Differential geometry

On the linearizability of 3-webs: End of controversy ☆

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A B S T R A C T

There are two theories describing the linearizability of 3-webs: one is developed in [10] and another in [8]. Unfortunately they cannot be both correct because on an explicit 3-web $\mathcal{W}_0$ they *contradict*: the first predicts that $\mathcal{W}_0$ is linearizable, while the second states that $\mathcal{W}_0$ is not linearizable. The essential question beyond this particular 3-web is: which theory describes correctly the linearizability condition? In this paper, we present a very short proof, due to J.-P. Dufour, that $\mathcal{W}_0$ is linearizable, confirming the result of [10].

R É S U M É

Il existe deux théories décrivant la linéarisabilité des 3-tissus : l'une est développée dans [10], l'autre dans [8]. Malheureusement, elles ne peuvent pas être correctes toutes les deux, car sur un 3-tissu $\mathcal{W}_0$ elles se contredisent : la première prédit que le tissu $\mathcal{W}_0$ est linéarisable, tandis que la seconde affirme que $\mathcal{W}_0$ n'est pas linéarisable. La question essentielle au-delà de ce 3-tissu particulier est : quelle théorie décrit correctement la condition de linéarisabilité ? Dans cet article, nous présentons une preuve très courte, due à J.-P. Dufour, de ce que le tissu $\mathcal{W}_0$ est linéarisable, confirmant le résultat de [10].

1. The linearizability problem for planar 3-webs

On a two-dimensional real or complex differentiable manifold M, a 3-web is given by three foliations of smooth curves in general position. Two webs $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ are locally equivalent at $p \in M$, if there exists a local diffeomorphism on a neighborhood of p that exchanges them. A 3-web is called *linear* if it is given by three foliations of straight lines. A web that is equivalent to a linear web is called linearizable.

The linearizability problem: *characterize the 3-webs on real or complex 2-dimensional manifolds that are equivalent, up to a local diffeomorphism, to linear webs, that is, webs such that the corresponding foliations are straight lines in a convenient coordinate system.*

In a fashion similar to the linearizability is the notion of parallelizability, a 3-web is called *parallelizable* if it is equivalent to three families of *parallel* lines. One can remark that for 1- and 2-webs, the notion of linearizability and parallelizability coincide: because of the inverse function theorem, any 1- and 2-webs are linearizable and also parallelizable. This is not true in general: the notion of parallelizability is much stronger than that of linearizability. A generic 3-web is non-linearizable, and even if a web is linearizable, it is in general non-parallelizable.

Basic examples of planar 3-webs comes from complex projective algebraic geometry. If $\mathcal{C} \subset \mathbb{P}^2$ is a not necessarily irreducible and possibly singular algebraic curve of degree 3 on the projective plane $\mathbb{P}^2$, then by duality in the Grassmannian manifold $\mathbb{G}r(1, \mathbb{P}^2)$, one can obtain a 3-web called the algebraic web associated with $\mathcal{C} \subset \mathbb{P}^2$ (cf. [14]). Graf and Sauer proved a theorem, which in web geometry language can be stated as follows: a linear web is parallelizable if and only if it is associated with an algebraic curve of degree 3, i.e. if its leaves are tangent lines to an algebraic curve of degree 3 [3, page 24]. This theorem is a special case of N.H. Abel's classical theorem and its converse: the general Lie–Darboux–Griffiths theorem [9].

Concerning the parallelizability of 3-webs, an elegant coordinate free characterization can be given in terms of the associated Chern connection: a 3-web is parallelizable if and only if the curvature of the Chern connection, called also Blaschke curvature, vanishes [5]. A new theoretical set-up of the problem can be found in [13].

Although the problem of finding a linearizability criterion is a very natural one, it is far from being trivial. T.H. Gronwall conjectured that if a non-parallelizable 3-web $\mathcal{W}$ is linearizable, then up to a projective transformation there is a unique diffeomorphism that maps $\mathcal{W}$ into a linear 3-web. G. Bol suggested in [4] a method to find a criterion of linearizability, but he was unable to carry out the computation. He showed that the number of projectively different linear 3-webs in the plane that are equivalent to a non-parallelizable 3-web is finite and less that 17. The formulation of the linearizability problem in terms of the Chern connection was suggested by M.A. Akivis in a lecture given in Moscow in 1973. In his approach, the linearizability problem is reduced to the solvability of a system of nonlinear partial differential equations on the components of the affine deformation tensor. Using Akivis' idea, V.V. Goldberg determined in [6] the first integrability conditions of the partial differential system.

2. The controversy

In 2001, J. Grifone, Z. Muzsnay, and J. Saab solved the linearizability problem by carrying out the computation [10]. They showed that, in the non-parallelizable case, there exists an algebraic submanifold $\mathcal{A}$ of the space of vector valued symmetric tensors $(S^2 T^* \otimes T)$ on a neighborhood of any point $p \in M$, expressed in terms of the curvature of the Chern connection and its covariant derivatives up to order 6, so that the affine deformation tensor is a section of $S^2 T^* \otimes T$ with values in $\mathcal{A}$. In particular, the web is linearizable if and only if $\mathcal{A} \neq \emptyset$, and there exists at most 15 projectively nonequivalent linearizations of a nonparallelizable 3-web. The expressions of the polynomials and their coefficients that define $\mathcal{A}$ can be found in [11]. The criteria of linearizability provides the possibility to make explicit computation on concrete examples to decide whether or not they are linearizable.

In 2006, V.V. Goldberg and V.V. Lychagin found results on the linearizability in [8]. Their results were different from that of [10] and they qualified [10] "incomplete because they do not contain all conditions" (see [7, page 171] and [8, page 70]) without pointing out any missing integrability condition or developing any further justification.

The GMS-approach developed in [10] and the GL-approach described in [8] *cannot be both correct* because there are cases where the two theories *contradict*.

Hence the small but dedicated scientific community working on the problems related to web geometry is in suspense (see for example [1, page 2], [2, page 2], or [15, page 40]). Therefore, the focus of this paper is to conclude which theory is describing correctly the linearizability condition.

3. Decisive example

The direct comparison of the two theories is not straightforward, since the formulas in both cases are long and complex containing the curvature tensor and its different derivatives. There is, however, a very specific case, where the two theories show clearly opposite results. This explicit example of 3-web was described in [10]. The particular 3-web $\mathcal{W}_0$ is determined by the web function $f(x, y) := (x + y)e^{-x}$, i.e. it is the 3-web given by the foliations

$$x = \text{const}, \qquad y = \text{const}, \qquad (x + y)\,e^{-x} = \text{const}, \tag{1}$$

on the domain $D := \{(x, y) \mid x + y \neq 1\} \subset \mathbb{R}^2$. Using the GMS-theory one gets that $\mathcal{W}$ is linearizable (page 2653, [10]) while GL-theory states the opposite (page 171, line 7–10, [8]). Evidently, *the correct theory should give a correct answer in that specific situation*. In the theorem below we show that the web $\mathcal{W}_0$ is linearizable, therefore the prediction of GMS-theory is correct. This result was obtained in [10] but the very short proof is due to J.-P. Dufour.

Theorem. *The 3-web $\mathcal{W}_0$ defined by the foliations $x = $ const, $y = $ const and $f(x, y) := (x + y)\,e^{-x} = $ const, is linearizable.*

Proof. The change of variable $\bar{x} = f(x, y)$, $\bar{y} = y$ clearly transforms the foliations $y = $ const and $f(x, y) = $ const into linear foliations. The line $x = c$ of the first foliation becomes the line $\bar{x} = (c + \bar{y})\,e^{-c}$. $\quad\square$

Remark. The statement of the theorem remains true if the function $f(x, y)$ has the form $f(x, y) = a(x)x + b(x)y$.

We note that the linearizability of $\mathcal{W}_0$ has already been investigated in [12] from a different point of view: it was showed, using the GMS approach, that $\mathcal{W}_0$ is linearizable by proving the existence of the affine deformation tensor. The lack of presenting the explicit linearization map, however, could maintain in some way the suspense. Now the suspense is over: using the Theorem, we can conclude that the prediction of GMS-theory is correct and the statement of GL-theory is wrong. One can also conclude that the criterion of linearizability of [10,11] provides effective tools to decide whether or not a 3-web is linearizable.

Acknowledgements

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