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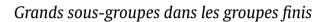# C. R. Acad. Sci. Paris, Ser. I

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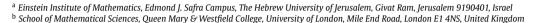Group theory

Large subgroups in finite groups

Grands sous-groupes dans les groupes finis

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A R T I C L E   I N F O

A B S T R A C T

Following Isaacs (see [6, p. 94]), we call a normal subgroup N of a finite group G *large*, if $\mathbf{C}_G(N) \leq N$, so that N has bounded index in G. Our principal aim here is to establish some general results for systematically producing large subgroups in finite groups (see Theorems A and C). We also consider the more specialised problems of finding large (non-abelian) nilpotent as well as abelian subgroups in soluble groups.

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R É S U M É

Suivant la terminologie introduite par Isaacs (voir [6], p. 94), nous disons qu'un sous-groupe distingué N d'un groupe fini G est *grand* si $\mathbf{C}_G(N) \leq N$, de sorte que N est d'indice borné dans G. Notre but principal est d'établir des résultats permettant de produire de façon systématique des grands sous-groupes dans les groupes finis (voir les théorèmes A et C). Nous considérons également les problèmes plus particuliers qui se posent pour trouver de grands sous-groupes nilpotents (non commutatifs) ainsi que de grands sous-groupes commutatifs dans les groupes résolubles.

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1. Introduction

Let G be a finite group and suppose that N is a normal subgroup of G such that $\mathbf{C}_G(N) \leq N$. Then we call N a *large* subgroup of G. The motivation for naming such a subgroup "large" stems from the observation that if $N \trianglelefteq G$, then the factor group $G/\mathbf{C}_G(N)$ is isomorphically embedded in $\mathbf{Aut}(N)$, and so $|G : \mathbf{C}_G(N)| \leq |\mathbf{Aut}(N)|$. Thus, if N is large, it follows that $|G : N| \leq |\mathbf{Aut}(N)|$, so that N has bounded index in G.

There are a number of examples of large subgroups in the literature. For instance, if G is soluble, then the (standard) Fitting subgroup $\mathbf{F}(G)$ of G is large. Also, if G is π-separable and $\mathbf{O}_{\pi'}(G) = 1$, then $\mathbf{O}_\pi(G)$ is large (this result is known as the Hall–Higman Lemma 1.2.3). Moreover, if G is any finite group, then the generalised Fitting subgroup $\mathbf{F}^*(G)$ of G is large.

We briefly recall the definition of Bender's subgroup $\mathbf{F}^*(G)$. Call a group G *quasi-simple*, if G is perfect and $G/\mathbf{Z}(G)$ is simple. A subnormal quasi-simple subgroup of an arbitrary group G is called a *component* of G and the *layer* of G, denoted by $\mathbf{E}(G)$, is the product of its components. It is known that, if A and B are components of G, then either $A = B$ or $[A, B] = 1$. The *generalised Fitting subgroup* of G, denoted by $\mathbf{F}^*(G)$, is the product of $\mathbf{E}(G)$ and $\mathbf{F}(G)$.

We shall reserve fraktur symbols for classes of finite groups; in particular, $\mathfrak{S}$ will denote the class of finite soluble groups, $\mathfrak{N}$ the class of finite nilpotent groups, and $\mathfrak{A}$ the class of finite abelian groups.

A formation $\mathfrak{F}$ with the property that $G/\Phi(G) \in \mathfrak{F}$ for G finite implies $G \in \mathfrak{F}$ is called saturated. We recall that if $\mathfrak{F}$ is a formation, then $\mathfrak{F}$ is called *solubly saturated* provided that $G \in \mathfrak{F}$ whenever $G/\Phi(G_{\mathfrak{S}}) \in \mathfrak{F}$, where $G_{\mathfrak{S}}$ is the soluble radical of G; that is, the largest soluble normal subgroup of G. If $\mathfrak{X}$ is a class of finite groups, then we shall write $\overline{\mathfrak{X}}$ for its extension closure; that is, the class of all (finite) groups having a subnormal series whose quotients lie in $\mathfrak{X}$.

2. Main results

Before establishing our main theorems (see Theorems A and C below), we need to briefly discuss some auxiliary results.

Lemma 1. *Let $\mathfrak{X}$ be a class of finite groups which is closed under taking normal subgroups and quotients. Then $\overline{\mathfrak{X}}$, the extension closure of $\mathfrak{X}$ (i.e. the class of poly-$\mathfrak{X}$ groups), enjoys the same closure properties.*

Proof. If $\mathfrak{X}$ is quotient-closed then so is $\overline{\mathfrak{X}}$ by Part (vi) of [1, Prop. 5.4]. Also, a straightforward modification of the proof of Part (vii) of [1, Prop. 5.4] shows that $\overline{\mathfrak{X}}$ inherits normal-subgroup-closure from $\mathfrak{X}$. $\square$

Lemma 2. *Let $\mathfrak{X}$ be a normal-subgroup-closed class of finite groups and let $\overline{\mathfrak{X}}$ be its extension closure. If $G \in \overline{\mathfrak{X}}$ and J is a simple subnormal subgroup of G, then $J \in \mathfrak{X}$.*

Proof. Since $\mathfrak{X}$ is closed under taking normal subgroups, a subnormal series for G with quotients in $\mathfrak{X}$ can be refined to yield a composition series with the same property. Since J can be made the first term of a composition series for G, the result follows by the Jordan–Hölder theorem. $\square$

Corollary 3. *Let $\mathfrak{X}$ be a class of finite groups which is closed under taking normal subgroups and direct products. If $G \in \overline{\mathfrak{X}}$ and N is a minimal normal subgroup of G, then $N \in \mathfrak{X}$.*

Proof. Since N is a minimal normal subgroup of G, we have that $N \cong J \times \cdots \times J$ for some simple group J. In particular, J is subnormal in G and thus Lemma 2 applies to yield that $J \in \mathfrak{X}$. Now direct-product-closure of $\mathfrak{X}$ shows that $N \in \mathfrak{X}$. $\square$

We are now in a position to prove our first main result.

Theorem A. *Let $\mathfrak{X}$ be a class of finite groups which is closed under taking normal subgroups, direct products, quotients, and central extensions, and let $\overline{\mathfrak{X}}$ be the extension closure of $\mathfrak{X}$. Suppose that $G \in \overline{\mathfrak{X}}$, and that H is a maximal normal $\mathfrak{X}$-subgroup of G. Then H is large in G.*

Proof. Let $G \in \overline{\mathfrak{X}}$, and let H be a maximal normal $\mathfrak{X}$-subgroup of G. We claim that H is large in G or, equivalently, that $\mathbf{Z}(H) = \mathbf{C}_G(H)$. Assume for a contradiction that $C > Z$, where $C := \mathbf{C}_G(H)$ and $Z := \mathbf{Z}(H)$.

Since H is a normal subgroup of G, so are C and Z, and thus C/Z is a non-trivial normal subgroup of G/Z. Note that $G \in \overline{\mathfrak{X}}$ implies that $G/Z \in \overline{\mathfrak{X}}$ by Lemma 1. Also, $C/Z \in \overline{\mathfrak{X}}$, again by Lemma 1. Now put $D/Z := \mathbf{Soc}(C/Z)$. By Corollary 3, we have $D/Z \in \mathfrak{X}$ and $1 < D/Z$ since $C/Z > 1$ by hypothesis. Also, $H/Z \in \mathfrak{X}$, since $\mathfrak{X}$ is quotient-closed. Now note that $(H/Z) \cap (D/Z) = 1$ since $Z \leq H \cap D \leq H \cap C = Z$. Since $\mathfrak{X}$ is closed under taking direct products, it follows that $HD/Z = (H/Z) \times (D/Z) \in \mathfrak{X}$, thus central-extension-closure of $\mathfrak{X}$ yields $HD \in \mathfrak{X}$. However, $HD > H$, and moreover HD is a normal subgroup of G, which contradicts the maximality of H. $\square$

Our next result guarantees the existence of a variety of natural classes of finite groups that are central-extension-closed.

Proposition B. *If $\mathfrak{X}$ is a solubly saturated formation of finite groups that is closed under taking normal subgroups and such that $\mathfrak{A} \subseteq \mathfrak{X}$, then $\mathfrak{X}$ is closed under taking central extensions.*

Proof. Let G be a finite group, $1 < Z \leq \mathbf{Z}(G)$, and suppose that $G/Z \in \mathfrak{X}$, with $\mathfrak{X}$ a class of finite groups as in the statement of the proposition.

First, we argue that, since Z is an abelian group, we can find an abelian group Y such that $\Phi(Y) \cong Z$. This follows from the structure theorem for abelian groups, the fact that $\Phi(C_{p^{\alpha+1}}) \cong C_{p^\alpha}$, and [4, Satz 6].

Now let $\Gamma = G \circ Y$ be the central product of G with Y, identifying Z with the Frattini subgroup of Y. Then $\Gamma/Z \cong (G/Z) \times (Y/Z)$, so $\Gamma/Z \in \mathfrak{X}$ since $Y/Z \in \mathfrak{A} \subseteq \mathfrak{X}$ and $\mathfrak{X}$ is a formation thus closed under taking direct products. Also, $Z \leq$

$\Phi(Y) \leq \Phi(\Gamma_{\mathfrak{S}})$, so $\Gamma/\Phi(\Gamma_{\mathfrak{S}}) \in \mathfrak{X}$ since $\mathfrak{X}$ is quotient-closed, and thus $\Gamma \in \mathfrak{X}$, as $\mathfrak{X}$ is solubly saturated. Since $G \trianglelefteq \Gamma$, it follows that $G \in \mathfrak{X}$, as $\mathfrak{X}$ is assumed to be normal-subgroup closed. $\square$

Remark 4. *Proposition B may also be established in a natural way by making use of Baer's local approach to solubly saturated formations; cf. Sec. 4 in [2, Chap. IV]. This method may also be used to construct concrete examples of formations, as discussed in the proposition. The idea to consider a central product in the proof of Proposition B is due to Isaacs. Since it leads to a shorter proof than our original one via the local approach, we have favoured it.*

Our next result is a consequence of Theorem A and Proposition B.

Theorem C. *Let $\mathfrak{X}$ be a solubly saturated formation of finite groups, which is closed under taking normal subgroups, and such that $\mathfrak{A} \subseteq \mathfrak{X}$. Let $G \in \overline{\mathfrak{X}}$ and let H be a maximal normal $\mathfrak{X}$-subgroup of G. Then H is a large subgroup of G.*

Note that if $\mathfrak{X} \subseteq \mathfrak{S}$, then $\mathfrak{X}$ is solubly saturated if, and only if, it is saturated. Moreover, if $\mathfrak{X}$ is additionally a Fitting class, then the unique maximal normal $\mathfrak{X}$-subgroup of a finite group is its $\mathfrak{X}$-radical.

Corollary D. *If G is a finite soluble group, then the Fitting subgroup $\mathbf{F}(G)$ of G is large in G.*

Proof. This follows from Theorem C by setting $\mathfrak{X} = \mathfrak{N}$, noting that $\overline{\mathfrak{N}} = \mathfrak{S}$, and that $\mathfrak{N}$ is a normal-subgroup-closed (solubly) saturated formation. $\square$

Corollary E. *If G is a finite group, then the generalised Fitting subgroup $\mathbf{F}^*(G)$ of G is large in G.*

Proof. Let $\mathfrak{B}$ denote the class of finite groups that induce inner automorphisms on each of their chief factors. This class comprises precisely those finite groups that are *quasi-nilpotent*, i.e. those finite groups G such that $G = \mathbf{F}^*(G)$. Since finite simple groups are quasi-nilpotent, we see that the extension closure of $\mathfrak{B}$ is the class of all finite groups. Further, it is known that $\mathfrak{B}$ is a solubly saturated Fitting formation that is closed under taking normal subgroups and that $\mathbf{F}^*(G)$ is the $\mathfrak{B}$-radical of the finite group G; cf. [2, p. 580]. Applying the conclusion of Theorem C completes the proof. $\square$

Corollary F. *If G is a finite π-separable group, then $\mathbf{O}_{\pi',\pi}(G)$ is large in G.*

Proof. Consider the class of groups which possess a normal Hall π'-subgroup. This class is easily seen to be a saturated Fitting formation which contains all abelian groups, and whose extension closure is the class of π-separable groups. Note that if G is π-separable, then $\mathbf{O}_{\pi',\pi}(G)$ is the radical associated with the class mentioned above. Thus Theorem C applies and gives what we want. $\square$

It is easy to see that Corollary F is just an equivalent formulation of [5, Lemma 1.2.3]; cf. also [6, Thm. 3.21].

3. Large nilpotent subgroups in soluble groups

Our next result, which generalises an observation communicated to us by Marty Isaacs, concerns the existence of large subgroups in finite soluble groups, which are of bounded nilpotency class. We note that Proposition G does not follow from our previous results, since the class $\mathfrak{X} = \mathfrak{N}_c$ of finite nilpotent groups of class at most c relevant here is not closed under central extensions.

Proposition G. *Suppose that G is finite and soluble, and let $H \trianglelefteq G$ be maximal in G with the property that it is nilpotent of class at most c, where $c \geq 2$ is a given integer. Then H is large in G.*

Proof. Assume that the assertion is false, and let (G, H) be a counterexample. Write $C := \mathbf{C}_G(H)$ and $Z := \mathbf{Z}(H)$. Let $1 = C_0 < \cdots < C_r = C$ be the derived series of C. Since C is not contained in H, there exists a smallest positive index j such that $C_j \not\leq H$. In this situation, we claim that

$$L_i(HC_j) \leq L_i(H), \quad \text{for } i \geq 2, \tag{3.1}$$

where $(L_i(K))_{i \geq 0}$ denotes the lower central series of the group K. We first note that

$$L_1(HC_j) = (HC_j)' = H'C_{j-1} \leq H'(C \cap H) = H'\mathbf{Z}(H),$$

where we have used the minimality of the index j to deduce that $C_{j-1} \leq H$. Thus, using the fact that C and Z are normal in G (because H is), we get

$$L_2(HC_j) = [HC_j, L_1(HC_j)] \le [HC_j, H'\mathbf{Z}(H)] = [H, H'] \cdot [H, \mathbf{Z}(H)] \cdot [C_j, H'] \cdot [C_j, \mathbf{Z}(H)] = [H, H'] = L_2(H).$$

Assuming inductively that $L_i(HC_j) \le L_i(H)$ for some $i \ge 2$, we now find that

$$L_{i+1}(HC_j) = [HC_j, L_i(HC_j)] \le [HC_j, L_i(H)] = [H, L_i(H)] \cdot [C_j, L_i(H)] = [H, L_i(H)] = L_{i+1}(H),$$

whence (3.1).

Next, we claim that

$$c(HC_j) \le c(H) + 1, \tag{3.2}$$

where $c(K)$ denotes the nilpotency class of K. In order to see this, suppose first that $c(H) \le 1$, i.e., that H is abelian. Then we need to show that $c(HC_j) \le 2$ or, equivalently, that $(HC_j)' \le \mathbf{Z}(HC_j)$. However, we have

$$(HC_j)' = H'C_{j-1} = C_{j-1} \le H \cap C = \mathbf{Z}(H) \le \mathbf{Z}(HC_j),$$

so that our claim (3.2) holds in this case. Next, let $c(H) \ge 2$. Then we may use (3.1) to deduce that

$$c(HC_j) \le c(H) \le c(H) + 1,$$

as required, whence (3.2).

We are now in a position to finish the proof. Since $C_j \not\le H$, we have $HC_j > H$, and also $HC_j \trianglelefteq G$, since $H \trianglelefteq G$ and C_j is characteristic in C, which in turn is normal in G. Moreover, in view of (3.2), if $c(H) < c$, we have $c(HC_j) \le c$, contradicting the maximality of H. Hence, we may assume that $c(H) = c \ge 2$. In this situation, however, we may apply (3.1) to conclude again that $c(HC_j) \le c$. This final contradiction to the maximality of H shows that no counterexample to our claim exists, and the proof is complete. $\square$

Remark 5. *A result analogous to Proposition G holds for the class $\mathfrak{S}_d$ of finite soluble groups of derived length at most d, where $d \ge 2$. The proof is similar to that of Proposition G, but is considerably easier due to the fact that only commutators play a role.*

4. Large abelian subgroups in soluble groups

We note that there is no analogue of Proposition G for maximal *abelian* normal subgroups of soluble groups. Of course, *some* soluble groups (like A_4) contain maximal abelian normal subgroups that are large (i.e. self-centralising), but not all of them do. In $\mathrm{SL}_2(3)$, the centre is a maximal abelian normal subgroup, but, of course, it does not contain its centraliser, which is the whole group. Our final result exhibits a class of finite soluble groups properly containing the class of finite supersoluble groups, for which an analogue of Proposition G does exist.

Denote by $\mathfrak{X}_0$ the class of those finite soluble groups G whose supersoluble residual is either trivial or a minimal normal subgroup of G. Note that $A_4 \in \mathfrak{X}_0$ but that A_4 is not supersoluble; in particular, $\mathfrak{X}_0$ is strictly larger than the class of all finite supersoluble groups.

Proposition H. *If $G \in \mathfrak{X}_0$, then every maximal abelian normal subgroup of G is large in G.*

Proof. First, we treat the case where G is supersoluble. Let H be a maximal abelian normal subgroup of the supersoluble group G, and write $C := \mathbf{C}_G(H)$. As $H \trianglelefteq G$, so $C \trianglelefteq G$ and since H is abelian we have $H \le C$. We claim that, in fact, $H = C$. We assume that this assertion is false, so that $H < C$, and work to derive a contradiction.

Consider the non-trivial group C/H, which is normal in the supersoluble group G/H. Since every non-trivial normal subgroup of a supersoluble group contains a non-trivial cyclic normal subgroup, it follows that there is an element $x \in C \setminus H$ such that $\langle xH \rangle = \langle H, x \rangle / H \trianglelefteq G/H$.

Now, consider the group $\langle H, x \rangle$. Since x centralises H and H is abelian, it follows that $\langle H, x \rangle$ is abelian, and it is normal in G since it is the full preimage of a normal subgroup of G/H. Moreover, $\langle H, x \rangle > H$ since $x \in C \setminus H$, contradicting the fact that H was chosen maximal with respect to being abelian and normal in G.

Next, let $G \in \mathfrak{X}_0$ be such that its supersoluble residual S is a minimal normal subgroup of G; in particular, S is abelian since G is soluble. Let A be a maximal abelian normal subgroup of G. Then we have $S \le A$, since otherwise, by minimality of S, $A \cap S = 1$ and thus $AS = A \times S$ is abelian (and normal in G), contradicting the maximality of A.

Now we argue as before. Suppose, for a contradiction, that $C := \mathbf{C}_G(A) > A$. Since $A \ge S$, it follows that G/A is isomorphic to a quotient of the supersoluble group G/S, thus is itself supersoluble. Then C/A is a non-trivial normal subgroup of G/A, hence contains some non-trivial normal cyclic subgroup, say K/A. As A is central in C, so A is central in K, and thus K is normal in G and abelian, since it is central-by-cyclic. This again contradicts the maximality of A, thus we have $C = A$, as desired. $\square$

The property of the class $\mathfrak{X}_0$ expressed in Proposition H does not characterise $\mathfrak{X}_0$, however, in that there exist soluble groups having every maximal abelian normal subgroup self-centralising (i.e. large), which are not contained in $\mathfrak{X}_0$. Indeed, as GAP [3] informs, the two smallest groups G such that $G \notin \mathfrak{X}_0$, while every maximal abelian normal subgroup is large in G, are of order 48 (SmallGroup[48, 3] and SmallGroup[48,50]). A larger example of this sort is the group $G = A_4 \times A_4$. Here, the supersoluble residual is the 2-Sylow subgroup of G, while the only minimal normal subgroups of G are the 2-Sylow subgroups of the individual factors. It would be of some interest to describe those groups in $\mathfrak{S} \setminus \mathfrak{X}_0$, for which every maximal abelian normal subgroup is large.

Acknowledgements

References

[1] S. Aivazidis, T. Müller, On residuals of finite groups, arXiv:1708.04224, 2017.
[2] K. Doerk, T. Hawkes, Finite Soluble Groups, de Gruyter Expositions in Mathematics, vol. 4, Walter de Gruyter & Co., Berlin, 1992.
[3] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.8.8, 2017.
[4] W. Gaschütz, Über die Φ-Untergruppe endlicher Gruppen, Math. Z. 58 (1953) 160–170.
[5] P. Hall, G. Higman, On the p-length of p-soluble groups and reduction theorems for Burnside's problem, Proc. London Math. Soc. (3) 6 (1956) 1–42.
[6] I.M. Isaacs, Finite Group Theory, Graduate Studies in Mathematics, vol. 92, American Mathematical Society, Providence, RI, USA, 2008.