



Differential geometry/Mathematical physics

Twist star products and Morita equivalence

*Produit étoile déformé et équivalence de Morita*

Francesco D'Andrea, Thomas Weber

Università di Napoli "Federico II" and I.N.F.N. Sezione di Napoli, Complesso MSA, Via Cintia, 80126 Napoli, Italy

ARTICLE INFO

Article history:

Received 17 August 2017

Accepted after revision 23 October 2017

Available online 6 November 2017

Presented by the Editorial Board

ABSTRACT

We present a simple no-go theorem for the existence of a deformation quantization of a homogeneous space M induced by a Drinfel'd twist: we argue that equivariant line bundles on M with non-trivial Chern class and symplectic twist star products cannot both exist on the same manifold M . This implies, for example, that there is no symplectic star product on the projective space $\mathbb{C}P^{n-1}$ induced by a twist based on $U(\mathfrak{gl}_n(\mathbb{C}))[[\hbar]]$ or any subalgebra, for every $n \geq 2$.

© 2017 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

R É S U M É

Nous exposons un théorème de non-existence concernant la quantification par déformation d'un espace homogène M , induite par un twist de Drinfel'd : nous montrons qu'un fibré en droites équivariant sur M avec une classe de Chern non triviale et un produit étoile symplectique ne peuvent coexister sur une même variété M . Ceci implique, par exemple, qu'il n'y a pas de produit étoile symplectique sur l'espace projectif complexe induit par un twist basé sur $U(\mathfrak{gl}_n(\mathbb{C}))[[\hbar]]$, ou sur toute sous-algèbre, pour tout $n \geq 2$.

© 2017 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

Drinfel'd twists [15] are powerful functorial tools to simultaneously deform bialgebras together with all of their modules and module algebras. Given an action of a Lie algebra \mathfrak{g} on a smooth manifold M by derivations, one can use a twist based on $U(\mathfrak{g})[[\hbar]]$ to obtain a formal deformation quantization of M (Definition 1). A star product obtained by a twist will be called a *twist star product*.

The idea of quantization induced by symmetries has always been appealing in mathematical physics. The approach to quantization via Drinfel'd twist was popularized by several mathematical physicists, among which Aschieri, Dimitrijevic, Fiore, Lizzi, Meyer, Vitale, Wess and many others (see, e.g., [1–4,19] and references therein), and its interest is testified by the number of papers on the subject.

E-mail addresses: francesco.dandrea@unina.it (F. D'Andrea), thomas.weber@unina.it (T. Weber).

The existence of star products for arbitrary Poisson manifolds is a celebrated result by Kontsevich [22], who improved previous results by DeWilde and Lecomte [14], Omori, Maeda and Yoshioka [25], and Fedosov [18], who proved the existence in the symplectic case (for a historical account see, e.g., the recent review [31]). While every Poisson manifold admits a deformation quantization, it is not clear when a manifold admits a deformation quantization by Drinfel'd twist. If M is a compact and connected symplectic manifold, we know that it admits a deformation quantization by a twist based on $U(\mathfrak{g})[[\hbar]]$ only if it is a homogeneous space [5, Thm. 1]. This condition unfortunately is only necessary, not sufficient: the symplectic 2-sphere is an example of homogeneous space that admits no deformation quantization induced by a twist [5, Cor. 3.12].

In this short note, we comment on how formal Morita equivalence can be used to prove a “no-go” theorem for the existence of twist star products. We will prove the following theorem.

Theorem. *Let G be a Lie group, \mathfrak{g} its Lie algebra, M a homogeneous G -space and ω a symplectic form on M . The following two properties are mutually exclusive:*

- (i) *there exists a G -equivariant smooth complex line bundle on M with non-trivial Chern class;*
- (ii) *there exists a deformation quantization of (M, ω) induced by a twist based on $U(\mathfrak{g})[[\hbar]]$.*

Using this theorem, one can show, for example, that there is no deformation quantization of a symplectic projective space $\mathbb{C}P^{n-1}$ induced by a twist based on $U(\mathfrak{gl}_n(\mathbb{C}))[[\hbar]]$ or any sub-bialgebra (for every $n \geq 2$). The symplectic 2-torus, on the other hand, provides an example where symplectic twist star products exist, and non-trivial \mathbb{R}^2 -equivariant line bundles do not.

It is worth noticing that if one works in the more general setting of bialgebroids, then any deformation quantization of any Poisson manifold (not only symplectic) is induced by a twist [32]: one can indeed interpret the formal Poisson bivector as a cocycle twist based on the topological bialgebroid of formal power series of differential operators on M (with base algebra $C^\infty(M)[[\hbar]]$).

Notations

In the following, algebras will be either over the field \mathbb{C} of complex numbers or over the ring $\mathbb{C}[[\hbar]]$ of formal power series in \hbar with complex coefficients. They are always assumed to be unital and associative. If V is a complex vector space, we denote by $V[[\hbar]]$ the $\mathbb{C}[[\hbar]]$ -module of formal power series in \hbar with coefficients in V .

2. Deformation quantization and Morita equivalence

Let us recall some basic definitions.

Definition 1 (*Star product*). A star product on a Poisson manifold M is a $\mathbb{C}[[\hbar]]$ -bilinear associative binary operation $*$ on $C^\infty(M)[[\hbar]]$ of the form:

$$f * g = \sum_{k=0}^{\infty} \hbar^k B_k(f, g), \quad \forall f, g \in C^\infty(M),$$

where each $B_k : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ is a bi-differential operator, $B_0(f, g) = fg$ is the pointwise multiplication,

$$B_1(f, g) - B_1(g, f) = i\{f, g\},$$

is the Poisson bracket, and the constant function 1 is the neutral element of this product:

$$f * 1 = 1 * f = f, \quad \forall f \in C^\infty(M).$$

The algebra $(C^\infty(M)[[\hbar]], *)$ is called a *deformation quantization* of the Poisson manifold M .

If M is a symplectic manifold, we call $*$ a *symplectic star product* (for short). From now on, we will be interested in the symplectic case.

Definition 2 (*Equivalence*). Two star products $*$ and $'$ on the same symplectic manifold M are *equivalent* if there are linear maps $T_k : C^\infty(M) \rightarrow C^\infty(M)$ such that $T := \text{Id} + \sum_{k=1}^{\infty} \hbar^k T_k$ satisfies $T(1) = 1$ and:

$$T(f * g) = T(f) *' T(g), \quad \forall f, g \in C^\infty(M).$$

The maps T_k in [Definition 2](#) are automatically differential operators [[21, Thm. 2.22](#)].

The map T gives an isomorphism of algebras $(C^\infty(M)[[h]], *) \rightarrow (C^\infty(M)[[h]], *')$ by extension of scalars, but notice that not every isomorphism is of this form. Any isomorphism that is continuous in the h -adic topology can be written as a combination of a change of parameter, an equivalence T as above, and the pullback of a symplectomorphism [[21, Prop. 9.4](#)].

A weaker notion is that of Morita equivalence of star products, inspired by the notion of (algebraic) Morita equivalence.

Definition 3 (Morita equivalence). Given two rings (resp. two algebras) A and B , a Morita equivalence A - B bimodule is a finitely generated projective right B -module N equipped with a ring (resp. algebra) isomorphism $\phi : A \rightarrow \text{End}_B(N)$. If such a module exists, we say that A and B are *Morita equivalent*.

One can see, e.g., [[23, §18](#)] for alternative equivalent definitions of Morita's equivalence.

Let $L \rightarrow M$ be a smooth complex line bundle, M a symplectic manifold, and $*$ a star product on M . The space $\Gamma^\infty(L)$ of smooth sections is a symmetric $C^\infty(M)$ -bimodule, with left and right module structure given by pointwise multiplication, and by extension of scalars $\Gamma^\infty(L)[[h]]$ is a $C^\infty(M)[[h]]$ -bimodule. It was proved in [[10](#)] (in the more general setting of formal deformations of \mathbb{C} -algebras and projective modules) that $\Gamma^\infty(L)[[h]]$ can be deformed into a right module for the algebra $(C^\infty(M)[[h]], *)$, i.e. there is a $\mathbb{C}[[h]]$ -bilinear map:

$$\bullet : \Gamma^\infty(L)[[h]] \times C^\infty(M)[[h]] \rightarrow \Gamma^\infty(L)[[h]] ,$$

unique modulo equivalences, such that

$$(s \bullet f) \bullet g = s \bullet (f \bullet g) , \quad s \bullet 1 = s , \quad s \bullet f = sf \pmod h ,$$

for all $f, g \in C^\infty(M)$, $s \in \Gamma^\infty(L)$. The line bundle determines a second star product $*'$ on M , unique modulo equivalences, such that there is an isomorphism of $\mathbb{C}[[h]]$ -algebras:

$$\phi : (C^\infty(M)[[h]], *') \rightarrow \text{End}_{(C^\infty(M)[[h]], *)}(\Gamma^\infty(L)[[h]], \bullet) , \tag{2.1}$$

where the latter is the set of all right module endomorphisms, with product given by composition. The isomorphism can be chosen in such a way that it deforms the action of functions on sections by pointwise multiplication [[9, §4](#)]:

$$\phi(f)s = fs \pmod h , \quad \forall f \in C^\infty(M), s \in \Gamma^\infty(L).$$

We stress that $*$ and $*'$ are deformation quantizations of the same symplectic structure on M [[9, Lemma 3.4](#)]. One can prove that the deformed right module above is projective and finitely generated [[10](#)], so that $(C^\infty(M)[[h]], *)$ and $(C^\infty(M)[[h]], *')$ are Morita equivalent in the ring-theoretic sense.

By [[11, Thm. 3.1](#)] the relative class $t(*, *')$ of the two-star products and the Chern class $c_1(L)$ of the line bundle are proportional: $t(*, *') = 2\pi i c_1(L)$. As a consequence,

Lemma 4. *The star products $*$ and $*'$ are equivalent if and only if $c_1(L) = 0$.*

3. Cocycle twists

We refer to [[24](#)] for general definitions about bialgebras and module algebras.

Definition 5 (Equivariant module). Let U be a \mathbb{C} -bialgebra and A and B two U -module algebras. An A - B bimodule N is called *U -equivariant* if it is equipped with a left action \triangleright of U such that:

$$x \triangleright (a\xi b) = (x_{(1)} \triangleright a)(x_{(2)} \triangleright \xi)(x_{(3)} \triangleright b) , \quad \forall x \in U, a \in A, b \in B, \xi \in N. \tag{3.1}$$

We will use the same symbol \triangleright for the actions of U on A , B and N , when there is no risk of confusion; we will also use the standard Sweedler notation for the coproduct; for example, above $x_{(1)} \otimes x_{(2)} \otimes x_{(3)}$ stands for $(\Delta \otimes \text{Id})\Delta(x)$.

It was shown in a seminal paper by Drinfel'd [[15](#)] that one can modify the coproduct of U by conjugation with an invertible 2-tensor $F \in U \otimes U$, thus getting a quasi-bialgebra. This is an ordinary bialgebra if the coassociator of F commutes with the image of the iterated coproduct. A special case, recalled below, is obtained when the coassociator is trivial. In this case, we talk about *cocycle twist*, since the defining condition can be interpreted as the vanishing of the coboundary of F in a suitable non-abelian cohomology associated with the bialgebra U (see, e.g., [[13, §2.3–2.4](#)] and references therein).

Definition 6 (Cocycle twist). Let U be a \mathbb{C} -bialgebra. An invertible element $F \in U \otimes U$ is called a *cocycle twist* (or simply a *twist*) based on U , if it satisfies:

$$(F \otimes 1)(\Delta \otimes \text{Id})(F) = (1 \otimes F)(\text{Id} \otimes \Delta)(F) \tag{cocycle condition} \tag{3.2a}$$

$$(\varepsilon \otimes \text{Id})(F) = (\text{Id} \otimes \varepsilon)(F) = 1 \tag{counitality} \tag{3.2b}$$

where Δ is the coproduct and ε the counit of U .

Remark 7. We adopt the convention of [12,24]. A different convention is that in [17,20], where (3.2a) is replaced by the condition $(\Delta \otimes \text{Id})(J)(J \otimes 1) = (\text{Id} \otimes \Delta)(J)(1 \otimes J)$, which is satisfied by the inverse $J = F^{-1}$ of any 2-cocycle F .

Cocycle twists have the advantage, over more general Drinfel'd twists, that they can be used to obtain associative deformations of U -module algebras.

Given a bialgebra U and a twist F based on U , we denote by U_F the new bialgebra that is given by U as an algebra, with the same counit, and with coproduct Δ_F given by

$$\Delta_F(x) := F\Delta(x)F^{-1}, \quad \forall x \in U.$$

Let now A and B be two U -module algebras and N a U -equivariant A - B bimodule like in Definition 5. Denote by $m_A : A \otimes A \rightarrow A$ the multiplication map of A , $\lambda_A : A \otimes N \rightarrow N$ the left A -module action, by m_B the multiplication map of B and $\rho_B : N \otimes B \rightarrow N$ the right B -module action.

Let A_F be the algebra given by A as a vector space, with the same unit element and with product:

$$m_{A_F} := m_A \circ F^{-1}.$$

That is $m_{A_F}(a \otimes b) = m_A(F^{-1}(\triangleright \otimes \triangleright)(a \otimes b))$ for all $a, b \in A$. Similarly, let B_F be the algebra given by B as a vector space, with the same unit and with product $m_{B_F} := m_B \circ F^{-1}$. Both A_F and B_F are U_F -module algebras (see, e.g., [24]), w.r.t. the undeformed action \triangleright .

Lemma 8. Let N be a U -equivariant A - B bimodule as in Definition 5. Then N is a U_F -equivariant A_F - B_F bimodule w.r.t. the actions:

$$\begin{aligned} \lambda_{A_F} : A_F \otimes N &\rightarrow N, & \lambda_{A_F}(a \otimes \xi) &:= \lambda_A(F^{-1}(\triangleright \otimes \triangleright)(a \otimes \xi)), \\ \rho_{B_F} : N \otimes B_F &\rightarrow N; & \rho_{B_F}(\xi \otimes b) &:= \rho_B(F^{-1}(\triangleright \otimes \triangleright)(\xi \otimes b)), \end{aligned}$$

for all $a \in A, b \in B, \xi \in N$.

Proof. Note that

$$\lambda_{A_F}(\text{Id} \otimes \lambda_{A_F}) = \lambda_A \circ F^{-1} \circ (\text{Id} \otimes \lambda_A) \circ (\text{Id} \otimes F^{-1}),$$

where we think of F^{-1} as a linear map on tensors, and the action symbol \triangleright is omitted (in this notation $1 \triangleright$ becomes the identity endomorphism Id). The equivariance of the module means that $F^{-1} \circ (\text{Id} \otimes \lambda_A) = (\text{Id} \otimes \lambda_A) \circ (\text{Id} \otimes \Delta)(F^{-1})$. Thus,

$$\begin{aligned} \lambda_{A_F}(\text{Id} \otimes \lambda_{A_F}) &= \lambda_A \circ (\text{Id} \otimes \lambda_A) \circ (\text{Id} \otimes \Delta)(F^{-1}) \circ (\text{Id} \otimes F^{-1}) \\ &= \lambda_A \circ (m_A \otimes \text{Id}) \circ (\Delta \otimes \text{Id})(F^{-1}) \circ (F^{-1} \otimes \text{Id}) \\ &= \lambda_A \circ F^{-1} \circ (m_A \otimes \text{Id}) \circ (F^{-1} \otimes \text{Id}) = \lambda_{A_F}(m_{A_F} \otimes \text{Id}), \end{aligned}$$

where we used the fact that λ_A is a left action, the cocycle property of F , and the module algebra property telling us that $(m_A \otimes \text{Id}) \circ (\Delta \otimes \text{Id})(F^{-1}) = F^{-1} \circ (m_A \otimes \text{Id})$.

From the property $x \triangleright 1_A = \varepsilon(x)1_A \forall x \in U$ and counitality of F , we deduce:

$$\lambda_{A_F}(1 \otimes \xi) = \lambda_A((\varepsilon \otimes \text{Id})(F^{-1})(\triangleright \otimes \triangleright)(1 \otimes \xi)) = \lambda_A(1 \otimes \xi) = \xi,$$

for all $\xi \in N$. The latter two equations prove that λ_{A_F} is a left action. Similarly, one proves that ρ_{B_F} is a right action, and that these left and right actions commute, that is

$$\lambda_{A_F}(\text{Id} \otimes \rho_{B_F}) = \rho_{B_F}(\lambda_{A_F} \otimes \text{Id}),$$

as linear maps $A \otimes N \otimes B \rightarrow N$. Finally, thinking of $x \in U$ as a linear map and omitting the action symbol \triangleright , one finds:

$$\begin{aligned} x \circ \lambda_{A_F} \circ (\text{Id} \otimes \rho_{B_F}) &= x \circ \lambda_A \circ F^{-1} \circ (\text{Id} \otimes \rho_B) \circ (\text{Id} \otimes F^{-1}) \\ &= \lambda_A \circ \Delta(x) \circ F^{-1} \circ (\text{Id} \otimes \rho_B) \circ (\text{Id} \otimes F^{-1}) \\ &= \lambda_A \circ F^{-1} \circ (F\Delta(x)F^{-1}) \circ (\text{Id} \otimes \rho_B) \circ (\text{Id} \otimes F^{-1}) \\ &= \lambda_A \circ F^{-1} \circ (\text{Id} \otimes \rho_B) \circ ((\text{Id} \otimes \Delta)(F)(\text{Id} \otimes \Delta)\Delta(x)(\text{Id} \otimes \Delta)(F^{-1})) \circ (\text{Id} \otimes F^{-1}) \\ &= \lambda_{A_F} \circ (\text{Id} \otimes \rho_{B_F}) \circ (\text{Id} \otimes \Delta_F)\Delta_F(x), \end{aligned}$$

where we used the U -equivariance of λ_A and ρ_B . This proves the U_F -equivariance of the actions λ_{A_F} and ρ_{B_F} . \square

We will denote by N_F the bimodule given by the vector space N with actions λ_{A_F} and ρ_{B_F} given in Lemma 8. Analogous definitions and constructions work for topological bialgebras over the ring $\mathbb{C}[[h]]$, with algebraic tensor products replaced by tensor products completed in the h -adic topology.

4. Twist star products

Let G be a Lie group, \mathfrak{g} its Lie algebra, $\pi : L \rightarrow M$ a G -equivariant line bundle over a smooth manifold M , i.e. both L and M are G -spaces, the action of G commutes with π , and is linear on fibers. Define an action α of G on $f \in C^\infty(M)$ and $s \in \Gamma^\infty(L)$ by

$$\alpha_g f(x) := f(g^{-1}x) ; \quad \alpha_g s(x) := g s(g^{-1}x) ,$$

for all $x \in M$ and $g \in G$. The equivariance condition of π guarantees that $\alpha_g s$ is still a section of L , indeed:

$$\pi(\alpha_g s(x)) = \pi(g s(g^{-1}x)) = g \pi(s(g^{-1}x)) = g(g^{-1}x) = x ,$$

which means that $\pi \circ \alpha_g s = \text{Id}_M$. Note that $\alpha_g(fs) = \alpha_g(f)\alpha_g(s)$ (equivariance condition for grouplike elements of a bialgebra).

By differentiating this action, we get an action of the bialgebra $U(\mathfrak{g})[[\hbar]]$ on $C^\infty(M)[[\hbar]]$ and on $\Gamma^\infty(L)[[\hbar]]$ that turns the latter into a $U(\mathfrak{g})[[\hbar]]$ -equivariant $C^\infty(M)[[\hbar]]$ -bimodule. It is a Morita equivalence bimodule, with isomorphism

$$\psi_0 : C^\infty(M)[[\hbar]] \rightarrow \text{End}_{C^\infty(M)[[\hbar]]}(\Gamma^\infty(L)[[\hbar]]) ,$$

given by pointwise multiplication: $\psi_0(f)s := fs \ \forall f \in C^\infty(M), s \in \Gamma^\infty(L)$.

Given a cocycle twist

$$F = \sum_{k=0}^{\infty} \hbar^k F_k ,$$

based on $U(\mathfrak{g})[[\hbar]]$ (here $F_k \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ for all $k \geq 0$), we can now apply the prescription in §3 and get a deformed multiplication on $A := C^\infty(M)[[\hbar]]$ (note that here, in the notations of §3, we have $A = B$) and a deformed bimodule structure on $N := \Gamma^\infty(L)[[\hbar]]$.

It is worth noticing that $U(\mathfrak{g})[[\hbar]]_F$ is a deformation of the bialgebra $U(\mathfrak{g})[[\hbar]]$ (in the sense, e.g., of [12, Def. 6.1.1])—that is

$$\Delta_F(x) = \Delta(x) \pmod{\hbar} ,$$

for all $x \in U(\mathfrak{g})$ —if and only if F_0 commutes with the image of Δ .

As one can easily check, this implies that $\tilde{F} := FF_0^{-1}$ is a cocycle twist, and $\Delta_{\tilde{F}} = \Delta_F$. Modulo a replacement of F by \tilde{F} , we can then assume that our twist is of the form

$$F = 1 \otimes 1 \pmod{\hbar} . \tag{4.1}$$

A byproduct of condition (4.1) is that $f * g := m_A \circ F^{-1}(\triangleright \otimes \triangleright)(f \otimes g)$ is equal to $fg \pmod{\hbar}$, i.e. a star product according to Definition 1.

As customary, we will include (4.1) in the definition of the formal twist (see, e.g., [17, §9.5]).

Definition 9 (Twist star product). A star product of the form

$$f * g := m \circ F^{-1}(\triangleright \otimes \triangleright)(f \otimes g) \quad \forall f, g \in C^\infty(M)[[\hbar]] , \tag{4.2}$$

with m the pointwise multiplication of $C^\infty(M)[[\hbar]]$ and F a twist satisfying (4.1), will be called *twist star product*.

Proposition 10. Let λ_{A_F} and ρ_{A_F} be the module actions of $A_F = (C^\infty(M)[[\hbar]], *)$ on the set $N = \Gamma^\infty(L)[[\hbar]]$ given in Lemma 8. The map

$$\psi : A_F \rightarrow \text{End}_{A_F}(N_F) , \tag{4.3}$$

from A_F into the algebra of right A_F -linear endomorphisms given by

$$\psi(f)s := \lambda_{A_F}(f \otimes s) , \quad \forall f \in C^\infty(M), s \in \Gamma^\infty(L) ,$$

is an algebra isomorphism.

Proof. Since $\psi(f)s = fs \pmod{\hbar}$, one has $\psi = \psi_0 + O(\hbar)$. Since ψ_0 is an invertible map, ψ is invertible as well (a formal power series is invertible iff its order zero term is invertible). Note that $\lambda_{A_F}(1 \otimes s) = s$, so that ψ is an isomorphism of unital algebras. \square

In the terminology of [9, Def. 4.2], N_F is a *bimodule quantization* of the line bundle L . We can now prove our main theorem. The technique employed is similar to that used in [8, Cor. 6.7] to prove the non-existence of formal deformations of “sufficiently non-trivial” principal bundles.

Theorem 11. *Let G be a Lie group, \mathfrak{g} its Lie algebra, M a homogeneous G -space and ω a symplectic form on M . The following two properties are mutually exclusive:*

- (i) *there exists a G -equivariant smooth complex line bundle on M with non-trivial Chern class;*
- (ii) *there exists a deformation quantization of (M, ω) induced by a twist based on $U(\mathfrak{g})[[\hbar]]$.*

Proof. Let us assume that (i) and (ii) both hold, and show that we arrive at a contradiction. In the notations above, let F be a twist, $*$ a symplectic star product induced by F , $L \rightarrow M$ an equivariant line bundle with $c_1(L) \neq 0$, A_F and N_F like in Proposition 10. The line bundle L induces a second star product $*'$ on M . On the other hand, composing (2.1) with the inverse of (4.3), we get an isomorphism (of unital algebras),

$$T := \psi^{-1} \circ \phi : (C^\infty(M)[[\hbar]], *') \rightarrow (C^\infty(M)[[\hbar]], *),$$

which satisfies $T(f) = f \pmod{\hbar} \forall f \in C^\infty(M)$. It is then an equivalence between $*$ and $*'$, in contradiction with Lemma 4, which states that $*$ and $*'$ are not equivalent. \square

5. Applications

5.1. Complex projective spaces

As a first example, let us consider the complex projective space $\mathbb{C}P^{n-1}$, $n \geq 2$. The tautological line bundle has total space:

$$L := \{(\ell, v) \in \mathbb{C}P^{n-1} \times \mathbb{C}^n : v \in \ell\},$$

where points $\ell \in \mathbb{C}P^{n-1}$ are lines through the origin in \mathbb{C}^n . The bundle map is simply $\pi : L \rightarrow \mathbb{C}P^{n-1}$, $\pi(\ell, v) = \ell$.

The action of $GL_n(\mathbb{C})$ on \mathbb{C}^n by row-by-column multiplication induces an action on $\mathbb{C}P^{n-1}$ (it sends 1-dimensional vector subspaces of \mathbb{C}^n into 1-dimensional vector subspaces); the diagonal action on $\mathbb{C}P^{n-1} \times \mathbb{C}^n$ induces an action on L commuting with π . It is then an equivariant line bundle. Since $c_1(L) \neq 0$, as a corollary of Theorem 11, we obtain the following.

Corollary 12. *There is no symplectic star product on $\mathbb{C}P^{n-1}$ induced by a twist based on $U(\mathfrak{gl}_n(\mathbb{C}))[[\hbar]]$ or any sub-bialgebra.*

Fuzzy projective spaces belong to this class of examples. From a mathematical point of view, fuzzy spaces are strict deformation quantizations of coadjoint orbits of connected compact semisimple Lie groups, obtained via covariant Berezin quantization (see, e.g., [26]). Alternatively, since on any such orbit there is a canonical invariant Kähler structure (see, e.g., [27]), they can also be obtained via Berezin–Toeplitz quantization [28]. It was shown by Schlichenmaier in [29] (see [30] for the original reference in German), using some estimates of [7], that one can associate a natural star product with the Berezin–Toeplitz quantization of any compact Kähler (hence symplectic) manifold, such as $\mathbb{C}P^{n-1}$. Corollary 12 can be applied to such star products on $\mathbb{C}P^{n-1}$ to conclude that they are not induced by a twist based on $U(\mathfrak{gl}_n(\mathbb{C}))[[\hbar]]$.

5.2. The noncommutative 2-torus

A prototypical example of symplectic twist star product is the Moyal–Weyl product on \mathbb{R}^{2n} , or its compact version: the (formal) noncommutative torus. Let $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, and denote by x, y the Cartesian coordinates on \mathbb{R}^2 . A global frame for vector fields on \mathbb{T}^2 is given by the partial derivatives ∂_x and ∂_y . The Lie algebra generated by such derivations will be identified with \mathbb{R}^2 .

Weyl's star product on \mathbb{T}^2 can be written in the form (4.2), with

$$F := \exp \frac{i\hbar}{2} (\partial_y \otimes \partial_x - \partial_x \otimes \partial_y), \tag{5.1}$$

a twist based on $U(\mathbb{R}^2)[[\hbar]]$. Together with the twists in [6,20] based on the $ax + b$ group, and to the construction in [16] relying on Fedosov techniques, (5.1) is one of the few instances of twist that can be written down explicitly.

Using (5.1), one gets a deformation quantization of \mathbb{T}^2 w.r.t. its standard symplectic structure. As a consequence of Theorem 11, we get the following corollary.

Corollary 13. *Every \mathbb{R}^2 -equivariant smooth complex line bundle L on \mathbb{T}^2 has $c_1(L) = 0$.*

Of course, it is not difficult to give a direct proof (not relying on Theorem 11) of this simple fact. Suppose $L \rightarrow \mathbb{T}^2$ is \mathbb{R}^2 -equivariant. Denote by

$$\triangleright : \mathbb{R}^2 \times \Gamma^\infty(L) \rightarrow \Gamma^\infty(L),$$

the corresponding action of the Lie algebra \mathbb{R}^2 on the module of sections. Then, the formula

$$\nabla_{a\partial_x + b\partial_y} s := a(\partial_x \triangleright s) + b(\partial_y \triangleright s), \quad \forall a, b \in C^\infty(\mathbb{T}^2), s \in \Gamma^\infty(L),$$

defines a *flat* connection ∇ on L . Indeed, property (3.1) guarantees that ∇ satisfies the Leibniz rule:

$$\nabla_X(fs) = X(f)s + f\nabla_X(s), \quad \forall X = a\partial_x + b\partial_y \in \mathfrak{X}(\mathbb{T}^2), f \in C^\infty(\mathbb{T}^2),$$

and clearly the connection 1-form of ∇ is zero, that means $c_1(L) = 0$.

Acknowledgements

We thank Chiara Esposito and Stefan Waldmann for many useful discussions and for their comments on a first version of the paper. This paper was written while F.D. was visiting for a semester Penn State University: F.D. is grateful to the institution for the excellent working conditions, and to Ping Xu for his constant support and help.

References

- [1] P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess, Noncommutative geometry and gravity, *Class. Quantum Gravity* 23 (2006) 1883–1912, arXiv:hep-th/0510059.
- [2] P. Aschieri, F. Lizzi, P. Vitale, Twisting all the way: from classical mechanics to quantum fields, *Phys. Rev. D* 77 (2008) 025037; arXiv:0708.3002 [hep-th].
- [3] P. Aschieri, Star product geometries, *Russ. J. Math. Phys.* 16 (2009) 371–383, arXiv:0903.2457 [math.QA].
- [4] P. Aschieri, Twisting all the way: from algebras to morphisms and connections, *Int. J. Mod. Phys. Conf. Ser.* 13 (2012) 1–19; arXiv:1210.1143 [math.QA].
- [5] P. Bieliavsky, C. Esposito, S. Waldmann, T. Weber, Obstructions for twist star products, arXiv:1607.06926 [math.QA].
- [6] P. Bieliavsky, X. Tang, Y. Yao, Rankin–Cohen brackets and formal quantization, *Adv. Math.* 212 (2007) 293–314, arXiv:math/0506506.
- [7] M. Bordemann, E. Meinrenken, M. Schlichenmaier, Toeplitz quantization of Kähler manifolds and $gl(N)$, $N \rightarrow \infty$ limits, *Commun. Math. Phys.* 165 (1994) 281–296, arXiv:hep-th/9309134.
- [8] M. Bordemann, N. Neumaier, S. Waldmann, S. Weiss, Deformation quantization of surjective submersions and principal fibre bundles, *J. Reine Angew. Math.* 639 (2010) 1–38, arXiv:0711.2965 [math.QA].
- [9] H. Bursztyn, Semiclassical geometry of quantum line bundles and Morita equivalence of star products, *Int. Math. Res. Not.* 16 (2002) 821–846, arXiv:math/0105001.
- [10] H. Bursztyn, S. Waldmann, Deformation quantization of Hermitian vector bundles, *Lett. Math. Phys.* 53 (2000) 349–365, arXiv:math/0009170.
- [11] H. Bursztyn, S. Waldmann, The characteristic classes of Morita equivalent star products on symplectic manifolds, *Commun. Math. Phys.* 228 (2002) 103–121, arXiv:math/0106178.
- [12] V. Chari, A.N. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, 1994.
- [13] F. D'Andrea, Topics in noncommutative geometry, in: *Lecture Notes for the School “From Poisson Geometry to Quantum Fields on Noncommutative Spaces”*, Würzburg, Germany, 2015, arXiv:1510.07271 [math.QA].
- [14] M. DeWilde, P.B.A. Lecomte, Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds, *Lett. Math. Phys.* 7 (1983) 487–496.
- [15] V.G. Drinfeld, Quasi-Hopf algebras, *Leningr. Math. J.* 1 (1990) 1419–1457.
- [16] C. Esposito, J. Schnitzer, S. Waldmann, An universal construction of universal deformation formulas, Drinfel'd twists and their positivity, arXiv:1608.00412 [math.QA].
- [17] P.I. Etingof, O. Schiffmann, *Lectures on Quantum Groups*, International Press, 2001.
- [18] B.V. Fedosov, A simple geometrical construction of deformation quantization, *J. Differ. Geom.* 40 (1994) 213–238.
- [19] G. Fiore, On second quantization on noncommutative spaces with twisted symmetries, *J. Phys. A* 43 (2010) 155401, arXiv:0811.0773 [hep-th].
- [20] A. Giaquinto, J.J. Zhang, Bialgebra actions, twists, and universal deformation formulas, *J. Pure Appl. Algebra* 128 (1998) 133–151, arXiv:hep-th/9411140.
- [21] S. Gutt, J. Rawnsley, Equivalence of star products on a symplectic manifold: an introduction to Deligne's Čech cohomology classes, *J. Geom. Phys.* 29 (1999) 347–392.
- [22] M. Kontsevich, Deformation quantization of Poisson manifolds, I, *Lett. Math. Phys.* 66 (2003) 157–216, arXiv:q-alg/9709040.
- [23] T.Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Mathematics, vol. 189, Springer, 1999.
- [24] S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, 1995.
- [25] H. Omori, Y. Maeda, A. Yoshioka, Weyl manifolds and deformation quantization, *Adv. Math.* 85 (1991) 224–255.
- [26] M.A. Rieffel, Matrix algebras converge to the sphere for quantum Gromov–Hausdorff distance, *Mem. Amer. Math. Soc.* 168 (2004) 67–91, arXiv:math/0108005.
- [27] M.A. Rieffel, Dirac operators for coadjoint orbits of compact Lie groups, *Münster J. Math.* 2 (2009) 265–298, arXiv:0812.2884 [math.DG].
- [28] M. Schlichenmaier, Berezin–Toeplitz quantization for compact Kähler manifolds. A review of results, *Adv. Math. Phys.* 2010 (2010) 927280, arXiv:1003.2523 [math.QA].
- [29] M. Schlichenmaier, Deformation quantization of compact Kähler manifolds by Berezin–Toeplitz quantization, in: G. Dito, D. Sternheimer (Eds.), *Proc. Conference Moshe Flato 1999*, Kluwer, 2000, pp. 289–306, arXiv:math.QA/9910137.
- [30] M. Schlichenmaier, *Zwei Anwendungen algebraisch-geometrischer Methoden in der theoretischen Physik: Berezin–Toeplitz-Quantisierung und globale Algebren der zweidimensionalen konformen Feldtheorie*, Habilitation Thesis, University of Mannheim, Germany, 1996.
- [31] S. Waldmann, Recent developments in deformation quantization, in: *Quantum Mathematical Physics: A Bridge Between Mathematics and Physics*, Springer, 2016, pp. 421–439, arXiv:1502.00097 [math.QA].
- [32] P. Xu, Quantum groupoids, *Commun. Math. Phys.* 216 (2001) 539–581, arXiv:math/9905192.