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On the irreducible action of PSL$(2,\mathbb{R})$ on the 3-dimensional Einstein universe ☆

Sur l'action irréductible de PSL$(2,\mathbb{R})$ sur l'univers d'Einstein de dimension 3

Masoud Hassani [a,b]

[a] *Université d'Avignon, Campus Jean-Henri-Fabre, 301, rue Baruch-de-Spinoza, BP 21239, 84916 Avignon cedex 9, France*
[b] *University of Zanjan, Faculty of Science, Department of Mathematics, University blvd, Zanjan, Iran*

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A B S T R A C T

In this paper, we study the irreducible representation of PSL$(2,\mathbb{R})$ in PSL$(5,\mathbb{R})$. This action preserves a quadratic form with signature $(2,3)$. Thus, it acts conformally on the 3-dimensional Einstein universe $\mathbb{E}\text{in}^{1,2}$. We describe the orbits induced in $\mathbb{E}\text{in}^{1,2}$ and its complement in $\mathbb{RP}^4$. This work completes the study in [2], and is one element of the classification of cohomogeneity one actions on $\mathbb{E}\text{in}^{1,2}$ [5].

R É S U M É

Dans cet article, nous étudions l'action irréductible de PSL$(2,\mathbb{R})$ dans PSL$(5,\mathbb{R})$. Cette action préserve une forme quadratique de signature $(2,3)$. Elle sur agit donc conformément sur l'univers d'Einstein $\mathbb{E}\text{in}^{1,2}$ de dimension 3, ainsi que sur son complément dans $\mathbb{RP}^4$. Ce travail complète l'étude préliminaire dans [2], et est un élément de la classification des actions sur $\mathbb{E}\text{in}^{1,2}$ de cohomogenéité un [5].

1. Introduction

1.1. The irreducible representation of PSL$(2,\mathbb{R})$

Let V denote an n-dimensional vector space. A subgroup of $GL(V)$ is **irreducible** if it preserves no proper subspace of V.

It is well known that, for every integer n, up to isomorphism, there is only one n-dimensional irreducible representation of PSL$(2,\mathbb{R})$. For $n=5$, this representation is the natural action of PSL$(2,\mathbb{R})$ on the vector space $\mathbb{V}=\mathbb{R}_4[X,Y]$ of homogeneous polynomials of degree 4 in two variables X and Y. This action induces three types of orbits in the 4-dimensional

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E-mail addresses: masoud.hasani@znu.ac.ir, masoud.hassani@alumni.univ-avignon.fr.

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projective space $\mathbb{RP}^4 = \mathbb{P}(\mathbb{V})$: an 1-dimensional orbit, three 2-dimensional orbits, and the orbits on which $\mathrm{PSL}(2,\mathbb{R})$ acts freely.

The irreducible action of $\mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{V}$ preserves the following quadratic form

$$\mathfrak{q}(a_4X^4 + a_3X^3Y + a_2X^2Y^2 + a_1XY^3 + a_0Y^4) = 2a_4a_0 - \frac{1}{2}a_1a_3 + \frac{1}{6}a_2^2.$$

The quadratic form $\mathfrak{q}$ is non-degenerate and has signature $(2,3)$. This induces an irreducible representation $\mathrm{PSL}(2,\mathbb{R}) \to O(2,3) \subset \mathrm{PSL}(5,\mathbb{R})$ [2]. On the other hand, by [3, Theorem 1], up to conjugacy, $SO_\circ(1,2) \simeq \mathrm{PSL}(2,\mathbb{R})$ is the only irreducible connected Lie subgroup of $O(2,3)$.

1.2. Einstein's universe

Let $\mathbb{R}^{2,3}$ denote a 5-dimensional real vector space equipped with a non-degenerate symmetric bilinear form $\mathfrak{q}$ with signature $(2,3)$. The null cone of $\mathbb{R}^{2,3}$ is

$$\mathfrak{N} = \{v \in \mathbb{R}^{2,3} \setminus \{0\} : \mathfrak{q}(v) = 0\}.$$

The 3-dimensional **Einstein universe** $\mathbb{E}\mathrm{in}^{1,2}$ is the image of the null cone $\mathfrak{N}$ under the projectivization:

$$\mathbb{P} : \mathbb{R}^{2,3} \setminus \{0\} \longrightarrow \mathbb{RP}^4.$$

The degenerate metric on $\mathfrak{N}$ induces a $O(2,3)$-invariant conformal Lorentzian structure on the Einstein universe. The group of conformal transformations on $\mathbb{E}\mathrm{in}^{1,2}$ is $O(2,3)$ [4].

A light-like geodesic in Einstein's universe is a **photon**. A photon is the projectivization of an isotropic 2-plane in $\mathbb{R}^{2,3}$. The set of photons through a point $p \in \mathbb{E}\mathrm{in}^{1,2}$ denoted by $L(p)$ is the **lightcone** at p. The complement of a lightcone $L(p)$ in Einstein's universe is the **Minkowski patch** at p and we denote it by $\mathrm{Mink}(p)$. A Minkowski patch is conformally equivalent to the 3-dimensional Minkoski space $\mathbb{E}^{1,2}$ [1].

The complement to the Einstein universe in $\mathbb{RP}^4$ has two connected components: the 3-dimensional Anti de-Sitter space $\mathrm{AdS}^{1,2}$ and the generalized hyperbolic space $\mathbb{H}^{2,2}$: the first (respectively the second) is the projection of the domain $\mathbb{R}^{2,3}$ defined by $\{\mathfrak{q} < 0\}$ (respectively $\{\mathfrak{q} > 0\}$).

An immersed submanifold S of $\mathrm{AdS}^{1,2}$ or $\mathbb{H}^{2,2}$ is of **signature** (p,q,r) (respectively $\mathbb{E}\mathrm{in}^{1,2}$) if the restriction of the ambient pseudo-Riemannian metric (respectively the conformal Lorentzian metric) is of signature (p,q,r), meaning that the radical has dimension r, and that maximal definite negative and positive subspaces have dimensions p and q, respectively. If S is nondegenerate, we forgot r and simply denote its signature by (p,q).

Theorem 1.1. *The irreducible action of PSL$(2,\mathbb{R})$ on the* 3*-dimensional Einstein universe* $\mathbb{E}\mathrm{in}^{1,2}$ *admits three orbits:*

- *a* 1*-dimensional light-like orbit, i.e. of signature* $(0,0,1)$*,*
- *a* 2*-dimensional orbit of signature* $(0,1,1)$*,*
- *an open orbit (hence of signature* $(1,2)$*) on which the action is free.*

The 1-dimensional orbit is light-like, homeomorphic to $\mathbb{RP}^1$, but not a photon. The union of the 1-dimensional orbit and the 2-dimensional orbit is an algebraic surface, whose singular locus is precisely the 1-dimensional orbit. It is the union of all projective lines tangent to the 1-dimensional orbit. Fig. 1 describes a part of the 1 and 2-dimensional orbits in the Minkowski patch $\mathrm{Mink}([Y^4])$.

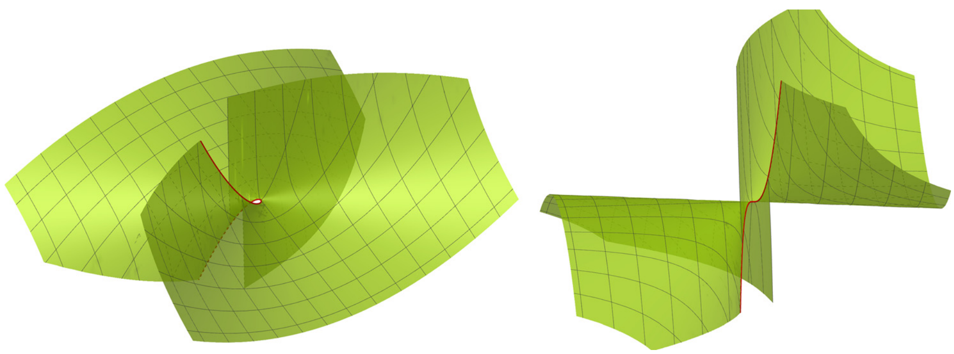

Fig. 1. Two partial views of the intersection of the 1 and 2-dimensional orbits in Einstein's universe with $\mathrm{Mink}([Y^4])$. **Red:** Part of the 1-dimensional orbit in Minkowski patch. **Green:** Part of the 2-dimensional orbit in Minkowski patch.

We will also describe the actions on the Anti de-Sitter space and the generalized hyperbolic space $\mathbb{H}^{2,2}$:

Theorem 1.2. *The orbits of PSL$(2,\mathbb{R})$ in the Anti de-sitter component* $\mathrm{AdS}^{1,3}$ *are Lorentzian, i.e. of signature* $(1,2)$. *They are the leaves of a codimension-*1 *foliation. In addition, PSL*$(2,\mathbb{R})$ *induces three types of orbits in* $\mathbb{H}^{2,2}$: *a* 2*-dimensional space-like orbit (of signature* $(2,0)$*) homeomorphic to the hyperbolic plane* $\mathbb{H}^2$, *a* 2*-dimensional Lorentzian orbit (i.e. of signature* $(1,1)$*) homeomorphic to the de-Sitter space* $\mathrm{dS}^{1,1}$, *and four kinds of* 3*-dimensional orbits where the action is free:*

- *a one-parameter family of orbits of signature* $(2,1)$, *consisting of elements with four distinct non-real roots,*
- *a one-parameter family of Lorentzian (i.e. of signature* $(1,2)$*) orbits consisting of elements with four distinct real roots,*
- *two orbits of signature* $(1,1,1)$,
- *a one-parameter family of Lorentzian (i.e. of signature* $(1,2)$*) orbits consisting of elements with two distinct real roots, and two distinct complex conjugate roots so that the cross-ratio of the four roots has an argument strictly between* $-\pi/3$ *and* $\pi/3$.

2. Proofs of the theorems

Let f be an element in $\mathbb{V}$. We consider it as a polynomial function from $\mathbb{C}^2$ into $\mathbb{C}$. Actually, by specifying $Y=1$, we consider f as a polynomial of degree at most 4. Such a polynomial is determined, up to a scalar, by its roots z_1, z_2, z_3, z_4 in $\mathbb{CP}^1$ (some of these roots can be ∞ if f can be divided by Y). It provides a natural identification between $\mathbb{P}(\mathbb{V})$ and the set $\widehat{\mathbb{CP}}^1_4$ made of 4-tuples (up to permutation) (z_1,z_2,z_3,z_4) of $\mathbb{CP}^1$ such that if some z_i is not in $\mathbb{RP}^1$, then its conjugate $\bar{z}_i$ is one of the z_j's. This identification is PSL$(2,\mathbb{R})$-equivariant, where the action of PSL$(2,\mathbb{R})$ on $\widehat{\mathbb{CP}}^1_4$ is simply the one induced by the diagonal action on $(\mathbb{CP}^1)^4$.

Actually, the complement of $\mathbb{RP}^1$ in $\mathbb{CP}^1$ is the union of the upper half-plane model $\mathbb{H}^2$ of the hyperbolic plane, and the lower half-plane. We can represent every element of $\widehat{\mathbb{CP}}^1_4$ by a 4-tuple (up to permutation) (z_1,z_2,z_3,z_4) such that:

- either every z_i lies in $\mathbb{RP}^1$,
- or z_1, z_2 lies in $\mathbb{RP}^1$, z_3 lies in $\mathbb{H}^2$ and $z_4=\bar{z}_3$,
- or z_1, z_2 lies in $\mathbb{H}^2$ and $z_3=\bar{z}_1$, $z_4=\bar{z}_2$.

Theorems 1.1 and 1.2 will follow from the proposition below.

Proposition 2.1. *Let* $[f]$ *be an element of* $\mathbb{P}(\mathbb{V})$. *Then:*

- *it lies in* $\mathbb{E}\mathrm{in}^{1,2}$ *if and only if it has a root of multiplicity at least* 3, *or two distinct real roots* z_1, z_2, *and two complex roots* z_3, $z_4=\bar{z}_3$, *with* z_3 *in* $\mathbb{H}^2$ *and such that the argument of the cross-ratio of* z_1,z_2,z_3,z_4 *is* $\pm\pi/3$;
- *it lies in* $\mathrm{AdS}^{1,3}$ *if and only it has two distinct real roots* z_1, z_2, *and two complex roots* z_3, $z_4=\bar{z}_3$, *with* z_3 *in* $\mathbb{H}^2$ *and such that the argument of the cross-ratio of* z_1,z_2,z_3,z_4 *has absolute value* $>\pi/3$;
- *it lies in* $\mathbb{H}^{2,2}$ *if and only if it has no real roots, or four distinct real roots, or a root of multiplicity exactly* 2, *or it has two distinct real roots* z_1, z_2, *and two complex roots* z_3, $z_4=\bar{z}_3$, *with* z_3 *in* $\mathbb{H}^2$ *and such that the argument of the cross-ratio of* z_1,z_2,z_3,z_4 *has absolute value* $<\pi/3$.

Proof of Proposition 2.1. Assume that f has no real root. Hence we are in the situation where z_1, z_2 lie in $\mathbb{H}^2$ and $z_3=\bar{z}_1$, $z_4=\bar{z}_2$. By applying a suitable element of PSL$(2,\mathbb{R})$, we can assume $z_1=i$, and $z_2=ri$ for some $r>0$. In other words, f is in the PSL$(2,\mathbb{R})$-orbit of $(X^2+Y^2)(X^2+r^2Y^2)$. The value of $\mathfrak{q}$ on this polynomial is $2\times 1\times r^2+\frac{1}{6}(1+r^2)^2>0$, hence $[f]$ lies in $\mathbb{H}^{2,2}$.

Hence, we can assume that f admits at least one root in $\mathbb{RP}^1$, and by applying a suitable element of PSL$(2,\mathbb{R})$, one can assume that this root is ∞, i.e. that f is a multiple of Y.

We first consider the case where this real root has multiplicity at least 2:

$$f=Y^2(aX^2+bXY+cY^2)$$

Then, $\mathfrak{q}(f)=\frac{1}{6}a^2$: it follows that if f has a root of multiplicity at least 3, it lies in $\mathbb{E}\mathrm{in}^{1,2}$, and if it has a real root of mulitplicity 2, it lies in $\mathbb{H}^{2,2}$.

We assume from now on that the real roots of f have multiplicity 1. Assume that all roots are real. Up to PSL$(2,\mathbb{R})$, one can assume that these roots are 0, 1, r and ∞ with $0<r<1$.

$$f(X,Y)=XY(X-Y)(X-rY)=X^3Y-(r+1)X^2Y^2+rXY^3.$$

Then, $\mathfrak{q}(f)=-\frac{1}{2}r+\frac{1}{6}(r+1)^2=\frac{1}{6}(r^2-r+1)>0$. Therefore, f lies in $\mathbb{H}^{2,2}$ once more.

The only remaining case is the case where f has two distinct real roots, and two complex conjugate roots z, $\bar{z}$ with $z\in\mathbb{H}^2$. Up to PSL$(2,\mathbb{R})$, one can assume that the real roots are 0, ∞, hence:

$$f(X,Y) = XY(X - zY)(X - \bar{z}Y) = XY(X^2 - 2|z|\cos\theta XY + |z|^2Y^2)$$

where $z = |z|\,\mathrm{e}^{\mathrm{i}\theta}$. Then:

$$\mathfrak{q}(f) = \frac{2|z|^2}{3}(\cos^2\theta - \frac{3}{4}).$$

Hence f lies in $\mathbb{E}\mathrm{in}^{1,2}$ if and only if $\theta = \pi/6$ or $5\pi/6$. The proposition follows easily.

Remark 1. F. Fillastre indicated to us that our description of the open orbit in $\mathbb{E}\mathrm{in}^{1,2}$ appearing in the first item of Proposition 2.1 has an alternative and more elegant description: this orbit corresponds to polynomials whose roots in $\mathbb{CP}^1$ are ideal vertices of a regular ideal tetraedra in $\mathbb{H}^3$.

Remark 2. In order to determine the signature of the orbits induced by $\mathrm{PSL}(2,\mathbb{R})$ in $\mathbb{P}(\mathbb{V})$, we consider the tangent vectors induced by the action of 1-parameter subgroups of $\mathrm{PSL}(2,\mathbb{R})$. We denote by E, P, and H, the 1-parameter elliptic, parabolic and hyperbolic subgroups stabilizing i, ∞ and $\{0,\infty\}$, respectively.

Proof of Theorem 1.1. It follows from Proposition 2.1 that there are precisely three $\mathrm{PSL}(2,\mathbb{R})$-orbits in $\mathbb{E}\mathrm{in}^{1,2}$:

- one orbit $\mathcal{N}$ comprising polynomials with a root of multiplicity 4, i.e. of the form $[(sY - tX)^4]$ with $s,t \in \mathbb{R}$. It is clearly 1-dimensional, and equivariantly homeomorphic to $\mathbb{RP}^1$ with the usual projective action of $\mathrm{PSL}(2,\mathbb{R})$. Since $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(Y - tX)^4 = -4XY^3$ is a $\mathfrak{q}$-null vector, this orbit is light-like (but cannot be a photon since the action is irreducible);
- one orbit $\mathcal{L}$ comprising polynomials with a real root of multiplicity 3, and another real root. These are the polynomials of the form $[(sY - tX)^3(s'Y - t'X)]$ with $s,t,s',t' \in \mathbb{R}$. It is 2-dimensional, and it is easy to see that it is the union of the projective lines tangent to $\mathcal{N}$. The vectors tangent to $\mathcal{L}$ induced by the 1-parameter subgroups P and E at $[XY^3] \in \mathcal{L}$ are $v_P = -Y^4$ and $v_E = 3X^2Y^2 + Y^4$. Obviously, v_P is orthogonal to v_E and $v_E + v_P$ is space-like. Hence $\mathcal{L}$ is of signature $(0,1,1)$;
- one open orbit comprising polynomials admitting two distinct real roots and a root in $\mathbb{H}^2$ such that the argument of the cross-ratio of the four roots is $\pi/3$. The stabilizers of points in this orbit are trivial, since an isometry of $\mathbb{H}^2$ preserving a point in $\mathbb{H}^2$ and one point in $\partial\mathbb{H}^2$ is necessarily the identity. □

Proof of Theorem 1.2. According to Proposition 2.1, the polynomials in $\mathrm{AdS}^{1,3}$ have two distinct real roots, and a complex root $\mathbb{H}^2$ (and its conjugate) such that the argument of the cross-ratio of the four roots has absolute value $> \pi/3$. It follows that the action in $\mathrm{AdS}^{1,3}$ is free, and that the orbits are the level sets of the function θ. Suppose that M is a $\mathrm{PSL}(2,\mathbb{R})$-orbit in $\mathrm{AdS}^{1,3}$. There exists $r \in \mathbb{R}$ such that $[f] = [Y(X^2+Y^2)(X - rY)] \in M$. The orbit induced by the 1-parameter subgroup E at $[f]$ is:

$$\gamma(t) = \big[(X^2+Y^2)\big((\sin t\cos t - r\sin^2 t)X^2 - (\sin t\cos t + r\cos^2 t)Y^2 + (\cos^2 t - \sin^2 t + 2r\sin t\cos t)XY\big)\big].$$

Then $\mathfrak{q}(\frac{d\gamma}{dt}|_{t=0}) = -2 - 2r^2 < 0$. This implies, as for any submanifold of a Lorentzian manifold admitting a time-like vector, that M is Lorentzian, i.e., of signature $(1,2)$.

The case of $\mathbb{H}^{2,2}$ is the richest one. According to Proposition 2.1, there are four cases to consider.

- *No real roots.* Then f has two complex roots z_1, z_2 in $\mathbb{H}^2$ (and their conjugates). It corresponds to two orbits: one orbit corresponding to the case $z_1 = z_2$: it is space-like and has dimension 2. It is the only maximal $\mathrm{PSL}(2,\mathbb{R})$-invariant surface in $\mathbb{H}^{2,2}$ described in [2, Section 5.3]. The case $z_1 \neq z_2$ provides a one-parameter family of 3-dimensional orbits on which the action is free (the parameter being the hyperbolic distance between z_1 and z_2). One may assume that $z_1 = i$ and $z_2 = ri$ for some $r > 0$. Denote by M the orbit induced by $\mathrm{PSL}(2,\mathbb{R})$ at $[f] = [(X^2+Y^2)(X^2+r^2Y^2)]$. The vectors tangent to M at $[f]$ induced by the 1-parameter subgroups H, P and E are:

  $$v_H = -4X^4 + 4r^2Y^4, \quad v_P = -4X^3Y - 2(r^2+1)XY^3, \quad v_E = 2(r^2-1)X^3Y + 2(r^2-1)XY^3,$$

  respectively. The time-like vector v_H is orthogonal to both v_P and v_E. It is easy to see that the 2-plane generated by $\{v_P, v_E\}$ is of signature $(1,1)$. Therefore, the tangent space $T_{[f]}M$ is of signature $(2,1)$.
- *Four distinct real roots.* This case provides a one-parameter family of 3-dimensional orbits on which the action is free – the parameter being the cross-ratio between the roots in $\mathbb{RP}^1$. Denote by M the $\mathrm{PSL}(2,\mathbb{R})$-orbit at $[f] = [XY(X - Y)(X - rY)]$ (here as explained in the proof of Proposition 2.1, we can restrict ourselves to the case $0 < r < 1$). The vectors tangent to M at $[f]$ induced by the 1-parameter subgroups H, P, and E are:

  $$v_H = -rY^4 + 2(r+1)XY^3 - 3X^2Y^2, \qquad v_P = -2X^3Y + 2rXY^3,$$
  $$v_E = X^4 - rY^4 + 3(r-1)X^2Y^2 + 2(r+1)XY^3 - 2(r+1)X^3Y,$$

respectively. A vector $x = av_H + bv_P + cv_E$ is orthogonal to v_P if and only if $2ra + b(r+1) + c(r+1)^2 = 0$. Set $a = \big(b(r+1) + c(r+1)^2\big)/-2r$ in

$$\mathfrak{q}(x) = 2ra^2 + \frac{3}{2}b^2 + \big(\frac{7}{2}(r^2+1) - r\big)c^2 + 2(r+1)ab + 2(r+1)^2 + ac(2r^2 - r + 5).$$

Consider $\mathfrak{q}(x) = 0$ as a quadratic polynomial F in b. Since $0 < r < 1$, the discriminant of F is non-negative and it is positive when $c \neq 0$. Thus, the intersection of the orthogonal complement of the space-like vector v_P with the tangent space $T_{[f]}M$ is a 2-plane of signature $(1,1)$. This implies that M is Lorentzian, i.e. of signature $(1,2)$.

– *A root of multiplicity* 2. Observe that if there is a non-real root of multiplicity 2, when we are in the first "no real root" case. Hence we consider here only the case where the root of multiplicity 2 lies in $\mathbb{RP}^1$. Then, we have three subcases to consider:
  – two distinct real roots of multiplicity 2: The orbit induced at X^2Y^2 is the image of the $\mathrm{PSL}(2,\mathbb{R})$-equivariant map

$$\mathrm{dS}^{1,1} \subset \mathbb{P}(\mathbb{R}_2[X,Y]) \longrightarrow \mathbb{H}^{2,2}, \qquad [L] \mapsto [L^2],$$

  where $\mathbb{R}_2[X,Y]$ is the vector space of homogeneous polynomials of degree 2 in two variables X and Y, endowed with discriminant as a $\mathrm{PSL}(2,\mathbb{R})$-invariant bilinear form of signature $(1,2)$ [2, Section 5.3]. (Here, L is the projective class of a Lorentzian bilinear form on $\mathbb{R}^2$.) The vectors tangent to the orbit at X^2Y^2 induced by the 1-parameter subgroups P and E are $v_P = -2XY^3$ and $v_E = 2X^3Y - 2XY^3$, respectively. It is easy to see that the 2-plane generated by $\{v_p, v_E\}$ is of signature $(1,1)$. Hence, the orbit induced at X^2Y^2 is Lorentzian.
  – three distinct real roots, one of them being of multiplicity 2; denote by M the orbit induced by $\mathrm{PSL}(2,\mathbb{R})$ at $[f] = [XY^2(X-Y)]$. The vectors tangent to M at $[f]$ induced by the 1-parameter subgroups H, P and E are:

$$v_H = -2XY^3, \quad v_P = Y^4 - 2XY^3, \quad v_E = Y^4 - X^4 - 2X^2Y^2 + X^3Y - XY^3,$$

  respectively. Obviously, the light-like vector $v_H + v_P$ is orthogonal to $T_{[f]}M$. Therefore, the restriction of the metric on $T_{[f]}M$ is degenerate. It is easy to see that the quotient of $T_{[f]}M$ by the action of the isotropic line $\mathbb{R}(v_H + v_P)$ is of signature $(1,1)$. Thus, M is of signature $(1,1,1)$.
  – one real root of multiplicity 2, and one root in $\mathbb{H}^2$: Denote by M the orbit induced by $\mathrm{PSL}(2,\mathbb{R})$ at $[f] = [Y^2(X^2 + Y^2)]$. The vectors tangent to M at $[f]$ induced by the 1-parameter subgroups H, P and E are $v_H = 4Y^4$, $v_P = -2XY^3$, and $v_E = 2X^3Y + 2XY^3$, respectively. Obviously, the light-like vector v_H is orthogonal $T_{[f]}M$. Therefore, the restriction of the metric on $T_{[f]}M$ is degenerate. It is easy to see that the quotient of $T_{[f]}M$ by the action of the isotropic line $\mathbb{R}(v_H)$ is of signature $(1,1)$. Thus M is of signature $(1,1,1)$.

– *Two distinct real roots, and a complex root in $\mathbb{H}^2$ (and its conjugate) such that the argument of the cross-ratio of the four roots has absolute value* $< \pi/3$. Denote by M the orbit induced by $\mathrm{PSL}(2,\mathbb{R})$ at $[f] = [Y(X^2+Y^2)(X - rY)]$. The vectors tangent to M at $[f]$ induced by the 1-parameter subgroups H, P and E are:

$$v_H = -4rY^4 - 2X^3Y + 2XY^3, \quad v_P = -3X^2Y^2 + 2rXY^3 - Y^4, \quad v_E = X^4 - Y^4 - 2rX^3Y - 2rXY^3,$$

respectively. The following set of vectors is an orthogonal basis for $T_{[f]}M$ where the first vector is time-like and the two others are space-like.

$$\{(7r + 3r^3)v_H + (6 - 2r^2)v_P + (5 + r^2)v_E, 4v_P + v_E, v_H\}.$$

Therefore, M is Lorentzian, i.e. of signature $(1,2)$. □

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