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Partial differential equations

On the local existence for the Euler equations with free boundary for compressible and incompressible fluids

Sur l'existence locale de solutions des équations d'Euler pour les fluides compressibles et incompressibles, avec frontière libre

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ABSTRACT

We consider the free boundary compressible and incompressible Euler equations with surface tension. In both cases, we provide a priori estimates for the local existence with the initial velocity in H^3, with the H^3 condition on the density in the compressible case. An additional condition is required on the free boundary. Compared to the existing literature, both results lower the regularity of initial data for the Lagrangian Euler equation with surface tension.

RÉSUMÉ

Nous considérons les équations d'Euler compressibles et incompressibles avec frontière libre et tension de surface. Dans les deux cas, nous fournissons des estimations *a priori* pour l'existence de solutions locales avec vitesse initiale dans H^3 et la condition H^3 sur la densité dans le cas compressible. Une condition supplémentaire est nécessaire sur la frontière libre. Par comparaison avec la littérature, les deux résultats abaissent la régularité des données initiales pour les équations d'Euler en coordonnées lagrangiennes, avec tension de surface.

1. Introduction

In this note, we address the water wave problem, which has been studied extensively. Our setting is rather general: we consider the Euler equations with a free surface and allow initial data to have nonzero curl, i.e. the initial data is rotational. The domain is assumed to be of finite depth, but the results can be easily adapted to the infinite depth as well.

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We consider both the compressible and incompressible cases; for the compressible Euler equations, we assume that the density is bounded from below, i.e. we consider the case of a liquid.

Our aim in this note is to announce two recent results on the local existence with non-zero surface tension. In both main theorems (the compressible and incompressible cases, respectively), we assume that the initial data is of H^3 regularity in the interior. This lowers the regularity of existing results for the compressible equations. In the incompressible setting, our result lowers the known regularity in Lagrangian coordinates, albeit it does not improve over what has been obtained in Eulerian coordinates.

The history of the Euler equations with free interface is rich—we refer the reader to [2,4,5] for a more complete account. We only mention a few important works dealing with non-zero surface tension. The first general well-posedness result for the incompressible free-boundary Euler equations with surface tension is [2], followed by [9,10] and [3]. Earlier, well-posedness had been established under the assumption that the initial vorticity vanishes on the boundary [8]. For the compressible equations with surface tension, the first result was [1].

In the first part of the note, we address the compressible Euler equations, while in the second part we treat the incompressible version.

2. Compressible case

In the Lagrangian setting, the free-surface compressible Euler equations read

$$R\partial_t v^\alpha + a^{\mu\alpha}\partial_\mu q = 0 \quad \text{in } [0,T)\times\Omega, \tag{1a}$$

$$\partial_t R + Ra^{\mu\alpha}\partial_\mu v_\alpha = 0 \quad \text{in } [0,T)\times\Omega, \tag{1b}$$

$$\partial_t a^{\alpha\beta} + a^{\alpha\gamma}\partial_\mu v_\gamma a^{\mu\beta} = 0 \quad \text{in } [0,T)\times\Omega, \tag{1c}$$

$$q = q(R) \quad \text{in } [0,T)\times\Omega, \tag{1d}$$

$$a^{\mu\alpha}N_\mu q + \sigma|a^T N|\Delta_g \eta^\alpha = 0 \quad \text{on } [0,T)\times\Gamma_1, \tag{1e}$$

$$v^\mu N_\mu = 0 \quad \text{on } [0,T)\times\Gamma_0, \tag{1f}$$

$$\eta(0,\cdot) = \mathrm{id}, \quad R(0,\cdot) = \varrho_0, \quad v(0,\cdot) = v_0. \tag{1g}$$

Above, v, R, and q denote the Lagrangian velocity, density, and the pressure, respectively; N is the unit outer normal to $\partial\Omega$, a is the inverse of $\nabla\eta$, $\sigma = \text{constant} > 0$ is the coefficient of surface tension, and Δ_g is the Laplacian of the metric g_{ij} induced on $\partial\Omega(t)$ by the embedding η, i.e. $g_{ij} = \partial_i\eta\cdot\partial_j\eta = \partial_i\eta^\mu\partial_j\eta_\mu$.

We consider the domain $\Omega_0 \equiv \Omega = \mathbb{T}^2\times(0,1)$. We note that using the straightening map in [7, Remark 4.2], it is easy to modify the approach to consider a general curved domain $\Omega' = \mathbb{R}^2\times(0,h(x_1,x_2))$. Applying the change of variable in [7], we get

$$R\partial_t v^\alpha + a^{\mu\alpha}b^\beta{}_\mu\partial_\beta q = 0$$

instead of (1a); here b is the cofactor inverse matrix of the straightening map. The other equations in the system (1a)–(1g) are modified similarly. The methods outlined here then easily carry over for the new system as well, provided Ω' is at least H^5 regular.

Denoting the coordinates on Ω by (x^1,x^2,x^3), we have $\Gamma_1 = \mathbb{T}^2\times\{x^3=1\}$ as the free boundary and $\Gamma_0 = \mathbb{T}^2\times\{x^3=0\}$ as the stationary one. On the pressure function, we assume

$$\left(\frac{q(R)}{R}\right)' \geq A_q = \text{constant} > 0, \tag{2}$$

which is satisfied by a large class of equations of state.

We denote by Π the canonical projection, on $\eta(\Gamma_1)$, from the tangent bundle of $\eta(\Omega)$ to its normal bundle, which is given by $\Pi^\alpha_\beta = \delta^\alpha_\beta - g^{kl}\partial_k\eta^\alpha\partial_l\eta_\beta$. We recall that initial data for (2) is required to satisfy compatibility conditions (cf. [5]). The following is the main result in the compressible case.

Theorem 2.1. *Let v_0 be a smooth vector field on Ω and ϱ_0 a smooth positive function on Ω bounded away from zero from below. Assume that v_0 and ϱ_0 satisfy the compatibility conditions. Let $q\colon (0,\infty)\to(0,\infty)$ be a smooth function satisfying (2) in a neighborhood of ϱ_0. Then, there exist a $T_* > 0$ and a constant C_*, depending only on $\sigma > 0$, $\|v_0\|_3$, $\|v_0\|_{3,\Gamma_1}$, $\|\varrho_0\|_3$, $\|\varrho_0\|_{3,\Gamma_1}$, and $\|(\Delta\operatorname{div} v_0)\restriction\Gamma_1\|_{-1,\Gamma_1}$ such that any smooth solution (v,R) to (1), with initial condition (v_0,ϱ_0) and defined on the time interval $[0,T_*)$, satisfies*

$$\begin{aligned}\mathscr{N}(t) = {} & \|v\|_3^2 + \|\partial_t v\|_2^2 + \|\partial_t^2 v\|_1^2 + \|\partial_t^3 v\|_0^2 + \|R\|_3^2 + \|\partial_t R\|_2^2 \\ & + \|\partial_t^2 R\|_1^2 + \|\partial_t^3 R\|_0^2 + \|\Pi\overline{\partial}\partial_t^2 v\|_{0,\Gamma_1}^2 + \|\Pi\overline{\partial}^2\partial_t v\|_{0,\Gamma_1}^2 \leq C_*,\end{aligned} \tag{3}$$

where $\overline{\partial}$ stands for the tangential derivative.

Next, we describe the strategy of the proof and discuss the treatment of difficult terms. The main part is based on the energy estimate for three time derivatives,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} R(0)\partial_t^3 v^{\beta}\partial_t^3 v_{\beta} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\frac{R(0)}{R}\bar{q}'(R)(\partial_t^3 R)^2 + \int_{\Gamma_1}\partial_t^3(Ja^{\alpha\beta}q)\partial_t^3 v_{\beta}N_{\alpha} \\
&\quad = -\int_{\Omega}\frac{R(0)}{R}\left(\partial_t^3(Ra^{\alpha\beta}\partial_{\alpha}v_{\beta}) - Ra^{\alpha\beta}\partial_t^3\partial_{\alpha}v_{\beta}\right)\partial_t^3\left(\frac{q}{R}\right) \\
&\qquad + \int_{\Omega} R(0)\left(\partial_t^3\left(a^{\alpha\beta}\frac{q}{R}\right) - a^{\alpha\beta}\partial_t^3\left(\frac{q}{R}\right)\right)\partial_t^3\partial_{\alpha}v_{\beta} \\
&\qquad - 3\int_{\Omega} R(0)\frac{\bar{q}''(R)}{R}\partial_t^4 R\partial_t^2 R\partial_t R - \int_{\Omega} R(0)\frac{\bar{q}'''(R)}{R}\partial_t^4 R(\partial_t R)^3 \\
&\qquad + \frac{1}{2}\int_{\Omega} R(0)\partial_t\left(\frac{\bar{q}'(R)}{R}\right)(\partial_t^3 R)^2
\end{aligned} \tag{4}$$

where

$$\bar{q}(R) = \frac{q(R)}{R}$$

and $J = \det(\nabla\eta)$. The first two terms on the left provide a coercive term $\|\partial_t^3 v\|_0^2 + \|\partial_t^3 R\|_0^2$. In the third term $I_1 = \int_{\Gamma_1}\partial_t^3(Ja^{\alpha\beta}q)\partial_t^3 v_{\beta}N_{\alpha}$ on the left side of (4), we use (1e) for the pressure, leading to $I_1 = \int_0^t\int_{\Gamma_1}\partial_t^3(\sqrt{g}\Delta_g\eta^{\alpha})\partial_t^3 v_{\alpha}$, where we set $\sigma = 1$ for simplicity. This may be rewritten as

$$\begin{aligned}
I_1 &= -\int_0^t\int_{\Gamma_1}\sqrt{g}g^{ij}\Pi^{\alpha}_{\mu}\partial_j\partial_t^2 v^{\mu}\partial_i\partial_t^3 v_{\alpha} - \int_0^t\int_{\Gamma_1}\partial_i(\sqrt{g}g^{ij}\Pi^{\alpha}_{\mu})\partial_j\partial_t^2 v^{\mu}\partial_t^3 v_{\alpha} \\
&= -\frac{1}{2}\int_{\Gamma_1}\sqrt{g}g^{ij}\Pi^{\alpha}_{\mu}\partial_j\partial_t^2 v^{\mu}\partial_i\partial_t^2 v_{\alpha} + \frac{1}{2}\int_0^t\int_{\Gamma_1}\partial_t\left(\sqrt{g}g^{ij}\Pi^{\alpha}_{\mu}\right)\partial_j\partial_t^2 v^{\mu}\partial_i\partial_t^2 v_{\alpha} \\
&\quad - \int_0^t\int_{\Gamma_1}\partial_i(\sqrt{g}g^{ij}\Pi^{\alpha}_{\mu})\partial_j\partial_t^2 v^{\mu}\partial_t^3 v_{\alpha} + \frac{1}{2}\int_{\Gamma_1}\sqrt{g}g^{ij}\Pi^{\alpha}_{\mu}\partial_j\partial_t^2 v^{\mu}\partial_i\partial_t^2 v_{\alpha}|_0 \\
&= I_{11} + I_{12} + I_{13} + I_{14}.
\end{aligned} \tag{5}$$

The integral I_{11} leads to a coercive term $\frac{1}{4}\|\Pi\bar{\partial}\partial_t^2 v\|_{0,\Gamma_1}^2$. The highest-order term results when ∂_i falls on $\sqrt{g}g^{ij}$, which gives the integral

$$I_{131} = -\int_0^t\int_{\Gamma_1}\partial_i(\sqrt{g}g^{ij})\hat{n}^{\alpha}\hat{n}_{\mu}\partial_j\partial_t^2 v^{\mu}\partial_t^3 v_{\alpha},$$

for which we need to explore its special structure. Note that we used the identity $\Pi^{\alpha}_{\mu} = \hat{n}^{\alpha}\hat{n}_{\mu}$, $\hat{n}$ being the unit outer normal to the moving boundary. It turns out that this integral cancels a part of the integral resulting from I_{14} (cf. [5] for complete details).

It remains to discuss the treatment of the right side of (4). Denote the terms on the right side of (4) by $\mathcal{J}_1$–$\mathcal{J}_5$. All the integrals resulting from $\mathcal{J}_1$, $\mathcal{J}_3$, $\mathcal{J}_4$, and $\mathcal{J}_5$ can be estimated using integration by parts in time and space and employing the Hölder and Sobolev inequalities. On the other hand, when expanding $\mathcal{J}_2$, there is a tricky term $T = \int_0^t\int_{\Omega}\partial_t^3 A^{\mu\alpha}\partial_t^3\partial_{\mu}v_{\alpha}q$ where $A = Ja$, which cannot be treated in this way, and special cancellations are needed to control it. We namely claim

$$|T| \lesssim \epsilon\mathcal{N} + \mathscr{P}_0 + \mathscr{P}\int_0^t\mathscr{P}, \tag{6}$$

where $\mathscr{P}$ denotes a generic polynomial in $\mathcal{N}$, $\mathscr{P}_0$ a generic polynomial of the norms of the initial data.

Here we sketch the main ideas used in proving (6). Using the formula for the cofactor matrix $A = Ja$, we rewrite

$$\begin{aligned} T &= \int_0^t \int_\Omega q\epsilon^{\alpha\lambda\tau}\partial_2\partial_t^2 v_\lambda \partial_3\eta_\tau \partial_1\partial_t^3 v_\alpha + \int_0^t \int_\Omega q\epsilon^{\alpha\lambda\tau}\partial_2\eta_\lambda \partial_3\partial_t^2 v_\tau \partial_1\partial_t^3 v_\alpha \\ &\quad - \int_0^t \int_\Omega q\epsilon^{\alpha\lambda\tau}\partial_1\partial_t^2 v_\lambda \partial_3\eta_\tau \partial_2\partial_t^3 v_\alpha - \int_0^t \int_\Omega q\epsilon^{\alpha\lambda\tau}\partial_1\eta_\lambda \partial_3\partial_t^2 v_\tau \partial_2\partial_t^3 v_\alpha \\ &\quad + \int_0^t \int_\Omega q\epsilon^{\alpha\lambda\tau}\partial_1\partial_t^2 v_\lambda \partial_2\eta_\tau \partial_3\partial_t^3 v_\alpha + \int_0^t \int_\Omega q\epsilon^{\alpha\lambda\tau}\partial_1\eta_\lambda \partial_2\partial_t^2 v_\tau \partial_3\partial_t^3 v_\alpha + L \\ &= T_1 + \cdots + T_6 + L \end{aligned} \tag{7}$$

where $A^{1\alpha} = \epsilon^{\alpha\lambda\tau}\partial_2\eta_\lambda\partial_3\eta_\tau$, $A^{2\alpha} = -\epsilon^{\alpha\lambda\tau}\partial_1\eta_\lambda\partial_3\eta_\tau$, $A^{3\alpha} = \epsilon^{\alpha\lambda\tau}\partial_1\eta_\lambda\partial_2\eta_\tau$, and L stands for the lower-order terms. Also, $\varepsilon^{\alpha\beta\gamma}$ is the totally anti-symmetric symbol with $\varepsilon^{123} = 1$. To explore the cancellation, we group the terms in (7) as $T_1 + T_3$, $T_2 + T_5$, and $T_4 + T_6$. All three pairs are treated by integrating by parts in time. When writing out $T_1 + T_3$, the highest-order terms are

$$\int_0^t \int_\Omega q\epsilon^{\alpha\lambda\tau}\partial_2\partial_t^2 v_\lambda \partial_3\eta_\tau \partial_1\partial_t^3 v_\alpha + \int_0^t \int_\Omega q\epsilon^{\alpha\lambda\tau}\partial_1\partial_t^3 v_\lambda \partial_3\eta_\tau \partial_2\partial_t^2 v_\alpha$$

and the leading terms cancel by relabeling the indices $\alpha \leftrightarrow \lambda$ in the second integral. The same cancellation occurs also in the other groupings $T_2 + T_5$ and $T_4 + T_6$. However, there we integrate the derivatives ∂_2 and ∂_3; the last sum produces the boundary term $\int_{\Gamma_1} q\partial_t^2 v_2 \partial_2\partial_t^2 v_3$. To bound this term, we use the projection identity $\Pi^\alpha_\beta = \delta^\alpha_\beta - g^{kl}\partial_k\eta^\alpha\partial_l\eta_\beta$, which allows us to relate $\Pi\overline{\partial}\partial_t^2 v$ and $\overline{\partial}\partial_t^2 v^3$ and write

$$\int_{\Gamma_1} q\partial_t^2 v_2 \partial_2\partial_t^2 v_3 = \int_{\Gamma_1} q\partial_t^2 v_2 (\Pi^3_\lambda \partial_2\partial_t^2 v^\lambda + g^{kl}\partial_k\eta_3\partial_l\eta_\lambda\partial_2\partial_t^2 v^\lambda).$$

The last expression can then be controlled using the coercive terms and the trace inequality. Collecting all the inequalities leads to

$$\|\partial_t^3 v\|_0^2 + \|\partial_t^3 R\|_0^2 + \|\Pi\overline{\partial}\partial_t^2 v\|_{0,\Gamma_1}^2 \leq \widetilde{\epsilon}\mathscr{N} + \mathscr{P}_0 + \mathscr{P}\int_0^t \mathscr{P}.$$

The energy estimates for two time derivatives and one time derivative are similar. For instance, using analogous methods, we obtain

$$\|\overline{\partial}\partial_t^2 v\|_0^2 + \|\overline{\partial}\partial_t^2 R\|_0^2 + \|\Pi\overline{\partial}^2\partial_t v\|_{0,\Gamma_1}^2 \leq \widetilde{\epsilon}\mathscr{N} + \mathscr{P}_0 + \mathscr{P}\int_0^t \mathscr{P} \tag{8}$$

and

$$\|\overline{\partial}^2\partial_t v\|_0^2 + \|\overline{\partial}^2\partial_t R\|_0^2 + \|\Pi\overline{\partial}^3 v\|_{0,\Gamma_1}^2 \leq \widetilde{\epsilon}\mathscr{N} + \mathscr{P}_0 + \mathscr{P}\int_0^t \mathscr{P}. \tag{9}$$

We emphasize however, that unlike [1], we cannot continue and perform the same estimate for $\|\overline{\partial}^3 v\|_0^2 + \|\overline{\partial}^3 R\|_0^2$. The reason is that the analog of the term (7) cannot be treated the same way due to lack of time derivatives.

Instead, we obtain the full control of $\mathscr{N}$ by using a new Cauchy invariance property for the compressible Euler equations, stated next. Note that the Cauchy invariance provides a 3D analog of the 2D vorticity preservation.

Theorem 2.2. *(Compressible Cauchy Invariance Formula) Let* (v, R) *be a smooth solution to* (1) *defined on* $[0, T)$*. Then*

$$\varepsilon^{\alpha\beta\gamma}\partial_\beta v^\mu \partial_\gamma\eta_\mu = \omega_0^\alpha + \int_0^t \varepsilon^{\alpha\beta\gamma} a^{\lambda\mu}\partial_\lambda q \partial_\gamma\eta_\mu \frac{\partial_\beta R}{R^2}, \tag{10}$$

for $0 \leq t < T$*, where* ω_0 *is the vorticity at time zero.*

Here, we provide a sketch of the proof (cf. [5] for complete details). First, we have

$$\begin{aligned}\partial_t(\varepsilon^{\alpha\beta\gamma}\partial_\beta v^\mu\partial_\gamma\eta_\mu) &= \varepsilon^{\alpha\beta\gamma}\partial_\beta v^\mu\partial_\gamma v_\mu + \varepsilon^{\alpha\beta\gamma}\partial_\beta\partial_t v^\mu\partial_\gamma\eta_\mu \\ &= -\frac{1}{R}\varepsilon^{\alpha\beta\gamma}\partial_\beta(a^{\lambda\mu}\partial_\lambda q)\partial_\gamma\eta_\mu + \frac{1}{R^2}\varepsilon^{\alpha\beta\gamma}a^{\lambda\mu}\partial_\lambda q\partial_\beta R\partial_\gamma\eta_\mu,\end{aligned}$$

by the anti-symmetry of $\varepsilon^{\alpha\beta\gamma}$ and (1a). Since $a = (\nabla\eta)^{-1}$, we get $\partial_\beta(a^{\lambda\mu}\partial_\gamma\eta_\mu) = \partial_\beta a^{\lambda\mu}\partial_\gamma\eta_\mu + a^{\lambda\mu}\partial_\gamma\partial_\beta\eta_\mu = 0$, and after some simple algebra, $\partial_t(\varepsilon^{\alpha\beta\gamma}\partial_\beta v^\mu\partial_\gamma\eta_\mu) = \frac{1}{R^2}\varepsilon^{\alpha\beta\gamma}a^{\lambda\mu}\partial_\lambda q\partial_\beta R\partial_\gamma\eta_\mu$. The formula (10) then follows by integrating in time.

The rest of the proof of Theorem 2.1 may now be performed similarly to [6]. Namely, control of the curl is provided by the Cauchy invariance formula (10). Then the divergence of v and its time derivatives are obtained from (1b) in an inductive fashion, using the fact that the third time derivatives are independently estimated from (4). Finally, the boundary integral is estimated using (5), (8), and (9). The exception is the missing estimate for v_3, which is obtained from the equation (1e) (cf. [5] for details).

Collecting all the estimates leads to

$$\mathscr{N}(t) \le C_0 P(\mathscr{N}(0)) + P(\mathscr{N}(t))\int_0^t P(\mathscr{N}(s))\,\mathrm{d}s,$$

where P is a polynomial, and the rest follows by a standard Gronwall argument.

3. Incompressible case

In this section, we describe the a priori estimates for the local existence for the incompressible Euler equations. We address the equations in the Lagrangian coordinates, where they take the form

$$\partial_t v^\alpha + a^{\mu\alpha}\partial_\mu q = 0 \quad \text{in } [0,T)\times\Omega, \tag{11a}$$

$$a^{\alpha\beta}\partial_\alpha v_\beta = 0 \quad \text{in } [0,T)\times\Omega, \tag{11b}$$

$$\partial_t a^{\alpha\beta} + a^{\alpha\gamma}\partial_\mu v_\gamma a^{\mu\beta} = 0 \quad \text{in } [0,T)\times\Omega, \tag{11c}$$

$$a^{\mu\alpha}N_\mu q + \sigma|a^T N|\Delta_g\eta^\alpha = 0 \quad \text{on } [0,T)\times\Gamma_1, \tag{11d}$$

$$v^\mu N_\mu = 0 \quad \text{on } [0,T)\times\Gamma_0, \tag{11e}$$

$$\eta(0,\cdot) = \mathrm{id}, \quad v(0,\cdot) = v_0. \tag{11f}$$

All the quantities are as in the compressible case, except for q, which is now a Lagrange multiplier enforcing the incompressibility constraint. Then we have the following a priori estimates supporting the local existence with the velocity H^3 in the interior, with a condition on the trace at the free boundary.

Theorem 3.1. *Assume that $\sigma > 0$ in (11). Let v_0 be a smooth divergence-free vector field on Ω. Then there exist $T_* > 0$ and a constant C_0, depending only on $\|v_0\|_3$, $\|v_0\|_{4,\Gamma_1}$, and $\sigma > 0$, such that any smooth solution (v, q) to (11) with initial condition v_0 and defined on the time interval $[0, T_*)$, satisfies*

$$\|v\|_3 + \|\partial_t v\|_{2.5} + \|\partial_t^2 v\|_{1.5} + \|\partial_t^3 v\|_0 + \|q\|_3 + \|\partial_t q\|_2 + \|\partial_t^2 q\|_1 \le C_0. \tag{12}$$

Observe that the Sobolev exponents in (3) and (12) differ. The proof of Theorem 3.1 is different than the one of Theorem 2.1; however, the main steps also rely on the energy inequality for the third time derivative of the velocity

$$\frac{1}{2}\|\partial_t^3 v\|_0^2 = \frac{1}{2}\|\partial_t^3 v(0)\|_0^2 - \int_0^t\int_\Omega \partial_t^3\partial_\mu(a^{\mu\alpha}q)\partial_t^3 v_\alpha.$$

The main difference with the compressible case are in the treatment of the pressure and in bounding the boundary integral

$$-\int_0^t\int_{\Gamma_1}\sqrt{g}(g^{ij}g^{kl} - g^{lj}g^{ik})\partial_j\eta^\alpha\partial_k\eta_\lambda\partial_t^2\partial_l v^\lambda\partial_t^3\partial_i v_\alpha. \tag{13}$$

For the pressure, the main inequalities are summarized in the following statement.

Lemma 3.2. *We have the estimates*

$$
\begin{aligned}
\|q\|_3 &\le C\|\nabla v\|_2\|v\|_2 + C\|\partial_t v\|_{1.5,\Gamma_1} + C,\\
\|\partial_t q\|_2 &\le C\|\nabla v\|_{1.5+\delta}(\|q\|_{2.5} + \|\partial_t v\|_{1.5}) + C(\|\nabla v\|_{1.5}\|\nabla v\|_{L^\infty} + \|\nabla\partial_t v\|_{1.5})\|v\|_{1.5+\delta}\\
&\quad + C\|\partial_t^2 v\|_{1,\Gamma_1} + C\|v\|_{2.5} + C\|\partial_t v\|_{2.5},\\
\|\partial_t^2 q\|_1 &\le C(\|v\|_{1.5}\|\nabla v\|_{L^\infty} + \|\partial_t v\|_{1.5})(\|q\|_2 + \|\partial_t v\|_1) + C\|\nabla v\|_{L^\infty}(\|\partial_t q\|_1 + \|\partial_t^2 v\|_0)\\
&\quad + C(\|v\|_{1.5}\|\nabla v\|_{L^\infty}^2 + \|\partial_t v\|_{1.5}\|\nabla v\|_{L^\infty} + \|\partial_t^2 v\|_{1.5})\|v\|_1 + C\|\partial_t^3 v\|_0\\
&\quad + C(\|\partial_t v\|_{2.5} + \|v\|_3\|\partial_t q\|_1) + C(1 + \|v\|_3^2 + \|\partial_t v\|_2)(\|\partial_t v\|_2 + \|v\|_3^2),
\end{aligned}
$$

for $t \in (0, T)$ and $\delta > 0$ a small number.

The lemma is obtained by solving the elliptic boundary value problems for q, $\partial_t q$, and $\partial_t^2 q$. For q and q_t, we consider the Neumann problem, while for the second derivative, we use the Dirichlet problem, together with an estimate

$$
\|\partial_t^2 q\|_{0,\Gamma_1} \le \epsilon\|\partial_t^2 q\|_1 + C(\|\partial_t v\|_{2.5} + \|v\|_3\|\partial_t q\|_1) + C(1 + \|v\|_3^2 + \|\partial_t v\|_2)(\|\partial_t v\|_2 + \|v\|_3^2)
$$

(cf. [4] for details). The treatment for the boundary integral (13) is involved and it relies on its determinant structure (cf. [2]). The rest of the proof is obtained by the H^1 estimate for $\partial_t^2 v$. However, in order to control $\|v\|_3$, we use the additional boundary regularity provided by the surface tension, together with the div-curl estimates provided by the Cauchy invariance [7].

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