C. R. Acad. Sci. Paris, Ser. I 355 (2017) 1041–1045

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Logic/Algebraic geometry

A proof of the integral identity conjecture, II

Une preuve de la conjecture de l'identité intégrale, II

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ARTICLE INFO

Article history:
Received 17 July 2017
Accepted after revision 12 October 2017
Available online 16 October 2017

Presented by Claire Voisin

ABSTRACT

In this note, using Cluckers–Loeser's theory of motivic integration, we prove the integral identity conjecture with framework a localized Grothendieck ring of varieties over an arbitrary base field of characteristic zero.

RÉSUMÉ

Dans cette note, en utilisant la théorie de l'intégration motivique de Cluckers et Loeser, nous prouvons la conjecture de l'identité intégrale dans le cadre d'un anneau de Grothendieck de variétés localisé sur un corps arbitraire de caractéristique nulle.

Version française abrégée

Dans [4], Kontsevich et Soibelman introduisent des invariants de Donaldson–Thomas motiviques, qui sont définis pour des variétés de Calabi–Yau de dimension 3 non commutatives. L'existence de ces invariants repose sur une identité intégrale conjecturale dans l'anneau de Grothendieck équivariant $\mathcal{M}_k^{\hat{\mu}}$ de variétés sur un corps k de caractéristique nulle, concernant la fibre de Milnor motivique d'un potentiel spécial. Nous nous intéressons à une version de l'identité intégrale pour la fibre de Milnor motivique d'une fonction régulière. Elle est vérifiée dans [5] quand le potentiel, soit a la forme d'une composition d'un polynôme à deux variables avec une paire de fonctions régulières, soit est de type de Steenbrink. Pour le cas où le corps k est algébriquement clos, Lê [6] montre que la conjecture est vraie dans $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$, un localisé de $\mathcal{M}_k^{\hat{\mu}}$. Plus récemment, Nicaise et Payne [7] prouvent la conjecture dans le cadre de $\mathcal{M}_k^{\hat{\mu}}$ pour le corps k qui contient toutes les racines de l'unité.

Le résultat principal de cette note est une preuve de la conjecture de l'identité intégrale dans le cadre de $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$ sur un corps *arbitraire* de caractéristique nulle. La théorie de l'intégration motivique de Cluckers et Loeser joue le rôle crucial dans notre preuve.

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https://doi.org/10.1016/j.crma.2017.10.005
1631-073X/

1. Statement of conjecture and main theorem

1.1. Equivariant Grothendieck ring of varieties

Let k be a field of characteristic zero, S an algebraic k-variety, and Var_S the category of S-varieties. Let $K_0(\mathrm{Var}_S)$ be the Grothendieck ring of Var_S, which is the quotient of the free abelian group generated by the S-isomorphism classes $[X \to S]$ in Var_S such that $[X \to S] = [Y \to S] + [X \setminus Y \to S]$ for any Zariski closed subvariety Y of X. It is a commutative ring with respect to fiber product.

We consider the projective system of $\mu_n = \mathrm{Spec}k[t]/(t^n - 1)$ with transitions $\mu_{mn} \to \mu_n$ given by $\lambda \mapsto \lambda^m$, and define $\hat{\mu} = \varprojlim \mu_n$. A good μ_n-action on an S-variety X is a group action each of whose orbits is contained in an affine k-subvariety of X, a good $\hat{\mu}$-action on S-variety X is a good μ_n-action for some n. The $\hat{\mu}$-equivariant Grothendieck group $K_0^{\hat{\mu}}(\mathrm{Var}_S)$ of S-varieties endowed with good $\hat{\mu}$-action is the quotient of the free abelian group generated by the $\hat{\mu}$-equivariant isomorphism classes $[X \to S, \sigma]$, σ being a good $\hat{\mu}$-action on S-variety X, modulo the conditions $[X \to S, \sigma] = [Y \to S, \sigma|_Y] + [X \setminus Y \to S, \sigma|_{X\setminus Y}]$, for Y σ-stable Zariski closed in X, and $[X \times_k \mathbb{A}_k^n \to S, \sigma] = [X \times_k \mathbb{A}_k^n \to S, \sigma']$, whenever σ and σ' lift the same $\hat{\mu}$-action on $X \to S$ to an affine action on $X \times \mathbb{A}_k^n \to S$. The structure of a commutative ring with unity on $K_0^{\hat{\mu}}(\mathrm{Var}_S)$ is given by fiber product. Let $\mathbb{L}$ be the class of the trivial line bundle over S, with trivial $\hat{\mu}$-action. We write $\mathcal{M}_S^{\hat{\mu}}$ for the localization of $K_0^{\hat{\mu}}(\mathrm{Var}_S)$ inverting $\mathbb{L}$, and $\mathcal{M}_{S,\mathrm{loc}}^{\hat{\mu}}$ for the localization of $\mathcal{M}_S^{\hat{\mu}}$ inverting the elements $1 - \mathbb{L}^{-n}$, for every n in $\mathbb{N}_{>0}$. The ring $\mathcal{M}_{\mathrm{Speck,loc}}^{\hat{\mu}}$ is rewritten simply by $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$.

Any morphism of k-varieties $g : S \to S'$ induces a ring morphism $g^* : \mathcal{M}_{S'}^{\hat{\mu}} \to \mathcal{M}_S^{\hat{\mu}}$ by fiber product, and induces a group morphism $g_! : \mathcal{M}_S^{\hat{\mu}} \to \mathcal{M}_{S'}^{\hat{\mu}}$ by composition. When S' is $\mathrm{Spec}k$, we replace the symbol $g_!$ by the symbol $\int_S$. Let loc denote the natural morphism $\mathcal{M}_k^{\hat{\mu}} \to \mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$.

Consider the ring $\mathcal{M}_S^{\hat{\mu}}[[T]]$, and its subset $\mathcal{M}_S^{\hat{\mu}}[[T]]_{\mathrm{sr}}$ of rational series, which consists of $\mathcal{M}_S^{\hat{\mu}}$-polynomials in variables $\frac{\mathbb{L}^p T^q}{(1-\mathbb{L}^p T^q)}$, with (p,q) in $\mathbb{Z} \times \mathbb{N}_{>0}$. There exists by [2] a unique $\mathcal{M}_S^{\hat{\mu}}$-linear morphism $\lim_{T\to\infty} : \mathcal{M}_S^{\hat{\mu}}[[T]]_{\mathrm{sr}} \to \mathcal{M}_S^{\hat{\mu}}$ such that $\lim_{T\to\infty} \frac{\mathbb{L}^p T^q}{(1-\mathbb{L}^p T^q)} = -1$, for every (p,q) in $\mathbb{Z} \times \mathbb{N}_{>0}$.

1.2. Motivic nearby cycles

Let X be a smooth algebraic k-variety of pure dimension d. For $n \geq 1$, let $\mathcal{L}_n(X)$ be the k-scheme of n-jets on X, which represents the functor sending a k-algebra A to the set of morphisms of k-schemes $\mathrm{Spec}(A[t]/(t^{n+1})) \to X$. These schemes together with morphisms $\mathcal{L}_m(X) \to \mathcal{L}_n(X)$ $(m \geq n)$ induced by truncation form a projective system, and we denote its limit by $\mathcal{L}(X)$. Let f be a regular function on X with nonempty zero locus X_0. For $n \geq 1$, let $\mathcal{X}_n(f)$ be the k-variety of n-jets φ in $\mathcal{L}_n(X)$ with $f(\varphi) = t^n \mod t^{n+1}$, which admits an obvious morphism to X_0 and the natural μ_n-action $(\lambda, \varphi(t)) \mapsto \varphi(\lambda t)$. Write $[\mathcal{X}_n(f)]$ for the class of $\mathcal{X}_n(f) \to X_0$ in $\mathcal{M}_{X_0}^{\hat{\mu}}$. By [2], the series $Z_f(T) := \sum_{n\geq 1} [\mathcal{X}_n(f)]\mathbb{L}^{-nd} T^n$ in $\mathcal{M}_{X_0}^{\hat{\mu}}[[T]]$ is rational, and the limit $\mathcal{S}_f := -\lim_{T\to\infty} Z_f(T)$ in $\mathcal{M}_{X_0}^{\hat{\mu}}[[T]]$ is called the motivic nearby cycles of f. If x is a closed point in X_0, we may consider the motivic Milnor fiber of f at x, $\mathcal{S}_{f,x} = i_x^* \mathcal{S}_f$ in $\mathcal{M}_k^{\hat{\mu}}[[T]]$, where i_x is the inclusion of $\{x\}$ in X_0.

1.3. Conjecture and main theorem

The integral identity conjecture plays a crucial role in Kontsevich–Soibelman's theory of motivic Donaldson–Thomas invariants for noncommutative Calabi–Yau threefolds [4]. We now state the version for regular functions of the conjecture (for the full version, see [4, Conjecture 4.4]).

Conjecture 1.1 *([4]). Let (x,y,z) be coordinates of the k-vector space $k^d = k^{d_1} \times k^{d_2} \times k^{d_3}$. Let f be in $k[x,y,z]$ such that $f(0,0,0) = 0$ and $f(\lambda x, \lambda^{-1} y, z) = f(x,y,z)$ for λ in $\mathbb{G}_{m,k}$. Then the identity $\int_{\mathbb{A}_k^{d_1}} i^* \mathcal{S}_f = \mathbb{L}^{d_1} \mathcal{S}_{\tilde{f},0}$ holds in $\mathcal{M}_k^{\hat{\mu}}$, with $\tilde{f}$ the restriction of f to $\mathbb{A}_k^{d_3}$, and i the inclusion of $\mathbb{A}_k^{d_1}$ in $f^{-1}(0)$.*

Here, we identify $\mathbb{A}_k^{d_1}$ with $\mathbb{A}_k^{d_1} \times \{0\} \times \{0\}$, hence by the homogeneity of f, we consider it as a subvariety of $f^{-1}(0)$. We also identify $\mathbb{A}_k^{d_3}$ with $\{0\} \times \{0\} \times \mathbb{A}_k^{d_3}$, thus by definition, $\tilde{f}(z) = f(0,0,z)$.

The conjecture was first proved in [5] in the case where f is either a function of Steenbrink type or the composition of a pair of regular functions with a polynomial in two variables. In [6, Theorem 1.2], we show that, if the field k is algebraically closed, Conjecture 1.1 holds in $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$. Recently, under the weaker assumption that the base field k contains all roots of unity, Nicaise and Payne [7] prove the conjecture with the full context $\mathcal{M}_k^{\hat{\mu}}$.

The main result of this note is the following theorem.

Theorem 1.2. *Conjecture 1.1 is true in $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$, namely,* $\mathrm{loc}\left(\int_{\mathbb{A}_k^{d_1}} i^*\mathcal{S}_f\right) = \mathrm{loc}\left(\mathbb{L}^{d_1}\mathcal{S}_{\tilde{f},0}\right)$.

Note that our proof for the theorem does not use the assumption that k is algebraically closed, that is, k may be any field of characteristic zero. The materials for the proof are in Cluckers–Loeser's motivic integration of constructible motivic functions [1].

2. Measurable subassignments

2.1. Definable subassignments

We consider the formalism of Cluckers and Loeser [1] with a concrete Denef–Pas language $\mathcal{L}_{\mathrm{DP,P}}$ consisting of the ring language $\{+,-,\cdot,0,1\}$ for valued fields, also the ring language for residue fields, and the Presburger language $\{+,-,0,1,\leq\}\cup\{\equiv_n | \, n\in\mathbb{N}_{>0}\}$ for value groups, where $\equiv_n$ is the equivalence relation modulo n. Let Field_k be the category of algebraically closed fields K containing k in the $\mathcal{L}_{\mathrm{DP,P}}$-language, where sentences take coefficients in k and $k((t))$, and morphisms of Field_k are field morphisms. The theory corresponding to Field_k is the theory of algebraically closed fields containing k, each model of this theory is a triple $(K((t)),K,\mathbb{Z})$ with K in Field_k. The valued fields $K((t))$ are endowed with a natural valuation map $\mathrm{ord}_t : K((t))^\times \to \mathbb{Z}$ augmented by $\mathrm{ord}_t(0)=+\infty$, and with a natural angular component map $\overline{\mathrm{ac}} : K((t)) \to K$, with convention $\overline{\mathrm{ac}}(0)=0$.

A basic affine definable subassignment has the form $h[m,n,r]$, where $h[m,n,r](K)=K((t))^m\times K^n\times\mathbb{Z}^r$. More generally, if $W=\mathcal{X}\times X\times\mathbb{Z}^r$ with $\mathcal{X}$ a $k((t))$-variety and X a k-variety, we define $h_W(K):=\mathcal{X}(K((t)))\times X(K)\times\mathbb{Z}^r$. An arbitrary definable subsassignment is a set of points in $h[m,n,r]$, or in h_W, satisfying a given formula φ.

Among a broad collection of definable subassignments, we now only consider the category Def_k of affine definable subassignments where objects are pairs $(\mathrm{Z},h[m,n,r])$ with Z being a definable subassignment of $h[m,n,r]$, and a morphism $(\mathrm{Z},h[m,n,r])\to(\mathrm{Z}',h[m',n',r'])$ is a definable morphism $\mathrm{Z}\to\mathrm{Z}'$. Due to [1], by a definable morphism $\mathrm{Z}\to\mathrm{Z}'$ one means a morphism of subassignments $\mathrm{Z}\to\mathrm{Z}'$ such that its graph is a definable subassignment of $h[m+m',n+n',r+r']$. Let RDef_k be the full subcategory of Def_k whose objects are definable subassignments of $h_{\mathbb{A}_k^n}$ for n in $\mathbb{N}$.

Let X be an affine algebraic k-variety. A *(good) μ_n-action* on h_X is a definable morphism of definable subassignments $h_{\mu_n\times X}\to h_X$ such that the corresponding morphism of k-varieties $\mu_n\times_k X\to X$ is a (good) μ_n-action. A *good $\hat{\mu}$-action* on h_X is a good μ_n-action on h_X for some integer $n\geq 1$. For an algebraic $k((t))$-variety $\mathcal{X}$, the definable subassignment $h_{\mathcal{X}}$ admits a natural μ_n-action $h_{\mu_n}\times h_{\mathcal{X}}\to h_{\mathcal{X}}$ induced by $(\lambda,t)\mapsto\lambda t$, for all $n\in\mathbb{N}_{>0}$. The profinite group scheme $\hat{\mu}$ acts naturally on $h_{\mathcal{X}}$ via μ_n for some integer $n\geq 1$.

The Grothendieck semiring and ring of the category RDef_k are defined in [1, Section 5.1.2]; however, in this note we only want to work with its $\hat{\mu}$-equivariant version. By definition, the *$\hat{\mu}$-equivariant Grothendieck group* $K_0^{\hat{\mu}}(\mathrm{RDef}_k)$ is the quotient of the free abelian group generated by definable $\hat{\mu}$-equivariant isomorphism classes $[\mathrm{X},\sigma]$, with X in RDef_k endowed with a good $\hat{\mu}$-action σ, modulo the relations: $[\mathrm{X},\sigma]=[\mathrm{Y},\sigma|_{\mathrm{Y}}]+[\mathrm{X}\setminus\mathrm{Y},\sigma|_{\mathrm{X}\setminus\mathrm{Y}}]$, for Y σ-stable definable subassignment of X, and $[\mathrm{X}\times h_{\mathbb{A}_k^m},\sigma]=[\mathrm{X}\times h_{\mathbb{A}_k^m},\sigma']$, whenever σ and σ' lift the same $\hat{\mu}$-action on X to an affine action on $\mathrm{X}\times h_{\mathbb{A}_k^m}$, for any integer $m\geq 0$. The Cartesian product of subassignments induces a commutative with unity ring structure on $K_0^{\hat{\mu}}(\mathrm{RDef}_k)$.

Put $\mathbb{A}:=\mathbb{Z}\left[\mathbb{L},\mathbb{L}^{-1},\frac{1}{1-\mathbb{L}^{-n}}\mid n\in\mathbb{N}_{>0}\right]$ where, by abuse of notation, $\mathbb{L}$ also stands for the class of $h_{\mathbb{A}_k^1}$ in $K_0^{\hat{\mu}}(\mathrm{RDef}_k)$. By quantifier elimination for the theory of algebraically closed fields containing k (i.e. the Chevalley constructibility), definable subassignments of $h_{\mathbb{A}_k^n}$, for n in $\mathbb{N}$, are defined by formulas without quantifiers, thus objects in RDef_k may be viewed as constructible sets. This correspondence is compatible with the $\hat{\mu}$-actions mentioned above. Hence, there are canonical isomorphisms of rings $K_0^{\hat{\mu}}(\mathrm{RDef}_k)\cong K_0^{\hat{\mu}}(\mathrm{Var}_k)$ and $K_0^{\hat{\mu}}(\mathrm{RDef}_k)\otimes_{\mathbb{Z}[\mathbb{L}]}\mathbb{A}\cong\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$.

2.2. Motivic measure

In view of Theorem 10.1.1 in the paper [1], there is a unique functor from Def_k to the category of abelian semigroups, $\mathrm{X}\mapsto\mathrm{IC}_+(\mathrm{X})$, which assigns to the projection $\mathrm{X}\to h_{\mathrm{Spec}k}$ a morphism of semigroups $\mu:\mathrm{IC}_+(\mathrm{X})\to\mathrm{IC}_+(h_{\mathrm{Spec}k})$, such that the eight axioms (A1) to (A8) in that theorem, characterizing an integration theory, are satisfied. By [1, Proposition 12.2.2], if X is a definable subassignment of $h[m,n,0]$ which is bounded, i.e. there exists an s in $\mathbb{N}$ such that X is contained in the subassignment of $h[m,n,0]$ defined by $\mathrm{ord}_t x_i\geq -s$ for $1\leq i\leq m$, then the characteristic function $\mathbf{1}_{\mathrm{X}}$ is in $\mathrm{IC}_+(\mathrm{X})$. In this case, we call X *motivically measurable* and its motivic measure $\mu(\mathrm{X}):=\mu(\mathbf{1}_{\mathrm{X}})\in\mathrm{IC}_+(h_{\mathrm{Spec}k})$. Also by [1], there is a canonical morphism from $\mathrm{IC}_+(h_{\mathrm{Spec}k})$ to $K_0(\mathrm{Var}_k)\otimes_{\mathbb{Z}[\mathbb{L}]}\mathbb{A}$, thus, by composition, we can consider $\mu(\mathrm{X})$ as an element of $K_0(\mathrm{Var}_k)\otimes_{\mathbb{Z}[\mathbb{L}]}\mathbb{A}$.

In the previous definition of boundedness, if we can take $s=0$, X is called *small* (see [1, Section 16.3] for a more general definition of *small definable subassignments*). There is a canonical action of $\hat{\mu}$ on $h[m,0,0]$ induced by $(\lambda,t)\mapsto\lambda t$. We say that the definable subassignment X is stable under this action if there exists a natural number $n\geq 1$ such that, for every $x=(x_1(t),\ldots,x_m(t))$ in X and λ in μ_n, the point $\lambda\cdot x=(x_1(\lambda t),\ldots,x_m(\lambda t))$ is in X. Since formulas defining X are in the language $\mathcal{L}_{\mathrm{DP,P}}$, by quantifier elimination for algebraically closed fields, they also define a semi-algebraic subset X of the arc space $\mathcal{L}(\mathbb{A}_k^m)$ of $\mathbb{A}_k^m$. The assignment $\mathsf{X} \mapsto X$ carries the canonical $\hat{\mu}$-action on $h[m,0,0]$ to the canonical $\hat{\mu}$-action on $\mathcal{L}(\mathbb{A}_k^m)$, and in that way, X is also stable for the action on $\mathcal{L}(\mathbb{A}_k^m)$. As in [1, Theorem 16.3.1, Remark 16.3.2], we can see that X is measurable as X is measurable, and that since (with the above action) $\mu'(X)$ is in $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$, the measure $\mu(\mathsf{X})$ of X is also in $K_0^{\hat{\mu}}(\mathrm{RDef}_k) \otimes_{\mathbb{Z}[\mathbb{L}]} \mathbb{A} \cong \mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$. Here, as explained in [1, Theorem 16.3.1], μ' stands for Denef–Loeser's motivic measure [3], and further by [1, Remark 16.3.2], we can consider that this measure takes value in $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$.

3. Sketch of proof of Theorem 1.2

Let us consider the motivic zeta function $Z_f(T)$ of the polynomial f in the theorem. Write the n-th coefficient of $\int_{\mathbb{A}_k^{d_1}} i^* Z_f(T)$ as follows:

$$\int_{\mathbb{A}_k^{d_1}} i^*[\mathcal{X}_n(f)]\mathbb{L}^{-nd} = [\mathcal{U}_n]\mathbb{L}^{-nd} + [\mathcal{W}_n]\mathbb{L}^{-nd},$$

where $\mathcal{U}_n$ is the set of jets in $\mathcal{X}_n(f) \cap \big(\mathcal{L}_n(\{0\} \times \mathbb{A}_k^{d_2+d_3}) \cup \mathcal{L}_n(\mathbb{A}_k^{d_1} \times \{0\} \times \mathbb{A}_k^{d_3})\big)$ originated in $\mathbb{A}_k^{d_1}$, $\mathcal{W}_n$ is the set of jets in $\mathcal{X}_n(f)$ that are originated in $\mathbb{A}_k^{d_1}$ and not contained in $\mathcal{U}_n$. The elements $\mathcal{U}_n$ and $\mathcal{W}_n$ are μ_n-stable, and they give rise to rational series with coefficients in $\mathcal{M}_k^{\hat{\mu}}$. Because of the hypothesis on f, taking $\lim_{T\to\infty}$ for the decomposition of the $\int_{\mathbb{A}_k^{d_1}} i^* Z_f(T)$ reduces the proof of Theorem 1.2 to checking that $\mathrm{loc}\big(\lim_{T\to\infty} \sum_{n\geq 1} [\mathcal{W}_n]\mathbb{L}^{-nd}T^n\big)$ vanishes in $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$. The latter has been a challenging problem, and the previous attempts [5], [6] and [7] for solving it had to use certain additional assumptions.

Now we write $\mathcal{W}_{n,m}$ for the set of $(\varphi_1, \varphi_2, \varphi_3)$ in $\mathcal{W}_n$ with $\mathrm{ord}_t\varphi_1 + \mathrm{ord}_t\varphi_2 = m$, and let us observe that it is still stable under the canonical μ_n-action. In what follows, we only consider the set $\mathcal{W}_{n,m}$ when it is nonempty. Suggested ideally from [6], with the hypothesis of Theorem 1.2, our approach is to construct a constructible set $\widetilde{\mathcal{W}}_{n,m}$ endowed with a good $\hat{\mu}$-action and a $\hat{\mu}$-equivariant constructible surjective morphism $\mathcal{W}_{n,m} \to \widetilde{\mathcal{W}}_{n,m}$ such that its fiber over a point of residue field k' is isomorphic to $\mathbb{A}_{k'}^{n+1} \setminus \mathbb{A}_{k'}^{n+1-m}$. Once we can do this, it follows (not obvious) that $[\mathcal{W}_{n,m}] = [\widetilde{\mathcal{W}}_{n,m}]\mathbb{L}^{n+1}(1-\mathbb{L}^{-m})$ in $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$, and then, Theorem 1.2 will be proved completely, because the rest of the proof is elementary.

Our idea is that we use Cluckers–Loeser's motivic integration [1], together to a slight development to $\hat{\mu}$-action context, as seen in Section 2. Clearly, $\mathbb{G}_{m,k((t))}$ is an algebraic group and the action of $\mathbb{G}_{m,k((t))}$ on the $k((t))$-variety

$$\mathcal{X} := \big(\mathbb{A}_{k((t))}^{d_1} \setminus \{0\}\big) \times_{k((t))} \big(\mathbb{A}_{k((t))}^{d_2} \setminus \{0\}\big) \times_{k((t))} \mathbb{A}_{k((t))}^{d_3}$$

given by

$$\tau \cdot (\varphi_1, \varphi_2, \varphi_3) := (\tau\varphi_1, \tau^{-1}\varphi_2, \varphi_3)$$

is free. It follows that the space of its orbits is an algebraic $k((t))$-variety and the canonical projection $\phi : \mathcal{X} \to \mathcal{Y} := \mathcal{X}/\mathbb{G}_{m,k((t))}$ is a surjective morphism of algebraic $k((t))$-varieties. This morphism ϕ induces a definable morphism $h_\phi : h_{\mathcal{X}} \to h_{\mathcal{Y}}$ of definable subassignments in the theory of Cluckers and Loeser. Take the preimage $\mathcal{W}_{n,m}^{\infty}$ of $\mathcal{W}_{n,m}$ under the canonical morphism $\mathcal{L}(\mathbb{A}_k^d) \to \mathcal{L}_n(\mathbb{A}_k^d)$. By [1, Section 16.3], $\mathcal{W}_{n,m}^{\infty}$ corresponds to a small definable subassignment $\mathsf{V}_{n,m}$ of the basic definable subassignment $h[d,0,0]$, both have the same measure $[\mathcal{W}_{n,m}]\mathbb{L}^{-nd}$ in $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}} \cong K_0^{\hat{\mu}}(\mathrm{RDef}_k) \otimes_{\mathbb{Z}[\mathbb{L}]} \mathbb{A}$ in Denef–Loeser's motivic measure and Cluckers–Loeser's motivic measure, endowed with $\hat{\mu}$-action, respectively. By the Denef–Pas quantifier elimination theorem, $h_\phi(\mathsf{V}_{n,m})$ is a definable subassignment and the restriction $h_\phi|_{\mathsf{V}_{n,m}}$ is a definable morphism. Moreover, we can prove that $h_\phi(\mathsf{V}_{n,m})$ and $h_\phi|_{\mathsf{V}_{n,m}}$ are small, and that the fiber of $h_\phi|_{\mathsf{V}_{n,m}}$ over $[(\varphi_1, \varphi_2, \varphi_3)] \in h_\phi(\mathsf{V}_{n,m})$ (of residue field k') equals

$$\big\{\tau \in h_{\mathbb{G}_{m,k'((t))}} \mid -\mathrm{ord}_t\varphi_1 \leq \mathrm{ord}_t\tau < \mathrm{ord}_t\varphi_2\big\} \cong \big\{\tau \in h_{\mathbb{G}_{m,k'((t))}} \mid 0 \leq \mathrm{ord}_t\tau < m\big\}.$$

By [1, Section 16.3], $h_\phi|_{\mathsf{V}_{n,m}}$ gives rise to a μ_n-equivariant semi-algebraic morphism of semi-algebraic sets in Denef–Loeser's framework $p : \mathcal{W}_{n,m}^{\infty} \to \widetilde{\mathcal{W}}_{n,m}^{\infty}$ with fiber over a point of residue field k' isomorphic to $\{\tau \in \mathcal{L}(\mathbb{A}_{k'}^1) \mid 0 \leq \mathrm{ord}_t\tau < m\}$. Finally, we can show that p induces a μ_n-equivariant constructible morphism of constructible sets $p_n : \mathcal{W}_{n,m} \to \widetilde{\mathcal{W}}_{n,m}$ with fiber $\{\tau \in \mathcal{L}_n(\mathbb{A}_{k'}^1) \mid 0 \leq \mathrm{ord}_t\tau < m\} \cong \mathbb{A}_{k'}^{n+1} \setminus \mathbb{A}_{k'}^{n+1-m}$, as desired.

Acknowledgements

This research is funded by the Vietnam National University, Hanoi (VNU) under project number QG.16.06. This research is also supported by ERCEA Consolidator Grant 615655 – NMST and by the Basque Government through the BERC 2014–2017 program and by Spanish Ministry of Economy and Competitiveness MINECO: BCAM Severo Ochoa excellence accreditation SEV-2013-0323.

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