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Complex analysis

The approximation of Laplace–Stieltjes transformations with finite order on the left half plane ☆

L'approximation des transformées de Laplace–Stieltjes d'ordre fini sur le demi-plan gauche

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A B S T R A C T

In this paper, we study the error in approximating the analytic function defined by a Laplace–Stieltjes transformation of finite order, which converges on the left half plane, and obtain the relation theorems between the error, the coefficients, and the proximate order of the Laplace–Stieltjes transformation.

R É S U M É

Dans cette Note, nous étudions l'erreur d'approximation d'une fonction analytique définie comme une transformée de Laplace–Stieltjes d'ordre fini, qui converge dans le demi-plan gauche. Nous obtenons des théorèmes reliant cette erreur, les coefficients et l'ordre d'approximation de la transformation de Laplace–Stieltjes.

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1. Introduction

For Dirichlet series

$$f(s)=\sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \qquad s=\sigma+it, \tag{1}$$

where

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$$0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \lambda_n \to \infty \ as \ n \to \infty; \tag{2}$$

$s = \sigma + it$ (σ, t are real variables), a_n are nonzero complex numbers. When a_n, λ_n, n satisfy some conditions, the series (1) can converge in the whole plane or the half plane, that is, $f(s)$ is an analytic function or entire function in the whole plane or the half plane. In the past few decades, many mathematicians studied the growth and value distribution of analytic (entire) functions defined by Dirichlet series, and obtained a lot of interesting results (see [7,12,19–21]).

As we know, the Dirichlet series is regarded as a special example of Laplace–Stieltjes transform. The Laplace–Stieltjes transform, named after Pierre-Simon Laplace and Thomas Joannes Stieltjes, is an integral transform similar to the Laplace transform. For real-valued functions, it is the Laplace transform of a Stieltjes measure; however, it is often defined for functions with values in a Banach space. It can be used in many fields of mathematics, such as functional analysis, and certain areas of theoretical and applied probability.

For Laplace–Stieltjes transforms,

$$G(s) = \int_0^{+\infty} e^{-sx} d\alpha(x), \qquad s = \sigma + it, \tag{3}$$

where $\alpha(x)$ is a bounded variation on any finite interval $[0, Y](0 < Y < +\infty)$, and σ and t are real variables. Let

$$B_n^* = \sup_{\lambda_n < x \le \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{x} e^{-ity} d\alpha(y) \right|,$$

where the sequence $\{\lambda_n\}_{n=1}^{\infty}$ satisfies (2) and

$$\limsup_{n \to +\infty} (\lambda_{n+1} - \lambda_n) = h < +\infty. \tag{4}$$

In 1963, Yu [18] proved the Valiron–Knopp–Bohr formula for the associated abscissas of bounded convergence, absolute convergence, and uniform convergence of Laplace–Stieltjes.

Theorem A. *Suppose that Laplace–Stieltjes transformations (3) satisfy (2), (4) and* $\limsup_{n \to +\infty} \frac{\log n}{\lambda_n} < +\infty$, *then*

$$\limsup_{n \to +\infty} \frac{\log B_n^*}{\lambda_n} \le \sigma_u^G \le \limsup_{n \to +\infty} \frac{\log B_n^*}{\lambda_n} + \limsup_{n \to +\infty} \frac{\log n}{\lambda_n},$$

where σ_u^F *is called the abscissa of uniformly convergent of* $F(s)$.

Moreover, Yu [18] first introduced the maximal molecule $M_u(\sigma, G)$, the maximal term $\mu(\sigma, G)$, the Borel line, and the order of analytic functions represented by Laplace–Stieltjes transformations convergent in the complex plane. After his works, considerable attention has been paid to the growth and the value distribution of the functions represented by Laplace–Stieltjes transformation convergent in the half-plane or whole complex plane in the field of complex analysis (see [1,2,4,5,9,13,14]).

In 2012, Luo and Kong [8] proposed the following form for the Laplace–Stieltjes transform,

$$F(s) = \int_0^{+\infty} e^{sx} d\alpha(x), \qquad s = \sigma + it, \tag{5}$$

where $\alpha(x)$ is stated as in (3), and $\{\lambda_n\}$ satisfies (2), (4). Set

$$A_n^* = \sup_{\lambda_n < x \le \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{x} e^{ity} d\alpha(y) \right|.$$

By using the same argument as in [18], we can get a similar result about the abscissa of uniformly convergent $F(s)$ easily. If

$$\limsup_{n \to +\infty} \frac{n}{\lambda_n} = D < \infty, \qquad \limsup_{n \to +\infty} \frac{\log A_n^*}{\lambda_n} = 0, \tag{6}$$

by (2), (4) and Theorem A, one can get that $\sigma_u^F = 0$, *i.e.*, $F(s)$ is analytic in the left half-plane. Set

$$\mu(\sigma, F) = \max_{n \in N} \{A_n^* e^{\lambda_n \sigma}\} \quad (\sigma < 0), \quad M(\sigma, F) = \sup_{-\infty < t < +\infty} |F(\sigma + it)|,$$

$$M_u(\sigma,F)=\sup_{0<x<+\infty,-\infty<t<+\infty}\left|\int_0^x \mathrm{e}^{(\sigma+\mathrm{i}t)y}\mathrm{d}\alpha(y)\right|,\quad (\sigma<0).$$

Remark 1.1. From (6), for any $\sigma<0$, we have

$$\limsup_{n\to+\infty}\frac{\log A_n^*+\lambda_n\sigma}{\lambda_n}=\sigma<0,\quad \textbf{or}\quad \limsup_{n\to+\infty}\log A_n^*\mathrm{e}^{\lambda_n\sigma}=-\infty.$$

This shows that $\mu(\sigma,F)$ exists.

We denote $\overline{L}_\alpha$ to be the class of all the functions $F(s)$ of the form (5) that are analytic in the half plane $\Re s<\alpha(-\infty<\alpha<\infty)$ and the sequence $\{\lambda_n\}$ satisfies (2) and (4), and denote L to be the class of all the functions $F(s)$ of the form (5) that are analytic in the half-plane $\Re s<0$; the sequence $\{\lambda_n\}$ satisfies (2), (4), and (6). Thus, if $-\infty<\alpha<0$ and $F(s)\in L$, then $F(s)\in\overline{L}_\alpha$; if $0<\alpha<+\infty$ and $F(s)\in\overline{L}_\alpha$, then $F(s)\in L$. If the L–S transform (5) satisfies, $A_n^*=0$ for $n\geq k+1$, and $A_n^*\neq 0$, then $F(s)$ will be called an exponential polynomial of degree k, usually denoted by p_k, *i.e.*, $p_k(s)=\int_0^{\lambda_k}\exp(sy)\,\mathrm{d}\alpha(y)$. When we choose a suitable function $\alpha(y)$, the function $p_k(s)$ may be reduced to a polynomial in terms of $\exp(s\lambda_i)$, that is, $\sum_{i=1}^k b_i\exp(s\lambda_i)$. Similarly, we give the following definition.

Definition 1.1. If the Laplace–Stieltjes transform (5) satisfies $\sigma_u^F=0$, the sequence $\{\lambda_n\}$ satisfies (2), (4) and (6) and

$$\limsup_{\sigma\to 0^-}\frac{\log^+\log^+ M_u(\sigma,F)}{\log(-\frac{1}{\sigma})}=\rho,$$

we say that $F(s)$ is of order ρ in $Res=\sigma<0$, where $\log^+ x=\max\{\log x,0\}$. If $\rho\in(0,+\infty)$, we say that $F(s)$ is an analytic function of finite order in the left half-plane.

Recently, many people studied some problems on analytic functions defined by L–S transformations and obtained a number of interesting results. Kong Y.Y., Sun D.C., Huo Y.Y., and Xu H.Y. investigated the growth of analytic functions with kinds of order defined by L–S transformations in the right half-plane (see [3,4,6,8,16]), and Shang L.N., Gao Z.S., etc., investigated the value distribution of such functions (see [10,11,15,17]).

For $F(s)\in\overline{L}_\alpha,-\infty<\alpha<+\infty$, we denote by $E_n(F,\alpha)$ the error in approximating the function $F(s)$ by exponential polynomials of degree n in uniform norm as

$$E_n(F,\alpha)=\inf_{p\in\Pi_n}\parallel F-p\parallel_\alpha,\quad n=1,2,\ldots,$$

where

$$\parallel F-p\parallel_\alpha=\max_{-\infty<t<+\infty}|F(\alpha+\mathrm{i}t)-p(\alpha+\mathrm{i}t)|.$$

In this paper, by using the properties on $E_n(F,\alpha)$, we will further investigate the growth of analytic functions defined by L–S transformations in the left half-plane. Moreover, we will deal with the relation between the order and type of $F(s)$ and $E_n(f,\alpha)$, λ_n. To state the results of this paper, we introduce some definitions and notations as follows.

Let $\rho\in(0,+\infty)$ and $\rho(r)(r>r_0)$ be a non-negative, continuous, monotonic function, which has left-hand derivative and right-hand derivative in every $r(>r_0)$, such that

$$\lim_{r\to+\infty}\rho(r)=\rho,\qquad \lim_{r\to+\infty}\rho'(r)r\log r=0,\tag{7}$$

and set $U(r)=r^{\rho(r)}$, which is a strictly increasing function of r in $r\geq r_0'>r_0$. Let

$$t=rU(r),\qquad r=W(t),\quad r>0,t>0,\tag{8}$$

be two reciprocally inverse functions. From Ref. [22], for any positive real number k, we have

$$\lim_{r\to+\infty}\frac{U(kr)}{U(r)}=k^\rho,\qquad \lim_{t\to+\infty}\frac{W(kt)}{W(t)}=k^{\frac{1}{\rho+1}}.\tag{9}$$

For the L–S transformation (5), if

$$\limsup_{\sigma\to 0^-}\frac{\log^+ M_u(\sigma,F)}{U(-\frac{1}{\sigma})}=1,\tag{10}$$

we call $\rho(-\frac{1}{\sigma})$ the proximate order of (5) and $U(-\frac{1}{\sigma})$ the type function of (5) in $Res=\sigma<0$.

The main results of this paper are as follows.

Theorem 1.1. *Suppose that the Laplace–Stieltjes transformations (5) of finite order ρ $(0<\rho<\infty)$ satisfy (2), (4), (6), and for any real number $-\infty<\alpha<0$. Then*

$$\limsup_{\sigma\to 0^-}\frac{\log^+ M_u(\sigma,F)}{U(-\frac{1}{\sigma})}=1 \Longleftrightarrow \limsup_{n\to+\infty}\Omega_n(F,\alpha,\lambda_n)=\frac{(1+\rho)^{1+\rho}}{\rho^\rho}, \tag{11}$$

where $U(r)$ is defined by (8) and

$$\Omega_n(F,\alpha,\lambda_n)=\frac{\log^+[E_{n-1}(F,\alpha)\exp\{-\alpha\lambda_n\}]}{U\left(\frac{\lambda_n}{\log^+[E_{n-1}(F,\alpha)\exp\{-\alpha\lambda_n\}]}\right)}.$$

Theorem 1.2. *Suppose that Laplace–Stieltjes transformations (5) of finite order ρ $(0<\rho<\infty)$ satisfy (2), (4), (6) and for any real number $-\infty<\alpha<0$. Then*

$$\lim_{\sigma\to 0^-}\frac{\log^+ M_u(\sigma,F)}{U(-\frac{1}{\sigma})}=1 \Longleftrightarrow (i)\quad \limsup_{n\to+\infty}\Omega_n(F,\alpha,\lambda_n)=\frac{(1+\rho)^{1+\rho}}{\rho^\rho};$$

(ii) There exists a non-decreasing positive integer sequence $\{n_\nu\}$ satisfying

$$\lim_{\nu\to+\infty}\Omega_{n_\nu}(F,\alpha,\lambda_{n_\nu})=\frac{(1+\rho)^{1+\rho}}{\rho^\rho},\qquad \lim_{\nu\to+\infty}\frac{\lambda_{n_{\nu+1}}}{\lambda_{n_\nu}}=1, \tag{12}$$

where $U(r)$ is stated in Theorem 1.1 and

$$\Omega_{n_\nu}(F,\alpha,\lambda_{n_\nu})=\frac{\log^+\left[E_{n_\nu-1}(F,\alpha)\exp\{-\alpha\lambda_{n_\nu}\}\right]}{U\left(\frac{\lambda_{n_\nu}}{\log^+\left[E_{n_\nu-1}(F,\alpha)\exp\{-\alpha\lambda_{n_\nu}\}\right]}\right)}.$$

2. Some lemmas

The process of proofs of Lemmas 2.1–2.3 are similar to those in [16,21,22]; for the convenience of the reader, we will give the complete proofs as follows.

Lemma 2.1. *Let δ and λ be any positive real numbers, then*

$$\varphi(\sigma)=\delta U\left(-\frac{1}{\sigma}\right)-\lambda\sigma,\quad \sigma<0,$$

attains the minimum

$$\delta^{\frac{1}{\rho+1}}\frac{\rho+1}{\rho^{\frac{\rho}{\rho+1}}}\frac{\lambda}{W(\lambda)}(1+o(1)),\lambda\to+\infty\quad\text{at}\quad \sigma=-\frac{(\delta\rho)^{\frac{1}{\rho+1}}}{W(\lambda)}(1+o(1)),\lambda\to+\infty.$$

Proof. For the definition of $U\left(-\frac{1}{\sigma}\right)$, we have

$$\varphi'(\sigma)=\delta U'\left(-\frac{1}{\sigma}\right)\cdot\frac{1}{\sigma^2}-\lambda.$$

Then we can get

$$\begin{aligned}\lambda&=-\frac{\delta}{\sigma}U\left(-\frac{1}{\sigma}\right)\left[\rho\left(-\frac{1}{\sigma}\right)+\rho'\left(-\frac{1}{\sigma}\right)\cdot\frac{1}{-\sigma}\log\frac{1}{-\sigma}\right]\\&=-\frac{\delta\rho}{\sigma}U\left(-\frac{1}{\sigma}\right)(1+o(1)),\qquad(\lambda\to+\infty),\end{aligned}$$

as $\varphi'(\sigma)=0$.

With the value of σ increased from $-\infty$ to 0, the value of $\varphi'(\sigma)$ changes from a negative one to a positive one. Then, from (7), (9) and the definition of $\varphi(\sigma)$, we can get that $\varphi(\sigma)$ attains a minimum when

$$\sigma = -\frac{(\delta\rho)^{\frac{1}{\rho+1}}}{W(\lambda)}(1+o(1)), \quad \lambda \to +\infty,$$

and the minimum is

$$\begin{aligned}
&\delta U\left(\frac{W(\lambda)}{(\delta\rho)^{\frac{1}{\rho+1}}(1+o(1))}\right) + \lambda\frac{(\delta\rho)^{\frac{1}{\rho+1}}}{W(\lambda)}(1+o(1))\\
&= \frac{1}{W(\lambda)}\left[\frac{\delta W(\lambda)U(W(\lambda))}{(\delta\rho)^{\frac{\rho}{\rho+1}}(1+o(1))} + \lambda(\delta\rho)^{\frac{1}{\rho+1}}(1+o(1))\right]\\
&= \frac{\lambda}{W(\lambda)}\left[\frac{\delta}{(\delta\rho)^{\frac{\rho}{\rho+1}}(1+o(1))} + (\delta\rho)^{\frac{1}{\rho+1}}(1+o(1))\right]\\
&= \delta^{\frac{1}{\rho+1}}\frac{\rho+1}{\rho^{\frac{\rho}{\rho+1}}}\frac{\lambda}{W(\lambda)}(1+o(1)), \qquad (\lambda \to +\infty).
\end{aligned}$$

Thus, we complete the proof of Lemma 2.1. □

Lemma 2.2. *Let b and σ be any negative real number, then*

$$\phi(x) = \frac{x}{W(bx)} + \sigma x,$$

attains the maximum

$$\frac{\rho^\rho}{b(\rho+1)^{\rho+1}}U\left(-\frac{1}{\sigma}\right)(1+o(1)), \quad \sigma \to 0^-$$

at

$$x = \frac{1}{b}\left(\frac{\rho}{\rho+1}\right)^{\rho+1}(-\frac{1}{\sigma})U\left(-\frac{1}{\sigma}\right)(1+o(1)), \quad \sigma \to 0^-.$$

Proof. From (9), we can get

$$\frac{\mathrm{d}t}{t} = \frac{U(r)+rU'(r)}{U(r)}\frac{\mathrm{d}r}{r}, \qquad \frac{\mathrm{d}r}{r} = \frac{tW'(t)}{W(t)}\frac{\mathrm{d}t}{t}.$$

Differentiating $U(r) = r^{\rho(r)}$ and applying (7) and the above two equalities, we can have

$$\frac{t\,W'(t)}{W(t)} = \frac{U(r)}{U(r)+rU'(r)} = \frac{1}{\rho+1} + o(1) \quad (t \to +\infty).$$

By (7), (9) and the above equality, we can have

$$\begin{aligned}
\phi'(x) &= \frac{W(bx) - bxW'(bx)}{W^2(bx)} + \sigma\\
&= \frac{1}{b^{\frac{1}{\rho+1}}}\frac{\rho}{\rho+1}\frac{1}{W(x)}(1+o(1)) + \sigma \qquad (x \to +\infty).
\end{aligned}$$

Then we can get that

$$W(x) = \frac{1}{b^{\frac{1}{\rho+1}}}\frac{\rho}{\rho+1}(-\frac{1}{\sigma})(1+o(1)) \quad (x \to +\infty)$$

as $\phi'(x) = 0$, *i.e.*,

$$\begin{aligned}
x &= \frac{1}{b^{\frac{1}{\rho+1}}}\frac{\rho}{\rho+1}(-\frac{1}{\sigma})(1+o(1))U\left(\frac{1}{b^{\frac{1}{\rho+1}}}\frac{\rho}{\rho+1}(-\frac{1}{\sigma})(1+o(1))\right)\\
&= \frac{1}{b}\left(\frac{\rho}{\rho+1}\right)^{\rho+1}(-\frac{1}{\sigma})U\left(-\frac{1}{\sigma}\right)(1+o(1)), \quad (\sigma \to 0^-)
\end{aligned}$$

as $\phi'(x) = 0$.

With the value of x increased from $-\infty$ to $+\infty$, the value of $\phi'(x)$ changes from a positive one to a negative one. Then, from (7), (9), and the definition of $\phi(x)$, we can get that $\phi(x)$ attains its maximum when

$$x=\frac{1}{b}\left(\frac{\rho}{\rho+1}\right)^{\rho+1}(-\frac{1}{\sigma})U\left(-\frac{1}{\sigma}\right)(1+o(1)),\quad (\sigma\to 0^+),$$

and the maximum is

$$\frac{\left(\frac{\rho}{\rho+1}\right)^{\rho+1}(-\frac{1}{\sigma})U\left(-\frac{1}{\sigma}\right)(1+o(1))}{bW\left[\left(\frac{\rho}{\rho+1}\right)^{\rho+1}(-\frac{1}{\sigma})U\left(-\frac{1}{\sigma}\right)(1+o(1))\right]}-\frac{1}{b}\left(\frac{\rho}{\rho+1}\right)^{\rho+1}U\left(-\frac{1}{\sigma}\right)(1+o(1))$$
$$=\frac{1}{b}\left(\frac{\rho}{\rho+1}\right)^{\rho}U\left(-\frac{1}{\sigma}\right)(1+o(1))-\frac{1}{b}\left(\frac{\rho}{\rho+1}\right)^{\rho+1}U\left(-\frac{1}{\sigma}\right)(1+o(1))$$
$$=\frac{1}{b}\frac{\rho^{\rho}}{(\rho+1)^{\rho+1}}U\left(-\frac{1}{\sigma}\right)(1+o(1)).$$

Thus, we complete the proof of this lemma. □

Lemma 2.3. *Let $A>0$ and $\{\lambda_{n_\nu}\}$ be a strictly increasing sequence tending to $\infty(\nu\to\infty)$ and satisfying $\lambda_{n_1}>Ar_0'U(r_0')$ where r_0' is stated as in Section 1. If $\lim_{\nu\to+\infty}\frac{\lambda_{n_{\nu+1}}}{\lambda_{n_\nu}}=1$, then there exists a monotone decreasing positive sequence $\{\sigma_\nu\}$ convergent to 0, satisfying*

$$\lambda_{n_\nu}=-\frac{A}{\sigma_\nu}U\left(-\frac{1}{\sigma_\nu}\right),\qquad \lim_{\nu\to\infty}\frac{-\frac{1}{\sigma_{\nu+1}}U\left(-\frac{1}{\sigma_{\nu+1}}\right)}{-\frac{1}{\sigma_\nu}U\left(-\frac{1}{\sigma_\nu}\right)}=1.$$

Lemma 2.4. *[20] If Dirichlet series $f(s)=\sum_{n=1}^{\infty}a_n\exp\{s\lambda_n\}$ is of order $\rho(0<\rho<+\infty)$, and satisfies (2), (4) and $\limsup_{n\to\infty}\frac{\log^+|a_n|}{\lambda_n}=0$, then*

$$\limsup_{\sigma\to 0^-}\frac{\log\log M(\sigma,f)}{-\log(-\sigma)}=\rho\Longleftrightarrow\limsup_{n\to+\infty}\frac{\log^+\log^+|a_n|}{\log\lambda_n}=\frac{\rho}{\rho+1},$$

where $M(\sigma,f)$ is the maximum modular of Dirichlet series $f(s)$.

Lemma 2.5. *[18] If $F(s)\in L$ is of the order ρ $(0<\rho<+\infty)$, and satisfies (2), (4), and (6), then*

$$\limsup_{\sigma\to 0^-}\frac{\log\log M_u(\sigma,F)}{-\log(-\sigma)}=\rho\Longleftrightarrow\limsup_{n\to+\infty}\frac{\log^+\log^+A_n^*}{\log\lambda_n}=\frac{\rho}{\rho+1}.$$

Lemma 2.6. *If the abscissa $\sigma_u^F=0$ of uniformly convergent to Laplace–Stieltjes transformation $F(s)$, and the sequence (2) satisfies (4), (6), then, for $\sigma(<0)$ sufficiently approaching 0, we have $\mu(\sigma,F)\le 4M_u(\sigma,F)$, and, for any real number γ, we have*

$$\left|\int_{\lambda_k}^{\infty}\exp\{(\gamma+\mathrm{i}t)y\}\,\mathrm{d}\alpha(y)\right|\le 2\sum_{n=k}^{+\infty}A_n^*\exp\{\gamma\lambda_{n+1}\},$$

where

$$A_n^*=\sup_{\lambda_n<x\le\lambda_{n+1},-\infty<t<+\infty}\left|\int_{\lambda_n}^{x}\mathrm{e}^{\mathrm{i}ty}\,\mathrm{d}\alpha(y)\right|.$$

Proof. Set

$$I(x;\mathrm{i}t)=\int_0^x\exp\{\mathrm{i}ty\}\,\mathrm{d}\alpha(y).$$

From (4), there exists $\eta>0$ satisfying $0<\lambda_{n+1}-\lambda_n\le\eta$ $(n=1,2,3,\ldots)$. When σ is sufficiently close to $0-$, it follows that $\mathrm{e}^{-\eta\sigma}<2$. When $x>\lambda_n$, we have

$$\int_{\lambda_n}^{x} \exp\{ity\}\,\mathrm{d}\alpha(y) = \int_{\lambda_n}^{x} \mathrm{e}^{-\sigma y}\,\mathrm{d}y\, I(y;\sigma+\mathrm{i}t)$$
$$= I(y;\sigma+\mathrm{i}t)\,\mathrm{e}^{-\sigma y}|_{\lambda_n}^{x} + \sigma \int_{\lambda_n}^{x} \mathrm{e}^{-\sigma y} I(y;\sigma+\mathrm{i}t)\,\mathrm{d}y.$$

Then, for $\sigma < 0$, it follows that

$$\left|\int_{\lambda_n}^{x} \exp\{ity\}\,\mathrm{d}\alpha(y)\right| \le M_u(\sigma,F)\left[|\mathrm{e}^{-\sigma x}+\mathrm{e}^{-\sigma\lambda_n}| + |\mathrm{e}^{-\sigma x}-\mathrm{e}^{-\sigma\lambda_n}|\right]$$
$$\le 2M_u(\sigma,F)\mathrm{e}^{-\sigma x}.$$

Thus, for any $\sigma < 0$ and any $x \in (\lambda_n, \lambda_{n+1}]$, it follows that

$$\left|\int_{\lambda_n}^{x} \exp\{ity\}\,\mathrm{d}\alpha(y)\right| \le 2M_u(\sigma,F)\mathrm{e}^{-\sigma\lambda_n}\mathrm{e}^{-\sigma\eta} \le 4M_u(\sigma,F)\mathrm{e}^{-\sigma\lambda_n},$$

that is,

$$\mu(\sigma,F) \le 4M_u(\sigma,F).$$

Further, for any real number γ, since $\int_{\lambda_k}^{\infty} \exp\{(\gamma+\mathrm{i}t)y\}\mathrm{d}\alpha(y) = \lim_{b\to+\infty} \int_{\lambda_k}^{b} \exp\{(\gamma+\mathrm{i}t)y\}\mathrm{d}\alpha(y)$, that is,

$$\left|\int_{\lambda_k}^{\infty} \exp\{(\gamma+\mathrm{i}t)y\}\mathrm{d}\alpha(y)\right| = \lim_{b\to+\infty}\left|\int_{\lambda_k}^{b} \exp\{(\gamma+\mathrm{i}t)y\}\mathrm{d}\alpha(y)\right|.$$

Set $I_{j+k}(b;it) = \int_{\lambda_{j+k}}^{b} \exp\{ity\}\,\mathrm{d}\alpha(y)$, $(\lambda_{j+k} < b \le \lambda_{j+k+1})$, then we have $|I_{j+k}(b;it)| \le A^*_{j+k}$. Thus, it follows that

$$\left|\int_{\lambda_k}^{b} \exp\{(\gamma+\mathrm{i}t)y\}\mathrm{d}\alpha(y)\right|$$
$$= \left|\sum_{j=k}^{n+k-1} \int_{\lambda_j}^{\lambda_{j+1}} \exp\{\gamma y\} d_y I_j(y;\mathrm{i}t) + \int_{\lambda_{n+k}}^{b} \exp\{\gamma y\} d_y I_{n+k}(y;\mathrm{i}t)\right|$$
$$= \left|\left[\sum_{j=k}^{n+k-1} \mathrm{e}^{\lambda_{j+1}\gamma} I_j(\lambda_{j+1};\mathrm{i}t) - \gamma \int_{\lambda_j}^{\lambda_{j+1}} \mathrm{e}^{\gamma y} I_j(y;\mathrm{i}t)\,\mathrm{d}y\right]\right.$$
$$\left. + \mathrm{e}^{\gamma b} I_{n+k}(b;\mathrm{i}t) - \gamma \int_{\lambda_{n+k}}^{b} \mathrm{e}^{\gamma y} I_j(y;\mathrm{i}t)\,\mathrm{d}y\right|$$
$$\le \sum_{j=k}^{n+k-1} \left[A^*_j \mathrm{e}^{\lambda_{j+1}\gamma} + A^*_j(\mathrm{e}^{\lambda_{j+1}\gamma} - \mathrm{e}^{\lambda_j\gamma})\right] + 2\,\mathrm{e}^{\gamma\lambda_{n+k+1}} A^*_{n+k} - \mathrm{e}^{\gamma\lambda_{n+k}} A^*_{n+k}$$
$$\le 2\sum_{j=k}^{n+k} A^*_n \mathrm{e}^{\lambda_{n+1}\gamma}.$$

When $n \to \infty$, we have $b \to \infty$, thus we have:

$$\left|\int_{\lambda_k}^{\infty} \exp\{(\gamma+\mathrm{i}t)y\}\mathrm{d}\alpha(y)\right| \le 2\sum_{n=k}^{+\infty} A^*_n \exp\{\gamma\lambda_{n+1}\}. \qquad \square$$

3. Proofs of Theorems 1.1 and 1.2

3.1. The proof of Theorem 1.1

Let

$$C_n = E_{n-1}(F,\alpha)\exp(-\alpha\lambda_n), \qquad (n=1,2,\cdots).$$

We first prove the necessity of Theorem 1.1. If $\limsup_{\sigma\to 0^-}\frac{\log^+ M_u(\sigma,F)}{U(-\frac{1}{\sigma})}=1$, then for $\forall\varepsilon>0$, there exists $\sigma_0<0$ such that

$$\log^+ M_u(\sigma,f)<(1+\varepsilon)U(-\frac{1}{\sigma}), \quad as \quad \sigma_0<\sigma<0. \tag{13}$$

Since $F(s)\in L$, thus, for any constant $\alpha(-\infty<\alpha<0)$, we have $F(s)\in\overline{L}_\alpha$. For $\alpha<\sigma<0$, it follows from the definitions of $E_n(F,\alpha)$ and p_n that

$$\begin{aligned} E_n(F,\alpha) &\le \| F-p_n \|_\alpha \le |F(\alpha+\mathrm{i}t)-p_n(\alpha+\mathrm{i}t)| \\ &\le \left|\int_0^{+\infty}\exp\{(\alpha+\mathrm{i}t)y\}\mathrm{d}\alpha(y)-\int_0^{\lambda_n}\exp\{(\alpha+\mathrm{i}t)y\}\mathrm{d}\alpha(y)\right| \\ &= \left|\int_{\lambda_n}^{\infty}\exp\{(\alpha+\mathrm{i}t)y\}\mathrm{d}\alpha(y)\right|. \end{aligned} \tag{14}$$

Thus, from the definition of A_n^* and $M_u(\sigma,F)$, and by Lemma 2.6, we have $A_n^*\le 4M_u(\sigma,F)\mathrm{e}^{-\sigma\lambda_n}$ for any $\sigma(\alpha<\sigma<0)$; it follows from (14) and Lemma 2.6 that

$$E_n(F,\alpha)\le 2\sum_{k=n+1}^{\infty}A_{k-1}^*\exp\{\alpha\lambda_k\}\le 8M_u(\sigma,F)\sum_{k=n+1}^{\infty}\exp\{(\alpha-\sigma)\lambda_k\}. \tag{15}$$

From (4), taking h' $(0<h'<h)$ such that $(\lambda_{n+1}-\lambda_n)\ge h'$ for $n\ge 0$, it follows from (15), for $\sigma\ge\frac{\alpha}{2}$, that

$$\begin{aligned} E_n(F,\alpha) &\le 8M_u(\sigma,F)\exp\{\lambda_{n+1}(\alpha-\sigma)\}\sum_{k=n+1}^{\infty}\exp\{(\lambda_k-\lambda_{n+1})(\alpha-\sigma)\} \\ &\le 8M_u(\sigma,F)\exp\{\lambda_{n+1}(\alpha-\sigma)\}\exp\{-\frac{\alpha}{2}h'(n+1)\}\sum_{k=n+1}^{\infty}(\exp\{\frac{\alpha}{2}h'k\}) \\ &= 8M_u(\sigma,F)\exp\{\lambda_{n+1}(\alpha-\sigma)\}\left(1-\exp\{\frac{\alpha}{2}h'\}\right)^{-1}, \end{aligned}$$

that is,

$$E_{n-1}(F,\alpha)\le KM_u(\sigma,F)\exp\{\lambda_n(\alpha-\sigma)\}, \tag{16}$$

where K is a constant. Thus, for any constant α $(-\infty<\alpha<0)$, it follows from (16) that

$$\log^+ C_n<(1+\varepsilon)U(-\frac{1}{\sigma})-\sigma\lambda_n+\log K. \tag{17}$$

By Lemma 2.1, there exists a positive integer $N_1\in N_+$, for any positive integer $n>N_1$, such that

$$\sigma=-\left[\frac{(a\rho)^{\frac{1}{\rho+1}}}{W(\lambda)}\right](1+o(1)),$$

then we have

$$\log^+ C_n\le(1+\varepsilon)^{\frac{1}{\rho+1}}\frac{\rho+1}{\rho^{\frac{\rho}{\rho+1}}}\frac{\lambda_n}{W(\lambda_n)}(1+\varepsilon)=\left(\frac{(1+\rho)^{1+\rho}}{\rho^\rho}(1+\varepsilon)\right)^{\frac{1}{\rho+1}}\frac{\lambda_n}{W(\lambda_n)}(1+\varepsilon),$$

that is,

$$W(\lambda_n) \le \frac{\lambda_n}{\log^+ C_n}\left(\frac{(1+\rho)^{1+\rho}}{\rho^\rho}(1+\varepsilon)\right)^{\frac{1}{\rho+1}}(1+\varepsilon).$$

Since $W(x)$ is a monotonous and increasing function for $x > x_0 = r_0' U(r_0')$, thus we have:

$$\begin{aligned}\lambda_n &\le \frac{\lambda_n}{\log^+ C_n}\left(\frac{(1+\rho)^{1+\rho}}{\rho^\rho}(1+\varepsilon)\right)^{\frac{1}{\rho+1}}(1+\varepsilon)U\left(\frac{\lambda_n}{\log^+ C_n}\left(\frac{(1+\rho)^{1+\rho}}{\rho^\rho}(1+\varepsilon)\right)^{\frac{1}{\rho+1}}(1+\varepsilon)\right)\\ &\le \frac{\lambda_n}{\log^+ C_n}\left(\frac{(1+\rho)^{1+\rho}}{\rho^\rho}(1+\varepsilon)\right)(1+\varepsilon)^{\rho+1}(1+o(1))U\left(\frac{\lambda_n}{\log^+ C_n}\right).\end{aligned}$$

Then it follows

$$\frac{\log^+ C_n}{U\left(\frac{\lambda_n}{\log^+ C_n}\right)} \le \frac{(1+\rho)^{1+\rho}}{\rho^\rho}(1+\varepsilon)^{\rho+2}(1+o(1)).$$

Hence we have:

$$\limsup_{n\to+\infty}\frac{\log^+ C_n}{U\left(\frac{\lambda_n}{\log^+ C_n}\right)} \le \frac{(1+\rho)^{1+\rho}}{\rho^\rho}. \tag{18}$$

Next, we prove that the following inequality

$$\limsup_{n\to+\infty}\frac{\log^+ C_n}{U\left(\frac{\lambda_n}{\log^+ C_n}\right)} < \frac{(1+\rho)^{1+\rho}}{\rho^\rho} \tag{19}$$

can not hold by using the method of reduction of absurdity.

If

$$\limsup_{n\to+\infty}\frac{\log^+ C_n}{U\left(\frac{\lambda_n}{\log^+ C_n}\right)} = \beta < \frac{(1+\rho)^{1+\rho}}{\rho^\rho}, \tag{20}$$

thus for any $\varepsilon > 0$ such that $\beta + 3\varepsilon < \frac{(1+\rho)^{1+\rho}}{\rho^\rho}$, there exists a positive integer $N_2 \in \mathbb{N}_+$ such that, for $n > N_2$, we have

$$\log^+ C_n < (\beta+\varepsilon)U\left(\frac{\lambda_n}{\log^+ C_n}\right);$$

then it follows

$$\lambda_n < (\beta+\varepsilon)\frac{\lambda_n}{\log^+ C_n}U\left(\frac{\lambda_n}{\log^+ C_n}\right).$$

Since $r = W(t)$ and $t = rU(r)$ are two reciprocally inverse and monotone increasing functions, it follows that

$$W(\frac{\lambda_n}{\beta+\varepsilon}) \le \frac{\lambda_n}{\log^+ C_n},$$

that is,

$$\log^+ C_n \le \frac{\lambda_n}{W\left(\frac{\lambda_n}{\beta+\varepsilon}\right)}.$$

So, there exists a constant $c > 0$ such that

$$C_n \le c\exp\left[\frac{\lambda_n}{W\left(\frac{\lambda_n}{\beta+\varepsilon}\right)}\right], \quad n = 0, 1, 2\cdots. \tag{21}$$

For any $\alpha < 0$, from the definition of $E_k(F,\alpha)$, there exists $p_1 \in \Pi_{n-1}$ such that

$$\|F - p_1\| \le 2E_{n-1}(F,\alpha). \tag{22}$$

And since

$$\begin{aligned}A_n^* \exp\{\alpha\lambda_n\} &= \sup_{\lambda_n < x \le \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{x} \exp\{ity\}\,\mathrm{d}\alpha(y) \right| \exp\{\alpha\lambda_n\} \\ &\le \sup_{\lambda_n < x \le \lambda_{n+1}, -\infty < t < +\infty} \left| \int_{\lambda_n}^{x} \exp\{(\alpha + \mathrm{i}t)y\}\mathrm{d}\alpha(y) \right| \\ &\le \sup_{-\infty < t < +\infty} \left| \int_{\lambda_n}^{\infty} \exp\{(\alpha + \mathrm{i}t)y\}\mathrm{d}\alpha(y) \right|,\end{aligned}$$

thus for any $p \in \Pi_{n-1}$, it follows

$$A_n^* \exp\{\alpha\lambda_n\} \le |F(\alpha + \mathrm{i}t) - p(\alpha + \mathrm{i}t)| \le \|F - p\|_\alpha. \tag{23}$$

Hence, from (22) and (23), for any $\alpha < 0$ and $F(s) \in L$, we have

$$A_n^* \exp\{\alpha\lambda_n\} \le 2E_{n-1}(F, \alpha). \tag{24}$$

Constructing Dirichlet series

$$f_\alpha(s) = \sum_{n=1}^{\infty} [E_{n-1}(F, \alpha)\mathrm{e}^{-\alpha\lambda_n}]\mathrm{e}^{s\lambda_n} = \sum_{n=1}^{\infty} C_n \mathrm{e}^{s\lambda_n},$$

where $\alpha < 0$ and $\{\lambda_n\}$ satisfy (2), (4) and (6). By using the Valiron–Knopp–Bohr formula of the associated abscissas of uniform convergence of Dirichlet series, it follows from (16) and (24) that $\sigma_u^{f_\alpha} = 0$, where $\sigma_u^{f_\alpha}$ is called the abscissas of uniform convergence of Dirichlet series $f_\alpha(s)$.

By Lemma 2.4 and Lemma 2.5, it follows from (16) and (24) that $\rho(f_\alpha(s)) = \rho$. Since

$$M(\sigma, f_\alpha) \le c \sup_{n \ge 1} \left\{ \exp\left[\frac{\lambda_n}{W\left(\frac{\lambda_n}{\beta+\varepsilon}\right)} + \lambda_n(1+\varepsilon)\sigma \right] \right\} \sum_{n=1}^{\infty} \exp\{-\varepsilon\sigma\lambda_n\}, \tag{25}$$

it follows from (21), (25) and Lemma 2.2 that

$$M(\sigma, f_\alpha) \le c \exp\left\{ (\beta + \varepsilon)(1 + o(1))U\left(-\frac{1}{(1+\varepsilon)\sigma}\right) \right\} O\left(U(-\frac{1}{\varepsilon\sigma}) \right). \tag{26}$$

From (24), we have

$$\begin{aligned}M_u(\sigma, F) &\le \left| \sum_{n=1}^{\infty} A_n^* \mathrm{e}^{\sigma\lambda_n} \right| \le A_0^* + 2\sum_{n=1}^{\infty} E_{n-1}(F, \alpha) \exp\{(\sigma - \alpha)\lambda_n\} \\ &= A_0^* + 2M(\sigma, f_\alpha),\end{aligned} \tag{27}$$

it follows from (9) and (27) that

$$\log^+ M_u(\sigma, F) \le (\beta + 3\varepsilon)(1 + o(1))U(-\frac{1}{\sigma}),$$

that is,

$$\limsup_{\sigma \to 0^-} \frac{\log^+ M_u(\sigma, F)}{U(-\frac{1}{\sigma})} \le \beta < \frac{(1+\rho)^{1+\rho}}{\rho^\rho}.$$

This is a contradiction. Thus, the necessity of Theorem 1.1 is completed.

The sufficiency of Theorem 1.1 can be easily proved in a fashion similar to the proof of the necessity.

3.2. The proof of Theorem 1.2

We first prove the sufficiency of Theorem 1.2. From the condition (i), (ii) of Theorem 1.2, for any $\varepsilon \in (0,1)$ and for sufficiently large ν, we have

$$\log^+ C_{n_\nu} > (1-\varepsilon)\frac{(1+\rho)^{1+\rho}}{\rho^\rho}U\left(\frac{\lambda_{n_\nu}}{\log^+ C_{n_\nu}}\right),$$

i.e.,

$$\frac{\rho^\rho}{(1+\rho)^{1+\rho}}\frac{\lambda_{n_\nu}}{1-\varepsilon} > \frac{\lambda_{n_\nu}}{\log^+ C_{n_\nu}}U\left(\frac{\lambda_{n_\nu}}{\log^+ C_{n_\nu}}\right).$$

Since $r = W(t)$ and $t = rU(r)$ are two reciprocally inverse functions and monotone increasing functions, then we can get

$$W\left(\frac{\lambda_{n_\nu}}{(1-\varepsilon)\frac{(1+\rho)^{1+\rho}}{\rho^\rho}}\right) > \frac{\lambda_{n_\nu}}{\log^+ C_{n_\nu}},$$

i.e.,

$$\log^+ C_{n_\nu} > \frac{\lambda_{n_\nu}}{W\left(\frac{\lambda_{n_\nu}}{(1-\varepsilon)\frac{(1+\rho)^{1+\rho}}{\rho^\rho}}\right)}.$$

We take a positive real sequence $\{\sigma_\nu\}$ satisfying

$$\lambda_{n_\nu} = \left(\frac{\rho}{\rho+1}\right)^{\rho+1}(1-\varepsilon)\frac{(1+\rho)^{1+\rho}}{\rho^\rho}(-\frac{1}{\sigma_\nu})U\left(-\frac{1}{\sigma_\nu}\right)(1+\varepsilon) = \rho(1-\varepsilon^2)(-\frac{1}{\sigma_\nu})U\left(-\frac{1}{\sigma_\nu}\right).$$

From Lemma 2.3, we have $\sigma_\nu \downarrow 0$, then for any sufficiently small $\sigma < 0$, there exists $\nu \in N_+$ such that $\sigma_\nu \le \sigma \le \sigma_{\nu+1}$. By Lemma 2.2, it follows from (16) that

$$\begin{aligned}
\log^+ M_u(\sigma, F) &\ge \log^+ C_{n_\nu} + \lambda_{n_\nu}\sigma - \log K \ge \log^+ C_{n_\nu} + \lambda_{n_\nu}\sigma_\nu - \log K \\
&\ge \frac{\lambda_{n_\nu}}{W\left(\frac{\lambda_{n_\nu}}{(1-\varepsilon)\frac{(1+\rho)^{1+\rho}}{\rho^\rho}}\right)} + \lambda_{n_\nu}\sigma_\nu - \log K \\
&= (1-\varepsilon)(1+o(1))U\left(-\frac{1}{\sigma_\nu}\right) = (1+o(1))\frac{\sigma_\nu}{\sigma_{\nu+1}}U\left(-\frac{1}{\sigma_{\nu+1}}\right) \\
&\ge (1+o(1))U\left(-\frac{1}{\sigma_{\nu+1}}\right) \ge (1+o(1))U\left(-\frac{1}{\sigma}\right),
\end{aligned}$$

that is,

$$\liminf_{\sigma\to 0^-}\frac{\log^+ M_u(\sigma, F)}{U(-\frac{1}{\sigma})} \ge 1.$$

Combining Theorem 1.1, we get

$$\lim_{\sigma\to 0^-}\frac{\log^+ M_u(\sigma, F)}{U(-\frac{1}{\sigma})} = 1.$$

We prove the necessity of the Theorem 1.2 in the following.

If $\lim\limits_{\sigma\to 0^-}\frac{\log^+ M_u(\sigma,F)}{U(-\frac{1}{\sigma})} = 1$, by Theorem 1.1, we can easily get (i) of Theorem 1.2. Then we will prove (ii) of Theorem 1.2 in the following. We take a positive decreasing sequence $\{\varepsilon_i\}$ $(0 < \varepsilon_i < 1)$, $\varepsilon_i \to 0 (i \to \infty)$. Let

$$E_i = \left\{ n : \frac{\log^+ C_n}{U\left(\frac{\lambda_n}{\log^+ C_n}\right)} > \frac{(1+\rho)^{1+\rho}}{\rho^\rho} - \varepsilon_i \right\}, \tag{28}$$

it follows that $\forall i, E_i \ne \Phi$ and $E_i \subset E_{i-1}$. For each i, we arrange the $n (\in E_i)$ in an increasing sequence $\{n_\nu^{(i)}\}_{\nu=1}^\infty$, then we consider the two cases in the following.

Case 1. Suppose that $\lim_{\nu\to+\infty}\frac{\lambda_{n^{(i)}_{\nu+1}}}{\lambda_{n^{(i)}_{\nu}}}=1$ for any i. Then there exists $N_i\in E_i (i\in N_+)$, such that, when $n^{(i)}_\nu\ge N_i$, we have

$$\frac{\lambda_{n^{(i)}_{\nu+1}}}{\lambda_{n^{(i)}_{\nu}}}\le 1+\varepsilon_k. \tag{29}$$

Note $E_{i+1}\subset E_i$, take $N_{i+1}>N_i$, denote E'_i the subset of E_i,

$$E'_i=\{n\in E_i : N_i\le n\le N_{i+1}\},$$

thus the elements of E'_i satisfy (28) and (29).

Therefore, let $E=\bigcup_{i=1}^{\infty}E'_i$ and arrange the $n(\in E'_i)$ in an increasing sequence $\{n_\nu\}$; (ii) is proved.

Case 2. If there exists $i\in N_+$ satisfying $\lim_{\nu\to+\infty}\frac{\lambda_{n^{(i)}_{\nu+1}}}{\lambda_{n^{(i)}_{\nu}}}\ne 1$, then, since $\lambda_{n^{(i)}_{\nu+1}}>\lambda_{n^{(i)}_{\nu}}$, we get $\lim_{\nu\to+\infty}\frac{\lambda_{n^{(i)}_{\nu+1}}}{\lambda_{n^{(i)}_{\nu}}}>1$. Hence, there exists $\{n^{(i)}_{\nu_k}\}\subseteq\{n^{(i)}_\nu\}$ (still marked with $\{n^{(i)}_\nu\}$) and $\delta\in(0,\frac{1}{2}(1+\frac{1}{\rho})^{-\rho})$, from which it follows that

$$\frac{\lambda_{n^{(i)}_{\nu+1}}}{\lambda_{n^{(i)}_{\nu}}}>1+\delta.\quad \nu=1,2,\cdots.$$

Let

$$n'_1=n^{(i)}_1, n'_2=n^{(i)}_3,\cdots,n'_\nu=n^{(i)}_{2\nu-1},\cdots$$
$$n''_1=n^{(i)}_1, n''_2=n^{(i)}_4,\cdots,n''_\nu=n^{(i)}_{2\nu},\cdots,$$

where $\{n'_\nu\}$, $\{n''_\nu\}$ are two increasing positive integer sequences, and

$$n''_\nu<n'_{\nu+1},\quad \lambda_{n''_\nu}>(1+\delta)\lambda_{n'_\nu},\quad \nu=1,2,\cdots.$$

Take $\gamma=\frac{1}{2}\varepsilon_i>0$ and from (28), for any sufficiently large ν, when $n\notin E_i$ satisfies $n'_\nu<n<n''_\nu$, we can get

$$\frac{\log^+ C_n}{U\left(\frac{\lambda_n}{\log^+ C_n}\right)}\le\frac{(1+\rho)^{1+\rho}}{\rho^\rho}-\varepsilon_i<\frac{(1+\rho)^{1+\rho}}{\rho^\rho}-\gamma,$$

thus, by using the same argument as in Theorem 1.1, we have

$$\log^+ C_n<\frac{\lambda_n}{W\left(\frac{\lambda_n}{\frac{(1+\rho)^{1+\rho}}{\rho^\rho}-\gamma}\right)},$$

and it follows from (24) that

$$\log^+(A^*_n\exp\{\lambda_n\sigma\})<\frac{\lambda_n}{W\left(\frac{\lambda_n}{\frac{(1+\rho)^{1+\rho}}{\rho^\rho}-\gamma}\right)}+\lambda_n\sigma.$$

Since $\sigma\to 0^-$, it follows by Lemma 2.2 that

$$\log^+(A^*_n\exp\{\lambda_n\sigma\})\le\left(\frac{(1+\rho)^{1+\rho}}{\rho^\rho}-\gamma\right)(1+o(1))U\left(-\frac{1}{\sigma}\right),\quad n'_\nu<n<n''_\nu. \tag{30}$$

Take $\mu>0$ and

$$\frac{1+\mu}{1+\delta}<1-\eta,\quad 0<\eta<1.$$

Let $\sigma_\nu=-\left[W\left(\frac{\lambda_{n''_\nu}}{\frac{(1+\rho)^{1+\rho}}{\rho^\rho}(1+\mu)}\right)\right]^{-1}$, then we have $\sigma_\nu\downarrow 0$ and

$$\lambda_{n''_\nu}=(1+\mu)\frac{(1+\rho)^{1+\rho}}{\rho^\rho}(-\frac{1}{\sigma_\nu})U\left(-\frac{1}{\sigma_\nu}\right). \tag{31}$$

For the above $\mu>0$ and from Theorem 1.2(i), there exists a positive integer $n_0\in N_+$ such that

$$\log^+(A_n^*\exp\{\lambda_n\sigma\}) < \frac{\lambda_n}{W\left(\frac{\lambda_n}{\frac{(1+\rho)^{1+\rho}}{\rho^\rho}(1+\mu)}\right)} + \lambda_n\sigma. \quad n \ge n_0. \tag{32}$$

When $n \ge n''_\nu > n_0$, then $\lambda_n \ge \lambda_{n''_\nu}$. Since $W(t)$ is a increasing function, from (31) and (32), we have

$$\log^+(A_n^*\exp\{\lambda_n\sigma_\nu\}) < \lambda_n(\frac{1}{W\left(\frac{\lambda_{n''_\nu}}{\frac{(1+\rho)^{1+\rho}}{\rho^\rho}(1+\mu)}\right)} + \sigma_\nu) = 0. \tag{33}$$

From Lemma 2.2 and for sufficiently large ν, when $n_0 \le n \le n'_\nu$, it follows that $\lambda_n \le \lambda_{n'_\nu} < \frac{1}{1+\delta}\lambda_{n''_\nu}$, then we have

$$\begin{aligned}\log^+(A_n^*\exp\{\lambda_n\sigma_\nu\}) &\le \frac{\frac{1}{1+\delta}\lambda_{n''_\nu}}{W\left(\frac{\frac{1}{1+\delta}\lambda_{n''_\nu}}{\frac{(1+\rho)^{1+\rho}}{\rho^\rho}(1+\mu)}\right)} + \frac{1}{1+\delta}\lambda_{n''_\nu}\sigma_\nu \\ &= \frac{1+\mu}{1+\delta}\frac{(1+\rho)^{1+\rho}}{\rho^\rho}(-\frac{1}{\sigma_\nu})U\left(-\frac{1}{\sigma_\nu}\right)\left[\frac{1}{W\left(\frac{-1}{(1+\delta)\sigma_\nu}U\left(-\frac{1}{\sigma_\nu}\right)\right)} + \sigma_\nu\right] \\ &\le \frac{1-\eta}{1+o(1)}\frac{(1+\rho)^{1+\rho}}{\rho^\rho}\left[(1+\delta)^{\frac{1}{1+\rho}} - 1 + o(1)\right]U\left(-\frac{1}{\sigma_\nu}\right) \\ &\le \frac{1-\eta}{1+o(1)}\left[\frac{\delta\frac{(1+\rho)^{1+\rho}}{\rho^\rho}}{1+\rho} + o(1)\right]U\left(-\frac{1}{\sigma_\nu}\right) \\ &= \frac{1-\eta}{1+o(1)}\left[\delta(1+\frac{1}{\rho})^\rho + o(1)\right]U\left(-\frac{1}{\sigma_\nu}\right) \\ &\le (1-\eta)(1+o(1))U\left(-\frac{1}{\sigma_\nu}\right),\end{aligned} \tag{34}$$

when $n \ge n_0$, from (30), (33), and (34), we have

$$\log^+(A_n^*\exp\{\lambda_n\sigma_\nu\}) < (1-\beta)(1+o(1))U\left(-\frac{1}{\sigma_\nu}\right), \quad 0 < \beta = \min\{\eta, \gamma\} < 1.$$

Hence we have

$$\mu(\sigma_\nu, F) \le K_1 \exp\left[(1-\beta)(1+o(1))U\left(-\frac{1}{\sigma_\nu}\right)\right], \tag{35}$$

where K_1 is a positive real number.

From (4) and (6), for any $\varepsilon > 0$ we have

$$M_u(\sigma_\nu, F) \le \sum_{n=0}^{\infty} A_n^* e^{\lambda_n\sigma_\nu} \le \mu((1-\varepsilon)\sigma_\nu, F)\sum_{n=0}^{\infty} e^{-\varepsilon\sigma_\nu\lambda_n} \le K_2\mu((1+\varepsilon)\sigma_\nu, F)(-\frac{1}{\sigma_\nu}),$$

it follows from (30) that

$$M_u(\sigma_\nu, F) \le K_2 \exp\left[(1-\beta)(1+o(1))U\left(-\frac{1}{(1+\varepsilon)\sigma_\nu}\right)\right]\left(-\frac{1}{\sigma_\nu}\right),$$

where K_1, K_2 are constants.

Therefore, when ν is sufficiently large, we have

$$\begin{aligned}\log^+ M_u(\sigma_\nu, F) &\le (1-\beta)(1+o(1))U\left(-\frac{1}{\sigma_\nu}\right) + \log^+\left(-\frac{1}{\sigma_\nu}\right) + K_3 \\ &\le (1-\frac{\beta}{2})(1+o(1))U(-\frac{1}{\sigma_\nu}),\end{aligned}$$

where K_3 is a constant.

Thus, it follows

$$\limsup_{\nu\to+\infty}\frac{\log^{+}M_u(\sigma_\nu,F)}{U\left(-\frac{1}{\sigma_\nu}\right)}\le 1-\frac{\beta}{2}.$$

This is contradictory with the condition of Theorem 1.2. Then the necessity of Theorem 1.2 is proved.

Therefore, we complete the proof of Theorem 1.2.

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