathrm{dR}}(\mathbb{C}^{(n)})$$

between the singular and (say) complex-valued smooth de Rham cohomologies in the following way: λ_0 is the image of $comp_{\mathbb{C}^{(n)}}$ under the map

$$H_1(\mathbb{C}^{(n)}, \mathbb{C}) \otimes H^1_{\mathrm{dR}}(\mathbb{C}^{(n)}) \longrightarrow\ \widehat{U\mathfrak{g}_0} \otimes \Omega^1(\mathbb{C}^{(n)})$$

defined by

$$s_{ij} \otimes [\frac{d(z_k - z_l)}{z_k - z_l}] \mapsto\ s_{ij} \otimes \frac{d(z_k - z_l)}{z_k - z_l} \qquad (i < j,\ k < l).$$

(Note that the s_{ij} with $i < j$ (resp. $[\frac{d(z_k - z_l)}{z_k - z_l}]$ with $k < l$) form a basis of $H_1(\mathbb{C}^{(n)}, \mathbb{C})$ (resp. $H^1_{\mathrm{dR}}(\mathbb{C}^{(n)})$.)

Now let $\overline{X}$ be a compact Riemann surface of genus $g > 0$, $S = \{Q_1, \ldots, Q_{|S|}\}$ a finite set of points in $\overline{X}$ (possibly empty), and $X = \overline{X} - S$. Let

$$X^{(n)} := \{(x_1, \ldots, x_n) : x_i \in X, x_i \ne x_j \text{ for } i \ne j\}.$$

Fix a base point $\underline{e} = (e_1, \ldots, e_n) \in X^{(n)}$ and let $\mathfrak{g}$ be the graded complex Lie algebra associated with the descending central series of $\pi_1(X^{(n)}, \underline{e})$. The goal of this note is to use the comparison isomorphism

$$comp_{X^{(n)}} : H^1(X^{(n)}, \mathbb{C}) \longrightarrow H^1_{\mathrm{dR}}(X^{(n)}) \tag{2}$$

to define an integrable connection ∇ on the trivial bundle $\widehat{U\mathfrak{g}} \otimes \mathcal{O}(X^{(n)})$.

1. Construction of the connection

We make three observations first.

(i) Since $g > 0$, the natural map

$$H^1_{\mathrm{dR}}(X^n) \longrightarrow H^1_{\mathrm{dR}}(X^{(n)}) \tag{3}$$

(induced by inclusion) is an isomorphism. Indeed, thanks to a theorem of Totaro [7, Theorem 1], one knows that the five-term exact sequence for the Leray spectral sequence for the constant sheaf $\mathbb{Z}$ and the inclusion $X^{(n)} \to X^n$ reads

$$0 \longrightarrow H^1(X^n, \mathbb{Z}) \xrightarrow{(3)} H^1(X^{(n)}, \mathbb{Z}) \longrightarrow \mathbb{Z}^{\{(a,b):1 \le a < b \le n\}} \xrightarrow{(*)} H^2(X^n, \mathbb{Z}) \longrightarrow H^2(X^{(n)}, \mathbb{Z}),$$

where the map $(*)$ sends 1 in the copy of $\mathbb{Z}$ corresponding to (a,b) $(a<b)$ to the class of the pullback of the diagonal $\Delta \subset X^2$ under the projection $p_{ab}: X^n \to X^2$ (defined in the obvious way). Since $g>0$, the class of Δ has a nonzero $H^1(X)\otimes H^1(X)$ Kunneth component (if $X=\overline{X}$ this is well known, and the noncompact case follows from the compact case in view of the functoriality of the class of the diagonal with respect to the inclusion $i: X^2 \to \overline{X}^2$ and injectivity of $i^*: H^2(\overline{X}^2) \to H^2(X^2)$ on $H^1\otimes H^1$ components). Thus the class of $p_{ab}^*(\Delta)$ has a nonzero $p_{ab}^*(H^1(X)\otimes H^1(X))$ component. Since every other $p_{a'b'}^*(\Delta)$ has a zero $p_{ab}^*(H^1(X)\otimes H^1(X))$ component, it follows that $(*)$ is injective.

(ii) Let $\Omega^1(\overline{X}\log S)$ be the space of differentials of the third kind on $\overline{X}$ with singularities in S. Then one has a distinguished isomorphism $\Omega^1(\overline{X}\log S) \cong F^1 H^1_{\mathrm{dR}}(X)$ given by $\omega \mapsto [\omega]$ (F being the Hodge filtration). (See [5, (3.2.13)(ii) and (3.2.14)], for instance.)

(iii) The cohomology $H^1_{\mathrm{dR}}(X)$ decomposes as an internal direct sum $F^1H^1_{\mathrm{dR}}(X) \oplus H^{0,1}$ (where $H^{0,1} \subset H^1_{\mathrm{dR}}(\overline{X}) \subset H^1_{\mathrm{dR}}(X)$). Indeed, this is simply the Hodge decomposition in $X=\overline{X}$ case. As for the noncompact case, strictness of morphisms of mixed Hodge structures with respect to the Hodge filtration implies that the two subspaces $F^1H^1_{\mathrm{dR}}(X)$ and $H^{0,1}$ of $H^1_{\mathrm{dR}}(X)$ have zero intersection, and by (ii) and the Riemann–Roch theorem $F^1H^1_{\mathrm{dR}}(X)$ has dimension $g+|S|-1$. The conclusion follows by a dimension count.

Let θ be the composition

$$H^1_{\mathrm{dR}}(X^{(n)}) \cong H^1_{\mathrm{dR}}(X^n) \overset{\text{Kunneth}}{\cong} H^1_{\mathrm{dR}}(X)^{\oplus n} \xrightarrow{(\dagger)} F^1H^1_{\mathrm{dR}}(X)^{\oplus n} \cong \Omega^1(\overline{X}\log S)^{\oplus n} \xrightarrow{(\ddagger)} \Omega^1(X^{(n)}),$$

where $(\dagger)$ is the sum of n copies of the natural projection, and $(\ddagger)$ is the sum of the pullbacks along projections $X^{(n)} \to X$. Note that the image of θ is contained in the subspace of closed forms, as it is contained in the subspace spanned by the pullbacks of holomorphic 1-forms on X along the aforementioned projections. Let ι be the composition of the inclusion $H_1(X^{(n)},\mathbb{C}) \subset \mathfrak{g}$ and the natural map $\mathfrak{g} \to \hat{U}\mathfrak{g}$. Denote by λ the image of the comparison isomorphism (2) under the map

$$\iota\otimes\theta: H_1(X^{(n)},\mathbb{C}) \otimes H^1_{\mathrm{dR}}(X^{(n)}) \longrightarrow \hat{U}\mathfrak{g} \otimes \Omega^1(X^{(n)}).$$

Define the connection

$$\nabla: \hat{U}\mathfrak{g} \otimes \mathcal{O}(X^{(n)}) \longrightarrow \hat{U}\mathfrak{g} \otimes \Omega^1(X^{(n)})$$

by

$$\nabla(f) = df - \lambda f.$$

(Note that λ multiplies with an element of $\hat{U}\mathfrak{g} \otimes \mathcal{O}(X^{(n)})$ through the multiplication in the universal algebra in the first factor and the algebra of differential forms in the second.)

2. Integrability

We prove that the connection ∇ is integrable. Since $\lambda \in \hat{U}\mathfrak{g} \otimes \Omega^1_{\mathrm{closed}}(X^{(n)})$, it is enough to show that

$$\lambda^2 \in \hat{U}\mathfrak{g} \otimes \Omega^2(X^{(n)})$$

is zero. For simplicity, denote $d = \dim H_1(X,\mathbb{Z})$ (thus $d=2g$ if $X=\overline{X}$ and $d=2g+|S|-1$ otherwise). Let $\{\alpha_i\}_{1\le i\le d}$ be a basis of $H_1(X,\mathbb{Z})$ such that for $i\le g$, α_i and α_{i+g} are (classes of) transversal loops around the i-th handle with $\alpha_i\cdot\alpha_{i+g}=1$ in $H_1(\overline{X},\mathbb{Z})$, and for $1\le i\le |S|-1$, α_{2g+i} is a simple loop going positively around the puncture Q_i, contractible in $X\cup\{Q_i\}$. Let $\{\omega_i\}_{1\le i\le d}$ be 1-forms such that $\{\omega_i\}_{i\le g}$ form a basis for holomorphic differentials on $\overline{X}$, $\omega_{g+i}=\overline{\omega_i}$ for $i\le g$, and ω_{2g+i} $(1\le i\le |S|-1)$ is a differential of the third kind with residual divisor $\frac{1}{2\pi i}(Q_i - Q_{|S|})$. With abuse of notation, we denote a differential form (resp. a loop) and its cohomology (resp. homology) class by the same symbol. Write the comparison isomorphism $comp_X \in H_1(X,\mathbb{C}) \otimes H^1_{\mathrm{dR}}(X)$ as $\sum_{i,j} \pi_{ij}\alpha_i\otimes\omega_j$. (Here and in all the sums in the sequel, unless otherwise indicated the indices run over all their possible values.) The matrix $(\pi_{ij})_{ij}$ (with ij-entry π_{ij}) is the inverse of the matrix whose ij-entry is $\int_{\alpha_j}\omega_i$, and is of the form

$$\begin{pmatrix} P^{-1} & 0 \\ & I_{|S|-1} \end{pmatrix},$$

where P is the matrix of periods of $\overline{X}$ with respect to the ω_i and α_j, and I denotes the identity matrix.

Let $\{\alpha_i^{(k)}\}_{\substack{1\le k\le n\\ 1\le i\le d}}$ be pure braids in X with n strands based at $\underline{e}$ (= loops in $X^{(n)}$ based at $\underline{e}$) such that the following hold:

(i) the only nonconstant strand in $\alpha_i^{(k)}$ is the one based at e_k;
(ii) for $i\le g$, the strands of $\alpha_i^{(k)}$ and $\alpha_{i+g}^{(k)}$ based at e_k are transversal loops around the i-th handle;
(iii) for $1\le i\le |S|-1$, the strand of $\alpha_{2g+i}^{(k)}$ based at e_k is a simple loop going around Q_i;
(iv) the k-th projection $X^{(n)}\to X$ sends $\alpha_i^{(k)}$ to α_i in homology.

Let $\omega_i^{(k)}$ be the pullback of ω_i under the k-th projection $X^{(n)}\to X$. Then $\{\alpha_i^{(k)}\}$ and $\{\omega_i^{(k)}\}$ are bases of $H_1(X^{(n)},\mathbb{C})$ and $H^1_{\mathrm{dR}}(X^{(n)})$, and

$$comp_{X^{(n)}}=\sum_{i,j,k}\pi_{ij}\alpha_i^{(k)}\otimes\omega_j^{(k)}.$$

Let $\mathcal{F}=\{1,\dots,d\}-\{g+1,\dots,2g\}$. Then

$$\lambda=\sum_{\substack{j\in\mathcal{F}\\ i,k}}\pi_{ij}\alpha_i^{(k)}\otimes\omega_j^{(k)}.$$

We have

$$\lambda^2=\sum_{\substack{j,j'\in\mathcal{F};\ i,i'\\ k,k'}}\pi_{ij}\pi_{i'j'}\alpha_i^{(k)}\alpha_{i'}^{(k')}\otimes\omega_j^{(k)}\wedge\omega_{j'}^{(k')}$$
$$=\sum_{\substack{j,j'\in\mathcal{F};\ i,i'\\ k<k'}}\pi_{ij}\pi_{i'j'}[\alpha_i^{(k)},\alpha_{i'}^{(k')}]\otimes\omega_j^{(k)}\wedge\omega_{j'}^{(k')}.$$

Simple calculations using Bellingeri's description of $\pi_1(X^{(n)})$ given in [1, Theorems 5.1 and 5.2] (also see [2] for a misprint corrected) show that in $\mathfrak{g}$, for arbitrary distinct k,k', one has $[\alpha_i^{(k)},\alpha_{i'}^{(k')}]=0$ unless $i,i'\le 2g$ and $|i-i'|=g$ (i.e. unless $\alpha_i^{(k)},\alpha_{i'}^{(k')}$ correspond to transversal loops going around the same handle), and moreover that

$$[\alpha_i^{(k)},\alpha_{i+g}^{(k')}]\qquad(i\le g)\tag{4}$$

only depends on the *set* $\{k,k'\}$. (Note that one can take $\alpha_i^{(k)}\in\pi_1(X^{(n)})$ to be Bellingeri's $A_{2i-1,d+k}$, $A_{2(i-g),d+k}$, or $A_{i,d+k}$ depending on whether $i\le g$, $g<i\le 2g$, or $2g<i\le d$ respectively.) Denoting (4) by $s_{kk'}$ $(=s_{k'k})$, we thus have

$$\lambda^2=\sum_{\substack{j,j'\le g\\ k<k'}}\left(\sum_{i\le g}\pi_{ij}\pi_{i+g,j'}-\pi_{i+g,j}\pi_{ij'}\right)s_{kk'}\otimes\omega_j^{(k)}\wedge\omega_{j'}^{(k')},$$

which is zero by Riemann period relations.

Remarks. (1) In the case $X=\overline{X}$, one can replace $\mathfrak{g}$ by the Lie algebra $\mathfrak{l}$ of the nilpotent completion of $\pi_1(X^{(n)})$. Thanks to a theorem of Bezrukavnikov [3] one knows similar relations to the ones in $\mathfrak{g}$ used above to prove integrability also hold in $\mathfrak{l}$.

(2) It would be interesting to relate the connection defined here with the one defined by Enriquez in [6] on configuration spaces of compact Riemann surfaces.

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