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# Existence and uniqueness of solutions to a model describing miscible liquids

## *Existence et unicité des solutions pour un modèle décrivant les liquides miscibles*

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ABSTRACT

The existence and the uniqueness of solutions to a problem of miscible liquids are investigated in this note. The model consists of Navier–Stokes equations with Korteweg stress terms coupled with the reaction–diffusion equation for the concentration. We assume that the fluid is incompressible and the Boussinesq approximation is adopted. The global existence and uniqueness of solutions is established for some optimal conditions on the reaction source term and the external force functions.



RÉSUMÉ

Dans cette note, nous étudierons un problème d'existence et d'unicité pour un modèle qui décrit l'interaction de deux fluides miscibles. Le modèle considéré prend la forme d'équations de Navier–Stokes avec contraintes de Korteweg couplées à l'équation de réaction–diffusion de la concentration. Nous supposons que le fluide est incompressible ; l'approximation de Boussinesq est adoptée. L'existence globale et l'unicité des solutions sont établies pour des conditions optimales sur le terme source de réaction et les forces externes.



### 1. Introduction

The theoretical and experimental investigation of the dynamics of two miscible liquids have attracted considerable attention from many researchers because of its significant applications in various fields such as enhanced oil recovery, hydrology, frontal polymerization, groundwater pollution, and filtration [1,2,6]. A detailed review of the results on the interfacial dy-

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namics of two miscible liquids is presented in [3]. In this paper, we are interested in studying the following miscible liquids model:

$$\frac{\partial C}{\partial t} + u.\nabla C = d\Delta C - Cg, \tag{1.1}$$

$$\frac{\partial u}{\partial t} + (u.\nabla)u = -\nabla p + \mu\Delta u + \nabla.T(C) + f, \tag{1.2}$$

$$\operatorname{div}(u) = 0, \tag{1.3}$$

with the following boundary conditions:

$$\frac{\partial C}{\partial n} = 0, \;\; u = 0 \;\; \text{on } \Gamma, \tag{1.4}$$

and the following initial conditions:

$$C(x,0) = C_0(x), \; u(x,0) = u_0(x), \; x \in \Omega. \tag{1.5}$$

Here $u$ is the velocity, $p$ is the pressure, $C$ is the concentration, $d$ is the coefficient of mass diffusion, $\mu$ is the viscosity, $\Gamma$ is a Lipschitz continuous boundary of the open bounded domain $\Omega$, $n$ is the unit outward normal vector to $\Gamma$, $f$ is the function describing the external forces such as gravity and buoyancy, while $g$ stands for the source term. The Korteweg stress tensor terms are given by the relations [4]:

$$T_{11} = k\left(\frac{\partial C}{\partial x_2}\right)^2, \; T_{12} = T_{21} = -k\frac{\partial C}{\partial x_1}\frac{\partial C}{\partial x_2}, \; T_{13} = T_{31} = -k\frac{\partial C}{\partial x_1}\frac{\partial C}{\partial x_3},$$
$$T_{23} = T_{32} = -k\frac{\partial C}{\partial x_2}\frac{\partial C}{\partial x_3}, \; T_{22} = k\left(\frac{\partial C}{\partial x_1}\right)^2, \; T_{33} = k\left(\frac{\partial C}{\partial x_3}\right)^2,$$

here $k$ is a nonnegative constant. We set

$$\nabla\cdot T(C) = \begin{pmatrix} \dfrac{\partial T_{11}}{\partial x_1} + \dfrac{\partial T_{12}}{\partial x_2} + \dfrac{\partial T_{13}}{\partial x_3} \\ \dfrac{\partial T_{21}}{\partial x_1} + \dfrac{\partial T_{22}}{\partial x_2} + \dfrac{\partial T_{23}}{\partial x_3} \\ \dfrac{\partial T_{31}}{\partial x_1} + \dfrac{\partial T_{32}}{\partial x_2} + \dfrac{\partial T_{33}}{\partial x_3} \end{pmatrix}.$$

In the absence of external and source forces ($f = g = 0$) and in the two-dimensional case, the existence and uniqueness of a solution to the model (1.1)–(1.5) is studied in [5]. The aim of this paper is to continue the investigations of miscible liquids by considering three-dimensional Navier–Stokes equations and by introducing the external forces in the equation of motion and the source terms in the equation for the concentration.

## 2. Existence of solutions

First, we will introduce our functional framework on which the study will be carried out. Let the following velocity and concentration spaces:

$$S_u = \{u \in H_0^1(\Omega); \operatorname{div}(u) = 0\}, \quad S_C = \{C \in H^2(\Omega); \frac{\partial C}{\partial n} = 0 \text{ on } \Gamma\}.$$

Hence, the variational formulation of the problem is to find $C \in S_C$ and also $u \in S_u$ such that, for all $B \in S_C$ and $v \in S_u$, we have the following equalities:

$$(\frac{\partial C}{\partial t}, B) + d(\nabla C, \nabla B) + (u\cdot\nabla C, B) + (gC, B) = 0, \tag{2.1}$$

$$(\frac{\partial u}{\partial t}, v) + \mu(\nabla u, \nabla v) - (\operatorname{div} T(C), v) - (f, v) = 0. \tag{2.2}$$

In what follows, we will assume that the function $f(x,t)$ is positive and bounded in $L^\infty(0,t;L^2(\Omega))$; we will assume also that $g(x)$ is positive and bounded in $L^\infty(\Omega)$. In other words, there exist two real positive constants $\bar{f}$ and $\bar{g}$, such that

$$\|f\|_{L^\infty(0,t;L^2(\Omega))} \leq \bar{f}, \;\; f(x,t) \geq 0 \tag{2.3}$$

and

$$\|g\|_{L^\infty(\Omega)} \leq \bar{g}, \;\; g(x) \geq 0. \tag{2.4}$$

In order to prove the existence of solutions, we will need the following lemmas.

**Lemma 2.1.** *The concentration C is bounded in* $L^\infty(0,t;L^2)$.

**Proof.** The proof is similar to that in [5]. □

**Lemma 2.2.** *The concentration C is bounded in* $L^\infty(0,t;H^1)$ *and* $L^2(0,t;H^2)$*, and the velocity u is bounded in* $L^\infty(0,t;L^2)$ *and* $L^2(0,t;H^1_0)$.

**Proof.** By choosing $-\Delta C$ as a test function in equation (2.1), we have:

$$(\frac{\partial C}{\partial t}, -\Delta C) + (u\cdot\nabla C, -\Delta C) = d(\Delta C, -\Delta C) + (gC, \Delta C).$$

Using the hypothesis (2.4), we get the following estimate:

$$\frac{1}{2}\frac{\partial}{\partial t}(\nabla C, \nabla C) + d(\Delta C, \Delta C) - (u\cdot\nabla C, \Delta C) \le \bar{g}(\nabla C, \nabla C).$$

Then

$$\frac{1}{2}\frac{\partial}{\partial t}(\nabla C, \nabla C) + d(\Delta C, \Delta C) \le (u\cdot\nabla C, \Delta C) + \bar{g}(\nabla C, \nabla C). \tag{2.5}$$

Besides, by choosing $u$ as a test function in (2.2), we obtain:

$$\frac{1}{2}\frac{\partial}{\partial t}(u, u) + \mu(\nabla u, \nabla u) - (\nabla\cdot T(C), u) = (f, u). \tag{2.6}$$

To have an explicit form of $\nabla\cdot T(C)$, we can calculate only the first component:

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} = 2k\frac{\partial C}{\partial x_2}\frac{\partial^2 C}{\partial x_1 \partial x_2} - k\frac{\partial^2 C}{\partial x_1 \partial x_2}\frac{\partial C}{\partial x_2} - k\frac{\partial C}{\partial x_1}\frac{\partial^2 C}{\partial x_2^2} - k\frac{\partial^2 C}{\partial x_1 \partial x_3}\frac{\partial C}{\partial x_2} - k\frac{\partial C}{\partial x_1}\frac{\partial^2 C}{\partial x_3^2}. \tag{2.7}$$

Then,

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} = k\frac{\partial C}{\partial x_1}\frac{\partial^2 C}{\partial x_1 \partial x_2} + k\frac{\partial C}{\partial x_1}\frac{\partial^2 C}{\partial x_1 \partial x_3} + k\frac{\partial C}{\partial x_1}\frac{\partial^2 C}{\partial x_1^2} - k\frac{\partial C}{\partial x_1}\Delta C.$$

Therefore,

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} = \frac{k}{2}\frac{\partial}{\partial x_1}(\nabla C)^2 - k\frac{\partial C}{\partial x_1}\Delta C.$$

Following the same reasoning for the other components, one concludes:

$$\nabla\cdot T = \frac{k}{2}\nabla(\nabla C)^2 - k\Delta C\nabla C.$$

Substituting this equality in (2.6) and using $u \in S_u$, we obtain:

$$\frac{1}{2}\frac{\partial}{\partial t}(u, u) + \mu(\nabla u, \nabla u) - k(\Delta C\nabla C, u) = (f, u). \tag{2.8}$$

Adding (2.8) to the inequality (2.5), and using the facts that $u \in S_u$ and $C \in S_c$, we have:

$$\frac{1}{2}\frac{\partial}{\partial t}((u, u) + (\nabla C, \nabla C)) + \mu(\nabla u, \nabla u) + d(\Delta C, \Delta C) \le (f, u) + \bar{g}(\nabla C, \nabla C),$$

$$\frac{1}{2}\frac{\partial}{\partial t}((u, u) + (\nabla C, \nabla C)) + \mu(\nabla u, \nabla u) + d(\Delta C, \Delta C) \le \frac{1}{2}(f, f) + \frac{1}{2}(u, u) + \bar{g}(\nabla C, \nabla C). \tag{2.9}$$

From the positivity of the third and the fourth terms in the left hand side of this inequality, we have:

$$\frac{\partial}{\partial t}((u, u) + (\nabla C, \nabla C)) \le (f, f) + (u, u) + 2\bar{g}(\nabla C, \nabla C).$$

Hence

$$\frac{\partial}{\partial t}((u, u) + (\nabla C, \nabla C)) \le (f, f) + \max(1; 2\bar{g})\left((u, u) + (\nabla C, \nabla C)\right).$$

By integrating over time, and using the hypothesis (2.3), we have:

$$\|u(x,t)\|_{L^2} + \|\nabla C(x,t)\|_{L^2} \le \|u_0\|_{L^2} + \|\nabla C_0\|_{L^2} + \bar{f} + \max(1;2\bar{g}) \int_0^t \left(\|u(x,s)\|_{L^2} + \|\nabla C(x,s)\|_{L^2}\right) \mathrm{d}s.$$

From the Gronwall's Lemma, it follows:

$$\|u(x,t)\|_{L^2} + \|\nabla C(x,t)\|_{L^2} \le (\|u_0\|_{L^2} + \|\nabla C_0\|_{L^2} + \bar{f}) \times \exp(\max(1;2\bar{g})t),$$

which means that $C$ is bounded in $L^\infty(0,t;H^1)$ and that $u$ is bounded in $L^\infty(0,t;L^2)$ for $t \in [0;\mathcal{T}]$ with $\mathcal{T} > 0$.

Integrating the inequality (2.9) over time, we obtain:

$$\|u(x,t)\|_{L^2} + \|\nabla C(x,t)\|_{L^2} + 2\mu\|u\|_{L^2(0,t;H_0^1)} + 2d\|\Delta C\|_{L^2(0,t;L^2)}$$
$$\le \|u_0\|_{L^2} + \|\nabla C_0\|_{L^2} + \|f\|_{L^2(0,t;L^2)} + \|u\|_{L^2(0,t;L^2)} + 2\bar{g}\|\nabla C\|_{L^2(0,t;L^2)}.$$

Therefore,

$$\|\nabla u\|_{L^2(0,t;H_0^1)} + \|\Delta C\|_{L^2(0,t;L^2)} \le \frac{1}{2\min(\mu;d)}\|u_0\|_{L^2} + \frac{1}{2\min(\mu;d)}\|\nabla C_0\|_{L^2}$$
$$+ \frac{1}{2\min(\mu;d)}\|f\|_{L^2(0,t;L^2)} + \frac{1}{2\min(\mu;d)}\|u\|_{L^2(0,t;L^2)} + \frac{\bar{g}}{\min(\mu;d)}\|\nabla C\|_{L^2(0,t;L^2)}.$$

From the boundedness of $C$ in $L^\infty(0,t;H^1)$, of $u$ in $L^\infty(0,t;L^2)$, and from hypothesis (2.3), we deduce that the solution $C$ is bounded in $L^2(0,t;H^2)$ and the solution $u$ is bounded in $L^2(0,t;H_0^1)$. □

**Lemma 2.3.** *The concentration time derivative* $\frac{\partial C}{\partial t}$ *is bounded in* $L^2(0,t;L^2)$.

**Proof.** The proof is similar to that in [5]. □

**Lemma 2.4.** *The velocity time derivative* $\frac{\partial u}{\partial t}$ *is bounded in* $L^2(0,t;L^2)$.

**Proof.** The proof is similar to that in [5]. □

We are now able to state the existence main result.

**Theorem 2.5.** *The problem (1.1)–(1.5) admits a global solution.*

**Proof.** From all the previous Lemmas, one can deduce easily the existence of solutions to the problem (1.1)–(1.5). □

## 3. Uniqueness of the solution

To prove the uniqueness of the solution, we first assume that the problem (1.1)–(1.5) has two different solutions $(C_1,u_1)$ and $(C_2,u_2)$. Hence, from (1.1), we have:

$$\frac{\partial}{\partial t}(C_1 - C_2) - d\Delta(C_1 - C_2) + u_1\nabla C_1 - u_2\nabla C_2 + g(C_1 - C_2) = 0, \tag{3.1}$$

also from (1.2), we get:

$$\frac{\partial}{\partial t}(u_1 - u_2) + (u_1.\nabla)u_1 - (u_2.\nabla)u_2 - \mu(\Delta u_1 - \Delta u_2) + \nabla(p_1 - p_2) = \frac{k}{2}\nabla\left((\nabla C_1)^2 - (\nabla C_2)^2\right)$$
$$- k(\Delta C_1\nabla C_1 - \Delta C_2\nabla C_2). \tag{3.2}$$

By multiplying (3.1) by $-k\Delta(C_1 - C_2)$ and by integrating, we obtain the following:

$$(\frac{\partial}{\partial t}(C_1 - C_2), -k\Delta(C_1 - C_2)) + dk(\Delta(C_1 - C_2), \Delta(C_1 - C_2))$$
$$+ (u_1\nabla C_1, -k\Delta(C_1 - C_2)) + (u_2\nabla C_2, k\Delta(C_1 - C_2)) + (g(C_1 - C_2), -k\Delta(C_1 - C_2)) = 0.$$

Now, multiplying (3.2) by $u_1 - u_2$, integrating and since $u_i \in S_u$ $(i = 1,2)$, we have the following:

$$(\frac{\partial}{\partial t}(u_1 - u_2), u_1 - u_2) + (((u_1 - u_2).\nabla)u_2, u_1 - u_2) + \mu(\nabla(u_1 - u_2), \nabla(u_1 - u_2))$$
$$= \frac{k}{2}(\nabla\left((\nabla C_1)^2 - (\nabla C_2)^2\right), u_1 - u_2) - k(\Delta C_1\nabla C_1 - \Delta C_2\nabla C_2, u_1 - u_2).$$

Adding the two last equalities, using Green's formula and the fact that $u_i \in S_u$ $(i = 1, 2)$, we conclude that

$$\frac{1}{2}\frac{\partial}{\partial t}(\|u_1 - u_2\|^2_{L^2} + k\|\nabla C_1 - \nabla C_2\|^2_{L^2}) + \mu\|u_1 - u_2\|^2_{H^1} + kd\|\Delta(C_1 - C_2)\|^2_{L^2} =$$
$$k(u_1\nabla(C_1 - C_2), \Delta(C_1 - C_2)) + k((u_1 - u_2)\nabla C_2, \Delta(C_1 - C_2)) -$$
$$k(\Delta C_1\nabla(C_1 - C_2), u_1 - u_2) + k(-\Delta C_1\nabla C_2 + \Delta C_2\nabla C_2, u_1 - u_2) -$$
$$(((u_1 - u_2).\nabla)u_2, u_1 - u_2) + k(g(C_1 - C_2), \Delta(C_1 - C_2)).$$

Therefore,

$$\frac{1}{2}\frac{\partial}{\partial t}(\|u_1 - u_2\|^2_{L^2} + k\|\nabla C_1 - \nabla C_2\|^2_{L^2}) + \mu\|u_1 - u_2\|^2_{H^1} + kd\|\Delta(C_1 - C_2)\|^2_{L^2} =$$
$$k(u_1\nabla(C_1 - C_2), \Delta(C_1 - C_2)) - k(\Delta C_1\nabla(C_1 - C_2), u_1 - u_2) -$$
$$(((u_1 - u_2).\nabla)u_2, u_1 - u_2) + k(g(C_1 - C_2), \Delta(C_1 - C_2)). \tag{3.3}$$

We will look now for an estimate for the right-hand side of the equality. We put $C = C_1 - C_2$ and $u = u_1 - u_2$. From the Hölder inequality, it follows that:

$$|(\Delta C_1\nabla C, u)| \le \|\Delta C_1\|_{L^2}\|\nabla C.u\|_{L^2} \le \|\Delta C_1\|_{L^2}\|\nabla C\|_{L^4}\|u\|_{(L^4)^2}.$$

From the Gagliardo–Nirenberg inequality, we obtain:

$$|(\Delta C_1\nabla C, u)| \le N_1\|\Delta C_1\|_{L^2}\|\nabla C\|^{1/4}_{L^2}\|\Delta C\|^{3/4}_{L^2}\|u\|^{1/4}_{L^2}\|\nabla u\|^{3/4}_{L^2}.$$

By Young's inequality, we obtain:

$$|(\Delta C_1\nabla C, u)| \le \frac{3N_1}{8}\|\Delta C\|^2_{L^2} + \frac{5N_1}{8}\|\Delta C_1\|^{8/5}_{L^2}\|\nabla C\|^{2/5}_{L^2}\|u\|^{2/5}_{L^2}\|\nabla u\|^{6/5}_{L^2}.$$

Following the same reasoning, we obtain the following inequality:

$$|(u_1\nabla C, \Delta C)| \le \|\Delta C\|_{L^2}\|\nabla C.u_1\|_{L^2} \le \|\Delta C\|_{L^2}\|\nabla C\|_{L^4}\|u_1\|_{L^4}.$$

Hence,

$$|(u_1\nabla C, \Delta C)| \le N_2\|\Delta C\|^{7/4}_{L^2}\|\nabla C\|^{1/4}_{L^2}\|u_1\|^{1/4}_{L^2}\|\nabla u_1\|^{3/4}_{L^2}.$$

We deduce:

$$|(u_1\nabla C, \Delta C)| \le \frac{7N_2}{8}\|\Delta C\|^2_{L^2} + \frac{N_2}{8}\|\nabla C\|^2_{L^2}\|u_1\|^2_{L^2}\|\nabla u_1\|^6_{L^2}.$$

Also, we have:

$$|((u.\nabla)u_2, u)| \le \frac{N_3}{2}\|\nabla u\|^2_{L^2} + \frac{N_3}{2}\|u\|^2_{L^2}\|\nabla u_2\|^2_{L^2}.$$

Finally, from (3.3) and assuming that $3N_1 + 7N_2 \le 8d$ and $N_3 \le 2\mu$, we have:

$$\frac{1}{2}\frac{\partial}{\partial t}(\|u\|^2_{L^2} + k\|\nabla C\|^2_{L^2}) \le \frac{5N_1k}{8}\|\Delta C_1\|^{8/5}_{L^2}\|\nabla C\|^{2/5}_{L^2}\|u\|^{2/5}_{L^2}\|\nabla u\|^{6/5}_{L^2}$$
$$+ \frac{N_2k}{8}\|\nabla C\|^2_{L^2}\|u_1\|^2_{L^2}\|\nabla u_1\|^6_{L^2} + \frac{N_3}{2}\|u\|^2_{L^2}\|\nabla u_2\|^2_{L^2} + k\bar{g}\|\nabla C\|^2_{L^2} \le (\|u\|^2_{L^2} + k\|\nabla C\|^2_{L^2}) \times$$
$$\left(\frac{5N_1k}{8}\|\Delta C_1\|^{8/5}_{L^2}\|\nabla C\|^{2/5}_{L^2}\|u\|^{-8/5}_{L^2}\|\nabla u\|^{6/5}_{L^2} + \frac{N_2}{8}\|u_1\|^2_{L^2}\|\nabla u_1\|^6_{L^2} + \frac{N_3}{2}\|\nabla u_2\|^2_{L^2} + k\bar{g}\right).$$

Denote:

$$\phi(t) = \|\Delta C_1\|^{8/5}_{L^2}\|\nabla C\|^{2/5}_{L^2}\|u\|^{-8/5}_{L^2}\|\nabla u\|^{6/5}_{L^2} + \|u_1\|^2_{L^2}\|\nabla u_1\|^6_{L^2} + \|\nabla u_2\|^2_{L^2} + 1$$

and

$$M = \max(\frac{5N_1k}{8}, \frac{N_2}{8}, \frac{N_3}{2}, k\bar{g}).$$

Then, we have the following estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}(\exp(M\int_0^t \phi(s)\mathrm{d}s)(\|u\|^2_{L^2} + k\|\nabla C\|^2_{L^2})) \le 0,$$

for all $t \geq 0$. From this we deduce that

$$\exp(M\int_0^t \phi(s)\mathrm{d}s)(\|u\|_{L^2}^2 + k\|\nabla C\|_{L^2}^2) \leq \|u(0)\|_{L^2}^2 + k\|\nabla C(0)\|_{L^2}^2.$$

Since $u(0) = C(0) = 0$, we conclude that the uniqueness of the solution is demonstrated. We can now state the theorem on the uniqueness of the solution.

**Theorem 3.1.** *The problem (1.1)–(1.5) admits a unique solution.*

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