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Complex analysis

On the Erdös–Lax inequality concerning polynomials

Sur l'inégalité d'Erdös–Lax concernant les polynômes

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ABSTRACT

Let $P(z)$ be a polynomial of degree n and for any complex number α, let $D_\alpha P(z) := nP(z) + (\alpha - z)P'(z)$ denote the polar derivative of $P(z)$ with respect to α. In this paper, we present an integral inequality for the polar derivative of a polynomial. Our theorem includes as special cases several interesting generalisations and refinements of Erdöx–Lax theorem.

RÉSUMÉ

Soit $P(z)$ un polynôme de degré n. Pour tout nombre complexe α, notons $D_\alpha P(z) := nP(z) + (\alpha - z)P'(z)$ la dérivée polaire de $P(z)$ relative à α. Dans cette Note, nous présentons une inégalité intégrale pour la dérivée polaire. Notre théorème contient comme cas particuliers plusieurs généralisations et raffinements intéressants du théorème d'Erdös et Lax.

1. Introduction

Let $\mathbb{P}_n$ be the class of polynomials $P(z) = \sum_{v=0}^{n} a_v z^v$ of degree n and $P'(z)$ be the derivative of $P(z)$. For a complex number α and for $P \in \mathbb{P}_n$, let

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z).$$

Note that $D_\alpha P(z)$ is a polynomial of degree $n-1$. This is the so-called polar derivative of $P(z)$ with respect to point α (see [12]). It generalizes the ordinary derivative in the following sense:

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} := P'(z).$$

For $P \in \mathbb{P}_n$, we have

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$$\max_{|z|=1}|P'(z)| \le n \max_{|z|=1}|P(z)| \tag{1.1}$$

and for every $r \ge 1$,

$$\left\{\int_0^{2\pi}\left|P'(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}} \le n\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{1.2}$$

The inequality (1.1) is a classical result of Bernstein [9], whereas the inequality (1.2) is due to Zygmund [15], who proved it for all trigonometric polynomials of degree n and not only for those of the form $P(e^{i\theta})$. Arestov [1] proved that (1.2) remains true for $0 < r < 1$ as well. If we let $r \to \infty$ in (1.2), we get (1.1).

The above two inequalities (1.1) and (1.2) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$. In fact, if $P \in \mathbb{P}_n$ and $P(z) \ne 0$ in $|z| < 1$, then (1.1) and (1.2) can be respectively replaced by

$$\max_{|z|=1}|P'(z)| \le \frac{n}{2} \max_{|z|=1}|P(z)| \tag{1.3}$$

and

$$\left\{\int_0^{2\pi}\left|P'(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}} \le nC_r\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}, \tag{1.4}$$

where

$$C_r = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|1+e^{i\gamma}\right|^r d\gamma\right\}^{\frac{-1}{r}}. \tag{1.5}$$

The inequality (1.3) was conjectured by Erdös and later verified by Lax [7], whereas (1.4) was proved by de Bruijn [6] for $r \ge 1$. Further, Rahman and Schmeisser [11] have shown that (1.4) holds for $0 < r < 1$ as well. If we let $r \to \infty$ in inequality (1.4), we get (1.3).

In the literature, there already exists various refinements and generalisations of (1.3) (for example, see Aziz [2], Aziz and Dawood [3], Mir and Baba [10], Zireh [14], etc).

Aziz was among the first to extend some of the above inequalities by replacing the derivative with the polar derivatives of polynomials. In fact, in 1988, Aziz [2] extended (1.3) to the polar derivative of a polynomial and proved that if $P \in \mathbb{P}_n$ and $P(z) \ne 0$ in $|z| < 1$, then for every complex number α with $|\alpha| \ge 1$,

$$\max_{|z|=1}|D_\alpha P(z)| \le \frac{n}{2}(|\alpha|+1) \max_{|z|=1}|P(z)|. \tag{1.6}$$

As an L_r analogue of (1.6), Aziz and Rather [4] proved that if $P \in \mathbb{P}_n$ and $P(z) \ne 0$ in $|z| < 1$, then for every complex number α with $|\alpha| \ge 1$ and $r \ge 1$,

$$\left\{\int_0^{2\pi}\left|D_\alpha P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}} \le n(|\alpha|+1)C_r\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}, \tag{1.7}$$

where C_r is defined in (1.5).

Further, the following more general result, which in particular provides refinements and generalizations of the inequalities (1.6) and (1.7) and also extends inequality (1.7) for $r \in (0, 1)$, was proved by Mir and Baba [10]. More precisely, they proved that if $P \in \mathbb{P}_n$ and $P(z) \ne 0$ in $|z| < 1$, then for every complex numbers α, δ with $|\alpha| \ge 1$, $|\delta| \le 1$ and $r > 0$,

$$\left\{\int_0^{2\pi}\left|D_\alpha P(e^{i\theta}) + \frac{mn}{2}\delta\big(|\alpha|-1\big)\right|^r d\theta\right\}^{\frac{1}{r}} \le nC_r(|\alpha|+1)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}, \tag{1.8}$$

where here and throughout $m = \min_{|z|=1}|P(z)|$ and C_r is defined in (1.5).

2. Main results

In this paper, we shall prove the following more general result, which as special cases gives interesting generalizations of (1.6), (1.7) and (1.8). More precisely, we prove the following theorem.

Theorem 1. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every complex numbers α, β, δ with $|\alpha| \geq 1$, $|\beta| \leq 1$, $|\delta| \leq 1$ and $r \geq 1$,*

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n\beta \frac{(|\alpha|-1)}{2} P(e^{i\theta}) + \frac{mn}{2}\delta\left(\left|\alpha + \beta\frac{(|\alpha|-1)}{2}\right| - \left|e^{i\theta} + \beta\frac{(|\alpha|-1)}{2}\right| \right) \right|^r d\theta \right\}^{\frac{1}{r}} \leq nC_r\Big((|\alpha|+1) + |\beta|(|\alpha|-1) \Big) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}}, \tag{2.1}$$

where C_r is defined in (1.5).

Remark 1. If we put $\beta = 0$ in the above result, we get (1.8) for $r \geq 1$. Dividing both sides of (2.1) by $|\alpha|$ and let $|\alpha| \to \infty$, we get the following result.

Corollary 1. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then, for every complex numbers β, δ with $|\beta| \leq 1$, $|\delta| \leq 1$ and $r \geq 1$,*

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} P'(e^{i\theta}) + \frac{n\beta}{2} P(e^{i\theta}) + \frac{mn}{2}\delta\left(\left|1 + \frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right| \right) \right|^r d\theta \right\}^{\frac{1}{r}} \leq nC_r\Big(1 + |\beta|\Big) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}}, \tag{2.2}$$

where C_r is defined in (1.5).

Letting $r \to \infty$ in (2.2) and choosing the argument of δ suitably with $|\delta| = 1$, we get the following result.

Corollary 2. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every complex number β with $|\beta| \leq 1$, we have*

$$\max_{|z|=1} |zP'(z) + \frac{n\beta}{2} P(z)| \leq \frac{n}{2}\left\{ (1 + |\beta|) \max_{|z|=1} |P(z)| - \left(\left|1 + \frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right| \right) m \right\}. \tag{2.3}$$

Remark 2. For $\beta = 0$, (2.3) reduces to a result of Aziz and Dawood [3].

Many other interesting results easily follow from Theorem 1. Here, we mention a few of these. Taking $\delta = 0$ in (2.1), we get the following interesting generalisation of (1.7).

Corollary 3. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every complex numbers α, β with $|\alpha| \geq 1$, $|\beta| \leq 1$ and $r \geq 1$,*

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n\beta \frac{(|\alpha|-1)}{2} P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}} \leq nC_r\Big((|\alpha|+1) + |\beta|(|\alpha|-1) \Big) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}}, \tag{2.4}$$

where C_r is defined in (1.5).

Remark 3. If we take $\beta = 0$ in (2.4), we get (1.7). Also if we let $r \to \infty$ and take $\beta = 0$ in (2.4), we get (1.6).

For the proof of Theorem 1, we need the following lemmas.

3. Lemmas

Lemma 1. *Let $Q \in \mathbb{P}_n$ and $Q(z)$ has all its zeros in $|z| \leq 1$ and $P(z)$ be a polynomial of degree at most n. If $|P(z)| \leq |Q(z)|$ for $|z| = 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq 1$, $|\beta| \leq 1$,*

$$\left| zD_\alpha P(z) + n\beta\left(\frac{|\alpha|-1}{2}\right) P(z) \right| \leq \left| zD_\alpha Q(z) + n\beta\left(\frac{|\alpha|-1}{2}\right) Q(z) \right|, \text{ for } |z| \geq 1.$$

The above lemma is due to Liman, Mohapatra and Shah [8].

By applying Lemma 1 to polynomials $P(z)$ and $z^n \min_{|z|=1} |P(z)|$, we get the following result.

Lemma 2. *If $P \in \mathbb{P}_n$ and $P(z)$ has all its zeros in $|z| \le 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \ge 1, |\beta| \le 1$,*

$$\left| zD_\alpha P(z) + \beta n\left(\frac{|\alpha|-1}{2}\right)P(z)\right| \ge n|z|^n \left|\alpha + \beta\left(\frac{|\alpha|-1}{2}\right)\right| \min_{|z|=1} |P(z)|, \text{ for } |z| \ge 1.$$

Lemma 3. *If $P \in \mathbb{P}_n$, then for every complex α and $r > 0$,*

$$\left\{ \int_0^{2\pi} \left| D_\alpha P(e^{i\theta})\right|^r d\theta \right\}^{\frac{1}{r}} \le n(|\alpha|+1) \left\{ \int_0^{2\pi} \left| P(e^{i\theta})\right|^r d\theta \right\}^{\frac{1}{r}}.$$

The above lemma is due to Rather [13].

Lemma 4. *If $P \in \mathbb{P}_n$ and $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then for every $r > 0$ and γ real,*

$$\int_0^{2\pi}\int_0^{2\pi} \left| P'(e^{i\theta}) + e^{i\gamma} Q'(e^{i\theta})\right|^r d\theta\, d\gamma \le 2\pi n^r \int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta.$$

The above lemma is due to Aziz and Rather [5].

Lemma 5. *If $P \in \mathbb{P}_n$ and $P(z) \ne 0$ in $|z| < 1$, then for all β, α with $|\beta| \le 1, |\alpha| \ge 1$ and $|z| = 1$,*

$$\left| zD_\alpha P(z) + \beta n\left(\frac{|\alpha|-1}{2}\right)P(z)\right| \le \left| zD_\alpha Q(z) + \beta n\left(\frac{|\alpha|-1}{2}\right)Q(z)\right| - mn\left\{ \left|\alpha + \beta\left(\frac{|\alpha|-1}{2}\right)\right| - \left|z + \beta\left(\frac{|\alpha|-1}{2}\right)\right| \right\},$$

where $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$.

Proof of Lemma 5. Since $m = \min_{|z|=1} |P(z)|$. If $P(z)$ has a zero on $|z| = 1$, then $m = 0$ and the result follows from Lemma 1 in this case. Henceforth, we suppose that all the zeros of $P(z)$ lie in $|z| > 1$ and so $m > 0$. We have $|\lambda m| < |P(z)|$ on $|z| = 1$ for any λ with $|\lambda| < 1$. It follows by Rouche's theorem that the polynomial $G(z) = P(z) - \lambda m$ has no zeros in $|z| < 1$. Therefore, the polynomial $H(z) = z^n\overline{G(\frac{1}{\bar{z}})} = Q(z) - \bar{\lambda} m z^n$ will have all its zeros in $|z| \le 1$. Also $|G(z)| = |H(z)|$ for $|z| = 1$. On applying Lemma 1, we get for any β, α with $|\beta| \le 1, |\alpha| \ge 1$,

$$\left| zD_\alpha G(z) + \beta n\left(\frac{|\alpha|-1}{2}\right)G(z)\right| \le \left| zD_\alpha H(z) + \beta n\left(\frac{|\alpha|-1}{2}\right)H(z)\right|, \text{ for } |z| \ge 1.$$

Equivalently,

$$\left| zD_\alpha P(z) + \beta n\left(\frac{|\alpha|-1}{2}\right)P(z) - \lambda mn\left(z + \beta\left(\frac{|\alpha|-1}{2}\right)\right)\right| \le \left| zD_\alpha Q(z) + \beta n\left(\frac{|\alpha|-1}{2}\right)Q(z) - \bar{\lambda} mnz^n\left(\alpha + \beta\left(\frac{|\alpha|-1}{2}\right)\right)\right|. \tag{3.1}$$

Since $Q(z)$ has all zeros in $|z| \le 1$ and $\min_{|z|=1} |Q(z)| = \min_{|z|=1} |P(z)| = m$, we have by Lemma 2, for $|z| \ge 1$,

$$\left| zD_\alpha Q(z) + \beta n\left(\frac{|\alpha|-1}{2}\right)Q(z)\right| \ge n\left|\alpha + \beta\left(\frac{|\alpha|-1}{2}\right)\right| m|z|^n. \tag{3.2}$$

Now, by choosing a suitable argument of λ in the right-hand side of (3.1), in view of (3.2), we get, for $|z| = 1$,

$$\left| zD_\alpha P(z) + \beta n\left(\frac{|\alpha|-1}{2}\right)P(z)\right| - |\lambda| mn\left|z + \beta\left(\frac{|\alpha|-1}{2}\right)\right| \le \left| zD_\alpha Q(z) + \beta n\left(\frac{|\alpha|-1}{2}\right)Q(z)\right| - |\lambda| mn\left|\alpha + \beta\left(\frac{|\alpha|-1}{2}\right)\right|.$$

Letting $|\lambda| \to 1$, we get for $|z| = 1$,

$$\left|zD_\alpha P(z)+\beta n\left(\frac{|\alpha|-1}{2}\right)P(z)\right|-mn\left|z+\beta\left(\frac{|\alpha|-1}{2}\right)\right|$$
$$\leq\left|zD_\alpha Q(z)+\beta n\left(\frac{|\alpha|-1}{2}\right)Q(z)\right|-mn\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|.$$

This implies

$$\left|zD_\alpha P(z)+\beta n\left(\frac{|\alpha|-1}{2}\right)P(z)\right|\leq\left|zD_\alpha Q(z)+\beta n\left(\frac{|\alpha|-1}{2}\right)Q(z)\right|$$
$$-mn\left\{\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|-\left|z+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right\},$$

which completes the proof of Lemma 5. □

Lemma 6. *If A, B and C are non-negative real numbers such that $B+C\leq A$, then for every real number α,*

$$\left|(A-C)\mathrm{e}^{\mathrm{i}\alpha}+(B+C)\right|\leq\left|A\mathrm{e}^{\mathrm{i}\alpha}+B\right|.$$

The above lemma is due to Aziz and Rather [4].

4. Proof of the Theorem

Proof of Theorem 1. Since $P\in\mathbb{P}_n$ and $P(z)\neq 0$ in $|z|<1$, then the polynomial $Q(z)=z^n\overline{P(\frac{1}{\bar{z}})}\in\mathbb{P}_n$, and it can be easily verified that, for $0\leq\theta<2\pi$,

$$nP(\mathrm{e}^{\mathrm{i}\theta})-\mathrm{e}^{\mathrm{i}\theta}P'(\mathrm{e}^{\mathrm{i}\theta})=\mathrm{e}^{\mathrm{i}(n-1)\theta}\overline{Q'(\mathrm{e}^{\mathrm{i}\theta})}$$

and

$$nQ(\mathrm{e}^{\mathrm{i}\theta})-\mathrm{e}^{\mathrm{i}\theta}Q'(\mathrm{e}^{\mathrm{i}\theta})=\mathrm{e}^{\mathrm{i}(n-1)\theta}\overline{P'(\mathrm{e}^{\mathrm{i}\theta})}.$$

Hence

$$nP(\mathrm{e}^{\mathrm{i}\theta})+\mathrm{e}^{\mathrm{i}\gamma}nQ(\mathrm{e}^{\mathrm{i}\theta})=\mathrm{e}^{\mathrm{i}\theta}P'(\mathrm{e}^{\mathrm{i}\theta})+\mathrm{e}^{\mathrm{i}(n-1)\theta}\overline{Q'(\mathrm{e}^{\mathrm{i}\theta})}+\mathrm{e}^{\mathrm{i}\gamma}\left(\mathrm{e}^{\mathrm{i}\theta}Q'(\mathrm{e}^{\mathrm{i}\theta})+\mathrm{e}^{\mathrm{i}(n-1)\theta}\overline{P'(\mathrm{e}^{\mathrm{i}\theta})}\right)$$
$$=\mathrm{e}^{\mathrm{i}\theta}\left(P'(\mathrm{e}^{\mathrm{i}\theta})+\mathrm{e}^{\mathrm{i}\gamma}Q'(\mathrm{e}^{\mathrm{i}\theta})\right)+\mathrm{e}^{\mathrm{i}(n-1)\theta}\left(\overline{Q'(\mathrm{e}^{\mathrm{i}\theta})}+\mathrm{e}^{\mathrm{i}\gamma}\overline{P'(\mathrm{e}^{\mathrm{i}\theta})}\right),$$

which gives

$$n\left|P(\mathrm{e}^{\mathrm{i}\theta})+\mathrm{e}^{\mathrm{i}\gamma}Q(\mathrm{e}^{\mathrm{i}\theta})\right|\leq\left|P'(\mathrm{e}^{\mathrm{i}\theta})+\mathrm{e}^{\mathrm{i}\gamma}Q'(\mathrm{e}^{\mathrm{i}\theta})\right|+\left|\overline{Q'(\mathrm{e}^{\mathrm{i}\theta})}+\mathrm{e}^{\mathrm{i}\gamma}\overline{P'(\mathrm{e}^{\mathrm{i}\theta})}\right|$$
$$=2\left|P'(\mathrm{e}^{\mathrm{i}\theta})+\mathrm{e}^{\mathrm{i}\gamma}Q'(\mathrm{e}^{\mathrm{i}\theta})\right|. \tag{4.1}$$

Also, we have

$$\left|D_\alpha P(\mathrm{e}^{\mathrm{i}\theta})+\mathrm{e}^{\mathrm{i}\gamma}D_\alpha Q(\mathrm{e}^{\mathrm{i}\theta})\right|=\left|nP(\mathrm{e}^{\mathrm{i}\theta})+(\alpha-\mathrm{e}^{\mathrm{i}\theta})P'(\mathrm{e}^{\mathrm{i}\theta})+\mathrm{e}^{\mathrm{i}\gamma}\left(nQ(\mathrm{e}^{\mathrm{i}\theta})+(\alpha-\mathrm{e}^{\mathrm{i}\theta})Q'(\mathrm{e}^{\mathrm{i}\theta})\right)\right|$$
$$=\left|\left(nP(\mathrm{e}^{\mathrm{i}\theta})-\mathrm{e}^{\mathrm{i}\theta}P'(\mathrm{e}^{\mathrm{i}\theta})\right)+\mathrm{e}^{\mathrm{i}\gamma}\left(nQ(\mathrm{e}^{\mathrm{i}\theta})-\mathrm{e}^{\mathrm{i}\theta}Q'(\mathrm{e}^{\mathrm{i}\theta})\right)\right.$$
$$\left.+\alpha\left(P'(\mathrm{e}^{\mathrm{i}\theta})+\mathrm{e}^{\mathrm{i}\gamma}Q'(\mathrm{e}^{\mathrm{i}\theta})\right)\right|$$
$$=\left|\left(\overline{Q'(\mathrm{e}^{\mathrm{i}\theta})}+\mathrm{e}^{\mathrm{i}\gamma}\overline{P'(\mathrm{e}^{\mathrm{i}\theta})}\right)\mathrm{e}^{\mathrm{i}(n-1)\theta}+\alpha\left(P'(\mathrm{e}^{\mathrm{i}\theta})+\mathrm{e}^{\mathrm{i}\gamma}Q'(\mathrm{e}^{\mathrm{i}\theta})\right)\right|$$
$$\leq\left|\overline{P'(\mathrm{e}^{\mathrm{i}\theta})+\mathrm{e}^{\mathrm{i}\gamma}Q'(\mathrm{e}^{\mathrm{i}\theta})}\right|+|\alpha|\left|P'(\mathrm{e}^{\mathrm{i}\theta})+\mathrm{e}^{\mathrm{i}\gamma}Q'(\mathrm{e}^{\mathrm{i}\theta})\right|$$
$$=\left(|\alpha|+1\right)\left|P'(\mathrm{e}^{\mathrm{i}\theta})+\mathrm{e}^{\mathrm{i}\gamma}Q'(\mathrm{e}^{\mathrm{i}\theta})\right|. \tag{4.2}$$

The above inequality (4.2), with the help of Lemma 4, gives, for each $r\geq 1$,

$$\begin{aligned}&\int_0^{2\pi}\int_0^{2\pi}\left|D_\alpha P(e^{i\theta})+e^{i\gamma}D_\alpha Q(e^{i\theta})\right|^r d\theta\, d\gamma\\&\le(|\alpha|+1)^r\int_0^{2\pi}\int_0^{2\pi}\left|P'(e^{i\theta})+e^{i\gamma}Q'(e^{i\theta})\right|^r d\theta\, d\gamma\\&\le 2\pi n^r(|\alpha|+1)^r\int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta. \qquad (4.3)\end{aligned}$$

Now by Lemma 5, for each $\theta, 0\le\theta<2\pi$ and for all β, α with $|\beta|\le 1, |\alpha|\ge 1$, we have

$$\begin{aligned}\left|e^{i\theta}D_\alpha P(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})\right|&\le\left|e^{i\theta}D_\alpha Q(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)Q(e^{i\theta})\right|\\&-mn\left(\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|-\left|e^{i\theta}+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right).\end{aligned}$$

This implies

$$\begin{aligned}&\left|e^{i\theta}D_\alpha P(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})\right|+\frac{mn}{2}\left(\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|-\left|e^{i\theta}+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right)\\&\le\left|e^{i\theta}Q(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)Q(e^{i\theta})\right|-\frac{mn}{2}\left(\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|-\left|e^{i\theta}+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right). \qquad (4.4)\end{aligned}$$

Take $A=\left|e^{i\theta}D_\alpha Q(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)Q(e^{i\theta})\right|$, $B=\left|e^{i\theta}D_\alpha P(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})\right|$,

and $C=\frac{mn}{2}\left(\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|-\left|e^{i\theta}+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right)$ in Lemma 6, we get

$B+C\le A-C\le A.$

Hence for every real γ, we get with the help of Lemma 6, that

$$\begin{aligned}&\Bigg|\left\{\left|e^{i\theta}D_\alpha Q(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)Q(e^{i\theta})\right|-\frac{mn}{2}\left(\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|-\left|e^{i\theta}+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right)\right\}e^{i\gamma}\\&+\left\{\left|e^{i\theta}D_\alpha P(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})\right|+\frac{mn}{2}\left(\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|-\left|e^{i\theta}+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right)\right\}\Bigg|\\&\le\left|\left|e^{i\theta}D_\alpha Q(e^{i\theta})+\beta\left(\frac{|\alpha|-1}{2}\right)Q(e^{i\theta})\right|e^{i\gamma}+\left|e^{i\theta}D_\alpha P(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})\right|\right|.\end{aligned}$$

This implies for each $r\ge 1$,

$$\begin{aligned}&\int_0^{2\pi}\left|F(\theta)+e^{i\gamma}G(\theta)\right|^r d\theta\\&\le\int_0^{2\pi}\left|\left|e^{i\theta}D_\alpha P(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})\right|+e^{i\gamma}\left|e^{i\theta}D_\alpha Q(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)Q(e^{i\theta})\right|\right|^r d\theta, \qquad (4.5)\end{aligned}$$

where

$$F(\theta)=\left|e^{i\theta}D_\alpha P(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})\right|+\frac{mn}{2}\left(\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|-\left|e^{i\theta}+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right)$$

and

$$G(\theta)=\left|e^{i\theta}D_\alpha Q(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)Q(e^{i\theta})\right|-\frac{mn}{2}\left(\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|-\left|e^{i\theta}+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right).$$

Integrating both sides of (4.5) with respect to γ from 0 to 2π, we get

$$\int_0^{2\pi}\int_0^{2\pi}\left|F(\theta)+e^{i\gamma}G(\theta)\right|^r d\theta\, d\gamma$$

$$\leq \int_0^{2\pi}\left\{\int_0^{2\pi}\left|\left|e^{i\theta}D_\alpha P(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})\right|+e^{i\gamma}\left|e^{i\theta}D_\alpha Q(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)Q(e^{i\theta})\right|\right|^r d\theta\, d\gamma\right\}$$

$$=\int_0^{2\pi}\left\{\int_0^{2\pi}\left|e^{i\theta}D_\alpha P(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})+e^{i\gamma}\left(e^{i\theta}D_\alpha Q(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)Q(e^{i\theta})\right)\right|^r d\gamma\right\}d\theta$$

$$=\int_0^{2\pi}\left\{\int_0^{2\pi}\left|e^{i\theta}\left(D_\alpha P(e^{i\theta})+e^{i\gamma}D_\alpha Q(e^{i\theta})\right)+\beta n\left(\frac{|\alpha|-1}{2}\right)\left(P(e^{i\theta})+e^{i\gamma}Q(e^{i\theta})\right)\right|^r d\theta\right\}d\gamma.$$

Therefore, it follows by Minkowski's inequality that, for $r\geq 1$,

$$\left\{\int_0^{2\pi}\int_0^{2\pi}\left|F(\theta)+e^{i\gamma}G(\theta)\right|^r d\theta\, d\gamma\right\}^{\frac{1}{r}}$$

$$\leq\left\{\int_0^{2\pi}\int_0^{2\pi}\left|e^{i\theta}\left(D_\alpha P(e^{i\theta})+e^{i\gamma}D_\alpha Q(e^{i\theta})\right)+\beta n\left(\frac{|\alpha|-1}{2}\right)\left(P(e^{i\theta})+e^{i\gamma}Q(e^{i\theta})\right)\right|^r d\theta\, d\gamma\right\}^{\frac{1}{r}}$$

$$\leq\left\{\int_0^{2\pi}\int_0^{2\pi}\left|D_\alpha P(e^{i\theta})+e^{i\gamma}D_\alpha Q(e^{i\theta})\right|^r d\theta\, d\gamma\right\}^{\frac{1}{r}}+|\beta|n\left(\frac{|\alpha|-1}{2}\right)\left\{\int_0^{2\pi}\int_0^{2\pi}\left|P(e^{i\theta})+e^{i\gamma}Q(e^{i\theta})\right|^r d\theta\, d\gamma\right\}^{\frac{1}{r}},$$

which gives on using (4.1), (4.3) and Lemma 4 that, for every β with $|\beta|\leq 1$, $r\geq 1$ and γ real,

$$\left\{\int_0^{2\pi}\int_0^{2\pi}\left|F(\theta)+e^{i\gamma}G(\theta)\right|^r d\theta\, d\gamma\right\}^{\frac{1}{r}}\leq(2\pi)^{\frac{1}{r}}n\Big((|\alpha|+1)+|\beta|(|\alpha|-1)\Big)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{4.6}$$

Now for any real γ and $t\geq 1$, it can be easily verified that $|t+e^{i\gamma}|\geq|1+e^{i\gamma}|$.

Observe that for every $r\geq 1$ and $a,b\in\mathbb{C}$ such that $|b|\geq|a|$, we have

$$\int_0^{2\pi}|a+e^{i\gamma}b|^r d\gamma\geq|a|^r\int_0^{2\pi}|1+e^{i\gamma}|^r d\gamma. \tag{4.7}$$

Indeed, if $a=0$ the above inequality is obvious. In the case of $a\neq 0$, we get

$$\int_0^{2\pi}\left|1+e^{i\gamma}\frac{b}{a}\right|^r d\gamma=\int_0^{2\pi}\left|1+e^{i\gamma}\left|\frac{b}{a}\right|\right|^r d\gamma=\int_0^{2\pi}\left|\left|\frac{b}{a}\right|+e^{i\gamma}\right|^r d\gamma\geq\int_0^{2\pi}\left|1+e^{i\gamma}\right|^r d\gamma.$$

Now, we can take

$$a=F(\theta)$$

$$b=G(\theta),$$

because $|b|\geq|a|$ from (4.4) we get from (4.7) that

$$\int_0^{2\pi}\left|F(\theta)+e^{i\gamma}G(\theta)\right|^r d\gamma\geq|F(\theta)|^r\int_0^{2\pi}\left|1+e^{i\gamma}\right|^r d\gamma. \tag{4.8}$$

Integrating both sides of (4.8) with respect to θ from 0 to 2π, we get from (4.6) that, for every $r\geq 1$,

$$\left\{\int_0^{2\pi}\left|1+e^{i\gamma}\right|^r d\gamma\int_0^{2\pi}\left\{\left|e^{i\theta}D_\alpha P(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})\right|\right.\right.$$
$$\left.\left.+\frac{mn}{2}\left(\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|-\left|e^{i\theta}+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right)\right\}^r d\theta\right\}^{\frac{1}{r}}$$
$$\leq (2\pi)^{\frac{1}{r}} n\Big((|\alpha|+1|)+|\beta|(|\alpha|-1)\Big)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{4.9}$$

Now using the fact that for every complex number δ with $|\delta|\leq 1$,

$$\left|e^{i\theta}D_\alpha P(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})+\frac{mn}{2}\left(\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|-\left|e^{i\theta}+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right)\delta\right|$$
$$\leq\left|e^{i\theta}D_\alpha P(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})\right|+\frac{mn}{2}\left(\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|-\left|e^{i\theta}+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right),$$

we get from (4.9), that

$$\left\{\int_0^{2\pi}\left|e^{i\theta}D_\alpha P(e^{i\theta})+\beta n\left(\frac{|\alpha|-1}{2}\right)P(e^{i\theta})+\frac{mn}{2}\delta\left(\left|\alpha+\beta\left(\frac{|\alpha|-1}{2}\right)\right|-\left|e^{i\theta}+\beta\left(\frac{|\alpha|-1}{2}\right)\right|\right)\right|^r d\theta\right\}^{\frac{1}{r}}$$
$$\leq (2\pi)^{\frac{1}{r}} n\Big((|\alpha|+1)+|\beta|(|\alpha|-1)\Big)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}\left\{\int_0^{2\pi}\left|1+e^{i\gamma}\right|^r d\gamma\right\}^{\frac{-1}{r}}$$
$$= n\Big((|\alpha|+1)+|\beta|(|\alpha|-1)\Big)C_r\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}},$$

which is (2.1), and this completes the proof of Theorem 1. □

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