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Number theory

On AP_3-covering sequences ☆

Sur les suites d'entiers AP_3

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A R T I C L E I N F O

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A B S T R A C T

Recently, motivated by Stanley's sequences, Kiss, Sándor, and Yang introduced a new type sequence: a sequence A of nonnegative integers is called an AP_k-covering sequence if there exists an integer n_0 such that, if $n > n_0$, then there exist $a_1 \in A, \dots, a_{k-1} \in A$, $a_1 < a_2 < \cdots < a_{k-1} < n$ such that $a_1, \dots, a_{k-1}, n$ form a k-term arithmetic progression. They prove that there exists an AP_3-covering sequence A such that $\limsup_{n\to\infty} A(n)/\sqrt{n} \le 34$. In this note, we prove that there exists an AP_3-covering sequence A such that $\limsup_{n\to\infty} A(n)/\sqrt{n} = \sqrt{15}$.

R É S U M É

Motivés par la définition des suites de Stanley, Kiss, Sándor et Yang ont récemment introduit un nouveau type de suites : une suite d'entiers positifs ou nuls A est dite AP_k s'il existe un entier n_0 tel que, pour tout $n > n_0$, il existe $a_1, \dots, a_{k-1} \in A$, $a_1 < a_2 < \cdots < a_{k-1} < n$ tels que $a_1, \dots, a_{k-1}, n$ soit une progression arithmétique à k termes. Ils démontrent qu'il existe une suite d'entiers A qui est AP_3 et satisfait $\limsup_{n\to\infty} A(n)/\sqrt{n} \le 34$. Nous montrons ici qu'il en existe une satisfaisant $\limsup_{n\to\infty} A(n)/\sqrt{n} = \sqrt{15}$.

1. Introduction

Given an integer $k \ge 3$ and a set $A_0 = \{a_1, \dots, a_t\}(a_1 < \cdots < a_t)$ of nonnegative integers such that $\{a_1, \dots, a_t\}$ does not contain a k-term arithmetic progression. Define $a_{t+1}, \dots$ by the greedy algorithm: for any $l \ge t$, a_{l+1} is the smallest integer $a > a_l$ such that $\{a_1, \dots, a_l, a\}$ does not contain a k-term arithmetic progression. The sequence $A = \{a_1, a_2, \dots\}$ is called the Stanley sequence of order k generated by A_0. It is known that if A is a Stanley sequence of order 3, then

$$\liminf_{n\to\infty} \frac{A(n)}{\sqrt{n}} \ge \sqrt{2}$$

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(see [3] and [5]) and

$$\limsup_{n\to\infty}\frac{A(n)}{\sqrt{n}}\ge 1.77$$

(see [1]). For related results, one may refer to [2] and [6]. Recently, Kiss, Sándor and Yang [4] introduced the following notation: a sequence A of nonnegative integers is called an AP_k-covering sequence if there exists an integer n_0 such that, if $n>n_0$, then there exist $a_1\in A,\dots,a_{k-1}\in A$, $a_1<a_2<\cdots<a_{k-1}<n$ such that $a_1,\dots,a_{k-1},n$ form a k-term arithmetic progression. They [4] observed that

$$\liminf_{n\to\infty}\frac{A(n)}{\sqrt{n}}\ge \sqrt{2},\quad \limsup_{n\to\infty}\frac{A(n)}{\sqrt{n}}\ge 1.77$$

hold for any AP_3-covering sequence A and proved that there exists an AP_3-covering sequence A such that

$$\limsup_{n\to\infty}\frac{A(n)}{\sqrt{n}}\le 34.$$

In this note, the following result is proved.

Theorem 1.1. *There exists an AP_3-covering sequence A such that*

$$\limsup_{n\to\infty}\frac{A(n)}{\sqrt{n}}=\sqrt{15}. \tag{1.1}$$

If A is a Stanley sequence of order k, then A does not contain a k-term arithmetic progression. If A is an AP_k-covering sequence of order k, then A contains infinitely many k-term arithmetic progressions. So none of the sequences is both a Stanley sequence of order k and an AP_k-covering sequence. We pose a problem here.

Problem 1.2. Is there a Stanley sequence of order $k+1$ that is also an AP_k-covering sequence?

We introduce a new notation here that generalizes both Stanley sequences of order k and AP_k-covering sequences. A sequence A of nonnegative integers is called a weak AP_k-covering sequence if there exists an integer n_0 such that, if $n>n_0$ and $n\notin A$, then there exist $a_1\in A,\dots,a_{k-1}\in A$, $a_1<a_2<\cdots<a_{k-1}<n$ such that $a_1,\dots,a_{k-1},n$ form a k-term arithmetic progression. Clearly, a Stanley sequence of order k is also a weak AP_k-covering sequence and an AP_k-covering sequence of order k is also a weak AP_k-covering sequence.

2. Proof of Theorem 1.1

Let

$$T_l=\left\{u4^l+\sum_{i=0}^{l-1}v_i4^i : u\in\{1,2,3,4\},\ v_i\in\{1,2\}\right\},\quad l=0,1,\dots$$

and

$$A=\bigcup_{l=0}^{\infty}T_l.$$

First, we prove that A is an AP_3-covering sequence.

Let $n\ge 32$. We will prove that there exist $a,b\in A$ with $a<b<n$ such that a,b,n form a 3-term arithmetic progression. By $n\ge 32$, there exists an integer $l\ge 2$ such that $2\cdot 4^l\le n<2\cdot 4^{l+1}=8\cdot 4^l$. Let m be the integer with $m4^l\le n<(m+1)4^l$. Then $2\le m\le 7$ and

$$0\le n-m4^l<4^l.$$

Thus $n-m4^l$ can be written as

$$n-m4^l=\sum_{i=0}^{l-1}m_i4^i,\quad m_i\in\{0,1,2,3\}.$$

If $m_i=0$, then we take $v_{1,i}=1$ and $v_{2,i}=2$. If $m_i\in\{1,2\}$, then we take $v_{1,i}=v_{2,i}=m_i$. If $m_i=3$, then we take $v_{1,i}=2$ and $v_{2,i}=1$. If $m=2$, then we take $u_1=1$ and $u_2=0$. If $m=3$, then we take $u_1=2$ and $u_2=1$. If $m=4$, then we take

$u_1 = 2$ and $u_2 = 0$. If $m = 5$, then we take $u_1 = 3$ and $u_2 = 1$. If $m = 6$, then we take $u_1 = 3$ and $u_2 = 0$. If $m = 7$, then we take $u_1 = 4$ and $u_2 = 1$. Let

$$a = u_2 4^l + \sum_{i=0}^{l-1} v_{2,i} 4^i, \quad b = u_1 4^l + \sum_{i=0}^{l-1} v_{1,i} 4^i.$$

It is clear that $1 \le a < b < n$, $a, b \in T_l \cup T_{l-1} \subseteq A$ and a, b, n form a 3-term arithmetic progression. Hence A is an AP_3-covering sequence.

Now we prove that (1.1) holds.

Let

$$A = \{n_1, n_2, \dots\}, \quad n_1 < n_2 < \cdots .$$

For $n_j < m < n_{j+1}$, we have

$$\frac{A(m)}{\sqrt{m}} = \frac{A(n_j)}{\sqrt{m}} < \frac{A(n_j)}{\sqrt{n_j}}.$$

It follows that

$$\limsup_{n\to\infty} \frac{A(n)}{\sqrt{n}} = \limsup_{j\to\infty} \frac{A(n_j)}{\sqrt{n_j}}.$$

Let

$$n_j = u4^l + \sum_{i=0}^{l-1} v_i 4^i, \quad u \in \{1, 2, 3, 4\}, v_i \in \{1, 2\} \ (0 \le i \le l-1).$$

Then

$$A(n_j) = (u-1)2^l + \sum_{i=1}^{l-1} (v_i - 1)2^i + v_0 + 4(2^{l-1} + \cdots + 2 + 1). \tag{2.1}$$

It is clear that

$$n_j \ge u4^l + v_{l-1}4^{l-1} + \frac{1}{3}(4^{l-1} - 1) = (4u + v_{l-1} + \frac{1}{3})4^{l-1} - \frac{1}{3},$$
$$A(n_j) \le (u-1)2^l + (v_{l-1} - 1)2^{l-1} + 2^{l-1} + 4(2^l - 1) = (2u + 6 + v_{l-1})2^{l-1} - 4.$$

Since

$$2u + 6 + v_{l-1} < 4\sqrt{4u + v_{l-1} + \frac{1}{3}}$$

for $u \in \{1, 2, 3, 4\}$ and $v_{l-1} \in \{1, 2\}$, it follows that

$$A(n_j) \le (2u + 6 + v_{l-1})2^{l-1} - 4 < 4\sqrt{n_j + \frac{1}{3}} - 4 < 4\sqrt{n_j}.$$

If $v_i = 1$ for some $0 \le i \le l-1$, then $n_j + 4^i \in A$ and by (2.1), we have $A(n_j + 4^i) = A(n_j) + 2^i$. Since $n_j > 4^l \ge 4^{i+1}$, it follows that $\sqrt{n_j + 4^i} + \sqrt{n_j} > 4 \cdot 2^i$. That is, $2^i > 4(\sqrt{n_j + 4^i} - \sqrt{n_j})$. By $A(n_j) < 4\sqrt{n_j}$, we have

$$A(n_j)(\sqrt{n_j + 4^i} - \sqrt{n_j}) < 4\sqrt{n_j}(\sqrt{n_j + 4^i} - \sqrt{n_j}) < 2^i\sqrt{n_j}.$$

So

$$(A(n_j) + 2^i)\sqrt{n_j} > A(n_j)\sqrt{n_j + 4^i}.$$

Hence

$$\frac{A(n_j + 4^i)}{\sqrt{n_j + 4^i}} = \frac{A(n_j) + 2^i}{\sqrt{n_j + 4^i}} > \frac{A(n_j)}{\sqrt{n_j}}.$$

So we need only consider those n_j with all $v_i = 2$. Let

$$q_{u,l} = u4^l + \sum_{i=0}^{l-1} 2 \cdot 4^i = (u + \frac{2}{3})4^l - \frac{2}{3}.$$

By (2.1), $A(q_{u,l}) = (u+4)2^l - 4$. It follows that

$$\lim_{l\to\infty} \frac{A(q_{u,l})}{\sqrt{q_{u,l}}} = \frac{u+4}{\sqrt{u+2/3}}.$$

Hence

$$\limsup_{n\to\infty} \frac{A(n)}{\sqrt{n}} = \limsup_{j\to\infty} \frac{A(n_j)}{\sqrt{n_j}} = \max\left\{ \frac{u+4}{\sqrt{u+2/3}} : u = 1, 2, 3, 4 \right\} = \sqrt{15}.$$

This completes the proof.

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