C. R. Acad. Sci. Paris, Ser. I 355 (2017) 991–1021

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Topology/Computer science

Simplicial complexes and closure systems induced by indistinguishability relations

Complexes simpliciaux et systèmes de clôture induits par les relations d'indistinguabilité

Giampiero Chiaselotti, Tommaso Gentile, Federico Infusino

Department of Mathematics and Informatics, University of Calabria, Via Pietro Bucci, Cubo 30B, 87036 Arcavacata di Rende (CS), Italy

ARTICLE INFO

Article history:
Received 27 June 2016
Accepted after revision 12 September 2017
Available online 27 September 2017

Presented by Claire Voisin

ABSTRACT

In this paper, we develop in a more general mathematical context the notion of *indistinguishability*, which in graph theory has recently been investigated as a symmetry relation with respect to a fixed vertex subset. The starting point of our analysis is to consider a set Ω of functions defined on a universe set U and to define an equivalence relation $\equiv_A$ on U for any subset $A \subseteq \Omega$ in the following way: $u \equiv_A u'$ if $a(u) = a(u')$ for any function $a \in A$. By means of this family of relations, we introduce the *indistinguishability relation* $\approx$ on the power set $\mathcal{P}(\Omega)$ as follows: for $A, A' \in \mathcal{P}(\Omega)$, we set $A \approx A'$ if the relations $\equiv_A$ and $\equiv_{A'}$ coincide. We use then the indistinguishability relation $\approx$ to introduce several set families on Ω that have interesting order, matroidal and combinatorial properties. We call the above set families the *indistinguishability structures* of the *function system* (U, Ω). Furthermore, we obtain a closure system and an abstract simplicial complex interacting each other by means of three hypergraphs having relevance in both theoretical computer science and graph theory. The first part of this paper is devoted to investigate the basic mathematical properties of the indistinguishability structures for arbitrary function systems. The second part deals with some specific cases of study derived from simple undirected graphs and the usual Euclidean real line.

RÉSUMÉ

Nous développons dans ce texte la notion d'*indistinguabilité* dans un contexte mathématique plus général. Cette notion a en effet été récemment étudiée en théorie des graphes, comme une relation de symétrie relativement aux sommets fixés. Le point de départ de notre analyse est de considérer un ensemble Ω de fonctions définies sur un ensemble univers U et de définir pour tout sous-ensemble $A \subset \Omega$ une relation d'équivalence $\equiv_A$ sur U par $u \equiv u'$ si $a(u) = a(u')$ pour toute fonction $a \in A$. Au moyen de cette famille de relations, nous introduisons la *relation d'indistinguabilité* $\approx$ sur l'ensemble puissance $\mathcal{P}(\Omega)$ de la façon suivante : pour $A, A' \in \mathcal{P}(\Omega)$, nous posons $A \approx A'$ si les relations $\equiv_A$ et $\equiv_{A'}$ coïncident. Nous utilisons cette relation d'indistinguabilité $\approx$ pour définir plusieurs familles d'ensembles sur Ω ayant d'intéressantes propriétés d'ordre, de matroïde et combinatoires.

E-mail addresses: giampiero.chiaselotti@unical.it (G. Chiaselotti), gentile@mat.unical.it (T. Gentile), f.infusino@mat.unical.it (F. Infusino).

http://dx.doi.org/10.1016/j.crma.2017.09.010

Nous appelons les familles d'ensembles ci-dessus les *structures indistinguables* du système de fonctions (U, Ω). De plus, nous obtenons un système de clôture et un complexe simplicial abstrait interagissant l'un l'autre au travers de trois hypergraphes, qui sont significatifs aussi bien en théorie des graphes qu'en informatique théorique. La première partie du texte est dédiée à l'étude les propriétés mathématiques élémentaires des structures d'indistinguabilité pour les systèmes de fonctions arbitraires. La seconde partie traite de quelques cas particuliers dérivés des graphes non orientés simples et de la droite euclidienne réelle usuelle.

1. Introduction

In graph theory, the symmetry notion is classically related to the properties of the automorphism group $Aut(G)$ of a graph G (see [34]), and it is also object of deep algebraic investigations (see [39]). More in general, since a graph can be considered as a particular type of metric space, the symmetry notion in this context has been investigated in [40,46,47,53]. Recently, in [26,29] a type of notion of symmetry for simple undirected graphs has been investigated. More in detail, when we have a simple undirected graph G and when we fix a vertex subset A of G, we can consider the following equivalence relation $\equiv_A$ (called *A-symmetry*) on the vertex set $V(G)$. For $v, v' \in V(G)$, we set

$$v \equiv_A v' :\Longleftrightarrow N_G(v) \cap A = N_G(v') \cap A,$$

where $N_G(v)$ is the usual open neighborhood of a vertex v of G. In [26,29], some new hypergraph families induced by the above A-symmetry have been introduced and studied in relation to motivations derived from database theory and related fields (see [26] for details). Based on some results obtained in [29], in [30] it has been introduced a new binary operation $\circ$ on a vertex subset family of G whose automorphism group (with respect to $\circ$) is isomorphic to a subgroup of $Aut(G)$ (for other works on similar topics, see also [16,17]).

On the other hand, there are actually many researches and studies in discrete mathematics whose main theoretical results are strictly related to motivations derived from computer science and affine disciplines. For example, in [43–45] several types of results concerning the Cayley graphs, graph algebras and directed graphs are connected with automata theory and data mining methods. In [41,52,55], matroids have been used in their connections with new topics in theoretical computer science. From another perspective, in [2–5,36] some classes of bipartite graphs have been studied with FCA methods [35]. Again, in [20–22] some types of homotopy relations for graphs that are strictly related with database theory have been introduced. In [12,13,15,19,25], some classes of lattices of signed integer partitions [1,42] have been investigated as sequential and discrete dynamical systems [50,51], with motivations related to some extremal combinatorial sum problems [23,24].

In this paper, we develop the above notion of A-symmetry (that in the study of data tables is usually called *indiscernibility relation*) in a purely mathematical context. To be more specific, let Ω be an arbitrary non-empty set. Usually, in mathematics, the elements of Ω do not have a well-specified nature. The basic idea of this paper is to assume that the elements of Ω are *functions*, whose domain is some *point* set U and whose codomain is some *value* set Λ. We say therefore that a *function system* is a triple $\mathcal{I} = \langle U, \Omega, \Lambda\rangle$, where U is a set of *points* $u, v, z, \ldots$ and Ω is a set of *functions* $a, b, c, \ldots$ that have domain U and codomain Λ. Let us note that a function system is a type of very general structure, which can be found in many different mathematical contexts, both in the finite and in the infinite cases.
We can uniquely associate with the function system $\mathcal{I}$ the *global map* $F : U \times \Omega \to \Lambda$ defined by

$$F(u, a) := a(u),$$

for any $u \in U$ and $a \in \Omega$. Let then $\mathcal{I} = \langle U, \Omega, F, \Lambda\rangle$ be a given function system with global map F.

If $A \subseteq \Omega$ and $u, u' \in U$, we set

$$u \equiv_A u' :\Longleftrightarrow a(u) = a(u') \qquad \forall a \in A.$$

In an approximation geometry for conceptual patterns [48], the equivalence relation $\equiv_A$ is called *A-indiscernibility relation* and it is used when the functions in Ω are substituted by specific *attributes* (or, equivalently, *properties*) of a set of *objects*. In this paper, we continue to use this terminology, and we call the set partition of U induced $\equiv_A$ the *A-indiscernibility partition* and denote it by $\pi(A)$.

We consider now the following equivalence relation $\approx$ defined on the power set $\mathcal{P}(\Omega)$ of Ω. If A and A' are any two subsets of Ω, we set

$$A \approx A' :\Longleftrightarrow \pi(A) = \pi(A'),$$

and we call $\approx$ the *indistinguishability relation* on $\mathcal{I}$. If $A \subseteq \Omega$, we denote by $[A]_{\approx}$ the equivalence class of A with respect to the equivalence relation $\approx$ and we call $[A]_{\approx}$ the *indistinguishability class* of A. In this paper, we will show that the indistinguishability relation induces a very rich mathematical structure on the power set $\mathcal{P}(\Omega)$, and the richness of this structure is a consequence of the way in which the set partitions $\pi(A)$ are interrelated between them and with others three specific subset families of Ω.

A first useful property of $\approx$ is that any indistinguishability class is a union-closed family [49]. By the union-closed property, it follows that any indistinguishability class $[A]_{\approx}$ has a maximum element $M(A)$ with respect to set-theoretic inclusion. Therefore, we are led to introduce the following subset family $MAXP(\mathcal{I}) := \{M(A) : A \in \mathcal{P}(\Omega)\}$. We call the members of the family $MAXP(\mathcal{I})$ the *maximum partitioners* of $\mathcal{I}$. Then it results that $MAXP(\mathcal{I})$ is a closure system [11], so it is also a complete lattice with respect to usual inclusion and, in the infinite case, a convexity structure [9,10]. Moreover, the dual of the above lattice is order isomorphic to the partially ordered set $(\Pi_{\mathrm{ind}}(\mathcal{I}), \preceq)$, where $\Pi_{\mathrm{ind}}(\mathcal{I}) := \{\pi(A) : A \in \mathcal{P}(\Omega)\}$ and $\preceq$ is the usual partial order between set partitions of U (see [11]). In other terms, we have that

$$\pi(A) \preceq \pi(A') \iff M(A) \subseteq^* M(A'), \tag{1}$$

for any $A, A' \in \mathcal{P}(\Omega)$. It is appropriate to note here that, by means of a specific characterization that we establish in Theorem 3.19, it is possible to investigate the notion of orthogonality between set partitions (see [6,7]) in the set $\Pi_{\mathrm{ind}}(\mathcal{I})$. On the other hand, we can also consider the partial order $\sqsubseteq$ on the quotient set $IND_{\approx}(\mathcal{I}) := \{[A]_{\approx} : A \in \mathcal{P}(\Omega)\}$ defined naturally as follows:

$$[A]_{\approx} \preceq [A']_{\approx} \iff M(A) \subseteq^* M(A'). \tag{2}$$

We obtain three isomorphic complete lattices $\mathbb{M}(\mathcal{I}) := (MAXP(\mathcal{I}), \subseteq^*)$, $\mathbb{P}_{\mathrm{ind}}(\mathcal{I}) := (\Pi_{\mathrm{ind}}(\mathcal{I}), \preceq)$ and $\mathbb{I}(\mathcal{I}) := (IND_{\approx}(\mathcal{I}), \sqsubseteq)$, which have interesting mathematical properties.

The first of these properties is the *global–local regularity*, briefly (GLR), which can be expressed as follows:

(GLR) $M(A) \subsetneqq M(A') \Longrightarrow Y \nsubseteq X$ for any $X \in [A]_{\approx}$ and $Y \in [A']_{\approx}$.

The global–local regularity property tells us that the inclusion between the maximum elements of two any indistinguishability classes preserves the same type of inclusion between any two members of these classes. In other terms, we can say that the *macro*-inclusion relations between the maximum partitioners of $\mathcal{I}$ have a direct influence also on the *micro*-inclusion relations between the subsets of their corresponding indistinguishability classes.

Let us note that the global–local regularity is a property whose main fundament consists of the possibility to consider the order structure $\mathbb{I}(\mathcal{I})$ as a *lattice of posets*, i.e. a lattice whose nodes are the posets $([A]_{\approx}, \subseteq)$, when A runs over $\mathcal{P}(\Omega)$. We call these partially ordered sets the *local indistinguishability posets*.

At this point, we consider the minimal subset family $\min([A]_{\approx})$ and introduce the following subset family:

$$MINP(\mathcal{I}) := \bigcup_{A \in MAXP(\mathcal{I})} \min([A]_{\approx}),$$

whose members are here called *minimal partitioners* of $\mathcal{I}$. Then $MINP(\mathcal{I})$ is an abstract simplicial complex that is related to $MAXP(\mathcal{I})$ in a similar way, in which the independent set family of a matroid is related to its closed set family. In fact, although in general, $MINP(\mathcal{I})$ is not a matroid, when this happens, then $MAXP(\mathcal{I})$ often coincides with the closed family of this matroid and, in those cases, the map $A \mapsto M(A)$ is exactly its closure operator. On the other hand, also when $MINP(\mathcal{I})$ is not a matroid, its structure has many similitudes with a matroid. For example, if $A \in \mathcal{P}(\Omega)$ we call any subset $B \subset A$ such that $\pi(A) = \pi(B)$ and $\pi(A) \neq \pi(B')$ for any $B' \subsetneqq B$ an *A-reduct* of $\mathcal{I}$. Let $RED(A)$ be the A-reduct family. We will then show that $RED(\mathcal{I}) := RED(\Omega)$ acquires properties similar to those satisfied by the basis family of a "matroidal-like structure"; in fact it results that

$$MINP(\mathcal{I}) \supseteq \bigcup_{A \in RED(\Omega)} \mathcal{P}(A).$$

In general, the complete determination of the reduct family $RED(A)$ is not an easy task, because the reducts of Ω are exactly the minimal transversals of the subset family $ESS(A)$ of all the subsets C, here called *essential subsets* of A, such that $\pi(A \setminus C) \neq \pi(A)$ and $\pi(A \setminus C') = \pi(A)$ for any $C' \subsetneqq C$. A particularly interesting case of function system $\mathcal{I}$ is that associated with the adjacency matrix of a simple graph G [26,29]. In this case, in [29] has been provided a complete geometric classification of the subgraphs induced by the members of $RED(G)$ when G is the Petersen graph.

In the literature, the *hypergraph transversal problem* for a *finite* hypergraph H is the problem of generating all the elements of $\mathrm{Tr}(H)$. In general, this is an important mathematical problem that has many applications in mathematics and in computer science [32,33,37,38].

Returning finally to the above discussion concerning the notion of indistinguishability, the relevant aspect is that in a very general situation, where we have simply a function system $\mathcal{I}$, we can construct several subset families, $MAXP(\mathcal{I})$, $MINP(\mathcal{I})$, $RED(\mathcal{I})$, $ESS(\mathcal{I})$ (and others that we introduce in the next sections), that have not trivial links between them and interesting mathematical properties (for an interpretation of the above subset families in computer science, see [14,27,28]). The most important relation linking $MAXP(\mathcal{I})$ and $MINP(\mathcal{I})$ is given in Theorem 7.3 by means of the set operator M, i.e. $MINP(\mathcal{I}) = \{A \in \mathcal{P}(\Omega) : a \notin M(A \setminus \{a\})\, \forall a \in A\}$. In particular, when the hypergraph $\mathfrak{m}(\mathcal{I}) := (\Omega, MINP(\mathcal{I}))$ is a matroid, we can define two closure operators, namely that naturally associated with the matroid and M. They are a priori different between each other, as it can be seen in Example 7.10.

We conclude this introductory section with a brief description of the sections in this paper.

In Section 2, we recall the basic notions and fix the notations used in the remaining part of the paper.

In Section 3, we introduce the basic notion of our work, which is the indistinguishability relation on the power set $\mathcal{P}(\Omega)$, and we determine the main properties of this equivalence relation. We introduce the family $MAXP(\mathcal{I})$ of all maximum partitioners of $\mathcal{I}$, and we prove some basic facts concerning the family $MAXP(\mathcal{I})$. In particular, we show that $MAXP(\mathcal{I})$ is an intersection closed family and that $\mathbb{M}(\mathcal{I})$ and $\Pi_{\text{ind}}(\mathcal{I})$ are isomorphic complete lattices. Finally, we characterize the indiscernibility partitions of $\mathcal{I}$ by means of a set operator $\Gamma : \mathcal{P}(U) \to \mathcal{P}(\Omega)$, and we also use this operator to provide a relativized version of the lattice $\mathbb{M}(\mathcal{I})$.

In Section 4, we try to develop the starting point of a non-trivial study of function systems where both the function set and the point set are infinite. In this regard, we introduce the notion of *separator* of a function subset in order to find some topological-like properties of the function systems, and we relate this notion to the problem of verifying when a chain of maximum partitioners is a union-closed family. We call the function systems having this property *union chain preserving* (abbreviated UCP) function systems. On the Euclidean real line, we build a concrete model of infinite UCP function system, and we establish several general results concerning the UCP function systems.

In Section 5, we introduce the set families $ESS(A)$ and $RED(A)$, and we show that $RED(A)$ is the transversal hypergraph of $ESS(A)$, for any $A \in \mathcal{P}(\Omega)$. We also show that the subset $\bigcap_{A\in\mathcal{P}(\Omega)} RED(A)$ has deep relations with the subset family $MAXP(A) := \{C \cap A : C \in MAXP(\mathcal{I})\}$.

In Section 6, our intent consists in studying in details the family $MINP(\mathcal{I})$, providing all the main properties it satisfies, such as the global–local regularity (see Proposition 6.13) and the inheritance (see Theorem 6.16). Finally, we analyze the strict relation occurring between the minimum partitioners and the relative reducts of any function subset.

In Section 7, we investigate some properties of matroidal type for $MINP(\mathcal{I})$ by means of an operator $\hat{\psi}$ defined similarly to the closure operator of a matroid. For example, we will see that if M is a closure operator having the Mac Lane–Steinitz exchange property, then the hypergraph $\mathfrak{m}(\mathcal{I})$ is a matroid.

In Section 8, we interpret graphs as if they were function systems. In particular, we devote our attention to the determination of the maximum and the minimal partitioners of the complete multipartite graphs and n-cycle C_n. We also provide an algebraic characterization for the reduct family of C_n.

2. Reviews and notations

If k is a positive integer, we set $[k] := \{1, \ldots, k\}$. If X is a set, we denote by $\mathcal{P}(X)$ the power set of X and by $|X|$ the cardinality of X. We set $\mathbb{B}(X) := (\mathcal{P}(X), \subseteq)$ and $\mathbb{B}^*(X) := (\mathcal{P}(X), \subseteq^*)$, where $\subseteq^*$ is the dual inclusion, i.e. $A \subseteq^* B$ if $B \subseteq A$. If we want to specify that Y is a finite subset of X, we will write $Y \subseteq_f X$. If n is a non-negative integer, we write $Y \subseteq_{\le n} X$ if $Y \subseteq X$ and $|Y| \le n$.

A family of subsets $\mathcal{F} \subseteq \mathcal{P}(X)$ is called a *union-closed family* if whenever $\mathcal{F}' \subseteq \mathcal{F}$ then $\cup\mathcal{F}' \in \mathcal{F}$. We say that $\mathcal{F}$ is a *chain union-closed family* if whenever $\mathcal{C}$ is a chain of elements of $\mathcal{F}$, then $\cup\mathcal{C}$ is again an element of $\mathcal{F}$. Obviously, each subset family having a finite number of elements is chain union-closed. If X is a finite set and $\mathcal{F} \subseteq \mathcal{P}(X)$ such that all its members have the same cardinality, we say that $\mathcal{F}$ has *uniform cardinality*. In this case, we denote by $||\mathcal{F}||$ the common cardinality of the members of $\mathcal{F}$.

Definition 2.1. [8] A *hypergraph* is a pair $H = (V(H), E(H))$, where $V(H) = \{v_1, \ldots, v_n\}$ is an arbitrary finite set and $E(H) = \{Y_1, \ldots, Y_m\}$ is a non-empty family of subsets $Y_1, \ldots, Y_m$ of $V(H)$. The elements $v_1, \ldots, v_n$ are called vertices of H and the subsets $Y_1, \ldots, Y_m$ are called hyperedges of H. A *hypergraph on* $V(H)$ is a hypergraph having $V(H)$ as vertex set.

The classical notion of *transversal* for a finite hypergraph [8] can be also given for an arbitrary hypergraph.

Definition 2.2. Let H be a hypergraph with vertex set $V(H)$ and hyperedge set $E(H)$. We say that a subset $Y \subseteq V(H)$ is a *transversal* of H if $Y \cap A \neq \emptyset$ for each non-empty hyperedge $A \in E(H)$. We say that a transversal A of H is *minimal* if no proper subset of A is a transversal of H. We denote by $\mathrm{Tr}(H)$ the family of all minimal transversals of H. We call the hypergraph $H^{\mathrm{tr}} := (V(H), \mathrm{Tr}(H))$ the *transversal* hypergraph of H.

2.1. Posets

A *partially ordered set* (abbreviated *poset*) is a pair $P = (X, \le)$, where X is a set and $\le$ is a binary relation on X that is reflexive, antisymmetric and transitive. If $P = (X, \le)$ is a partially ordered set and $x, y \in X$, we also write $x < y$ if $x \le y$ and $x \neq y$. If x, y are two distinct elements of X, we say that y *covers* x, denoted by $x \prec y$ (or, equivalently, by $y \succ x$), if $x \le y$ and there is no element $z \in X$ such that $x < z < y$. If $x \in X$ we set $[x|P \uparrow] := \{z \in X : x \prec z\}$ and $[x|P \downarrow] := \{y \in X : y \prec x\}$. We call the elements of $[x|P \uparrow]$ the *covers* of x and the elements of $[x|P \downarrow]$ the *co-covers* of x. An element $x \in X$ is called *minimal* in P if $z \le x$ implies $z = x$, and, in a similar way, one defines a *maximal* element in P. We denote respectively by

$\min(P)$ and $\mathrm{Max}(P)$ the sets of all minimal and maximal elements of P. In particular, when $\mathcal{F}$ is a subset family of some set X, we denote respectively by $\min(\mathcal{F})$ and $\mathrm{Max}(\mathcal{F})$ the minimal and the maximal subsets of $\mathcal{F}$ with respect to the set inclusion. We also set $\min^*(\mathcal{F}) := \mathrm{Max}(\mathcal{F})$ and $\max^*(\mathcal{F}) := \min(\mathcal{F})$. If there is an element $\hat{0}_X \in X$ such that $\hat{0}_X \le x$, then $\hat{0}_X$ is unique, and it is called the *minimum* of P. In a similar way, the *maximum* of P, usually denoted by $\hat{1}_X$, can be defined. A *chain* C of P is a subset $C \subset X$ such that for all $x, y \in C$ we have $x \le y$ or $y \le x$.

A poset $P = (X_1, \le_1)$ is said *isomorphic* to another poset $P_2 = (X_2, \le_2)$ if there exists a bijective map $\phi : X_1 \to X_2$ such that $x \le_1 y \iff \phi(x) \le_2 \phi(y)$, for all $x, y \in X_1$. The *dual poset* of P is the poset $P^* := (X, \le^*)$, where $\le^*$ is the partial order on X defined by $x \le^* y : \iff y \le x$, for all $x, y \in X$. A poset P is called *self-dual* if P is isomorphic to its dual poset P^*.

2.2. Set partitions

If X is an arbitrary set and π is a set partition on X, we usually denote by $\{B_i : i \in I\}$ the block family of π. If $x \in X$, we denote by $\pi[x]$ the block of π that contains the element x. If $Y \subseteq X$ and $Y \subseteq B_i$, for some index $i \in I$, we say that Y is a *sub-block* of π and we write $Y \preccurlyeq \pi$. When X is finite, we use the standard notation $\pi := B_1|\dots|B_{|\pi|}$, where $|\pi|$ is the number of distinct blocks of π. We denote by $\Pi(X)$ the set of all set-partitions of X. It is well known that on the set $\Pi(X)$, we can consider a partial order $\preceq$ defined as follows: if $\pi, \pi' \in \Pi(X)$, then

$$\pi \preceq \pi' :\iff (\forall B \in \pi)\,(\exists B' \in \pi') : B \subseteq B', \tag{3}$$

which is equivalent to

$$\pi \preceq \pi' \iff (\forall x \in X)\,(\pi[x] \subseteq \pi'[x]). \tag{4}$$

We will write $\pi \prec \pi'$ when $\pi \preceq \pi'$ and $\pi \ne \pi'$.

The pair $\mathbb{P}(X) = (\Pi(X), \preceq)$ is a complete lattice, which is called *partition lattice* of the set X. We now recall the basic facts about the meet and the join of this lattice.

Let $\pi_1 = A_1|\dots|A_m$ and $\pi_2 = B_1|\dots|B_n$ be two partitions on the same finite universe X, i.e., $\pi_1, \pi_2 \in \Pi(X)$. We set

$$\mathcal{S}_{\pi_1,\pi_2} := \{C \subseteq X : \text{ if } x \in C, \text{ then } \pi_1[x] \subseteq C \text{ and } \pi_2[x] \subseteq C\}$$

Then the meet and the join of π_1 and π_2 in $\mathbb{P}(X)$ are given respectively by $\pi_1 \wedge \pi_2 := \{A_i \cap B_j : i = 1, \dots m;\ j = 1, \dots, n\}$ and $\pi_1 \vee \pi_2 := C_1|\dots|C_k$, where $C_1, \dots, C_k$ are the minimal elements of the set family $\mathcal{S}_{\pi_1,\pi_2}$ with respect to the inclusion.

2.3. Closure systems and matroids

Let S be a set.

Definition 2.3. [11] A family $\mathcal{F}$ on S is called a *closure system* on S if:

(i) $S \in \mathcal{F}$;
(ii) whenever $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

We denote by $CLSY(S)$ the set of all closure systems on S.

Definition 2.4. [11] A *closure operator* on S is a set map $\varphi : \mathcal{P}(S) \to \mathcal{P}(S)$ having the following properties:

(i) if $A \subseteq B$ then $\varphi(A) \subseteq \varphi(B)$, i.e. φ is isotone;
(ii) $A \subseteq \varphi(A)$, i.e. φ is extensive;
(iii) $\varphi(\varphi(A)) = \varphi(A)$, i.e. φ is idempotent.

We denote by $CLOP(S)$ the set of all closure operators on S.

Definition 2.5. A set map $\varphi : \mathcal{P}(S) \to \mathcal{P}(S)$ having the following MacLane–Steinitz exchange property: for any $A \in \mathcal{P}(S)$ and for all elements $a \in \Omega \setminus \varphi(A)$, $b \in \varphi(A \cup \{a\}) \setminus \varphi(A)$, it results that $a \in \varphi(A \cup \{b\})$ is called a *MLS operator* on S. We call a closure operator that is also a MLS operator a *MLS closure operator*.

If $\mathcal{F} \in CLSY(S)$, we define a map $\varphi_{\mathcal{F}} : \mathcal{P}(S) \to \mathcal{P}(S)$ in the following way:

$$C \in \mathcal{P}(S) \mapsto \varphi_{\mathcal{F}}(C) := \bigcap \{A \in \mathcal{F} : C \subseteq A\} \in \mathcal{P}(S). \tag{5}$$

Then it is immediate to verify that $\varphi_{\mathcal{F}} \in CLOP(S)$.
On the other hand, if $\varphi \in CLOP(S)$, we set

$$\mathcal{F}_\varphi := \{A \in \mathcal{P}(S) : \varphi(A) = A\}. \tag{6}$$

Then, also in this case, it is immediate to verify that $\mathcal{F}_\varphi \in CLSY(S)$.

Theorem 2.6. *The maps $\mathcal{F} \in CLSY(S) \mapsto \varphi_{\mathcal{F}} \in CLOP(S)$ and $\varphi \in CLOP(S) \mapsto \mathcal{F}_{\varphi} \in CLSY(S)$ are inverses of each other. In other terms, if $\mathcal{F} \in CLSY(S)$ and $\varphi \in CLOP(S)$, then $\mathcal{F}_{\varphi_{\mathcal{F}}} = \mathcal{F}$ and $\varphi_{\mathcal{F}_{\varphi}} = \varphi$. Moreover, if $\mathcal{F} \in CLSY(S)$, then it is a complete lattice under set-inclusion in which meet means intersection.*

Proof. See [11]. $\square$

By Theorem 2.6, we can consider as equivalent the notions of closure system and closure operator on the same set S (clearly this equivalence is described by means of (5) and (6)).

For general results on matroids, see [54]. In what follows, we provide the main results on matroid theory that we use in the sequel. Let X be a finite set and $\mathcal{F}$ be a family of subsets of X.

Definition 2.7. The pair $(X, \mathcal{F})$ is a *matroid* if:

$(M1)$ $\emptyset \in \mathcal{F}$;
$(M2)$ if $Y \in \mathcal{F}$ and $Z \subseteq Y$, then $Z \in \mathcal{F}$.
$(M3)$ if $U, V \in \mathcal{F}$ are such that $|U| = |V| + 1$, then there exists $x \in U \setminus V$ such that $V \cup \{x\} \in \mathcal{F}$.

In this case, any element of $\mathcal{F}$ is called *independent set*.

We finally characterize the condition for a subset family $\mathcal{F}$ is a matroid in terms of closure operators.

Theorem 2.8. *A function $\sigma : \mathcal{P}(X) \to \mathcal{P}(X)$ is the closure operator of a matroid on X if and only if σ is a MLS closure operator on X. In this case, the family of the independent sets is $\mathcal{F} = \{A \subseteq X : a \notin \sigma(A \setminus \{a\})\ \forall a \in A\}$.*

3. Indistinguishability relation and maximum partitioners

In this section, we introduce the basic notions that we will use in the remaining part of this paper. We first formalize the notion of *function system* and related concepts.

Definition 3.1. We call a structure $\mathcal{I} = \langle U, \Omega, F, \Lambda \rangle$, where U, Ω and Λ are three non-empty sets and $F : U \times \Omega \to \Lambda$ is a map, a *function system*. We call the map F the *pairing* of $\mathcal{I}$. If Λ is a set with two elements (usually denoted by 0 and 1), we say that $\mathcal{I}$ is a *Boolean function system*. We call the elements of U the *points* of $\mathcal{I}$, the elements of Ω the *functions* of $\mathcal{I}$ and the elements of Λ the *values* of $\mathcal{I}$. When both the sets $U = \{u_1, \dots, u_m\}$ and $\Omega = \{a_1, \dots, a_n\}$ are finite, we say that $\mathcal{I}$ is a *finite function system*. For a finite function system, we denote by $T[\mathcal{I}]$ the $m \times n$ rectangular table having on the i-th row the point u_i, on the j-th column the function a_j and the value $F(u_i, a_j)$ in the place (i, j). It is clear that, in the finite case, we can identify $\mathcal{I}$ with $T[\mathcal{I}]$, and therefore in such a case we speak equivalently of *functional table*. Finally, we say that $\mathcal{I}$ is a *symmetric function system* if $U = \Omega$ and $F(u, a) = F(a, u)$ for any $u \in U$ and $a \in \Omega$.

In what follows, we assume that $\mathcal{I} = \langle U, \Omega, F, \Lambda \rangle$ is a given function system and that A is an arbitrary element of $\mathcal{P}(\Omega)$. We set

$$u \equiv_A u' :\Longleftrightarrow F(u, a) = F(u', a),\ \forall a \in A. \tag{7}$$

Then $\equiv_A$ is an equivalence relation on U that in database theory and related contexts is usually called *A-indiscernibility relation* (for details see [18,26]). In this paper, we continue to use such terminology. For any $u \in U$, we denote by $[u]_A$ the equivalence class of u with respect to $\equiv_A$ and we call $[u]_A$ the *A-indiscernibility class* of u.

We introduce now the map $\pi : \mathcal{P}(\Omega) \to \Pi(U)$ defined by

$$A \in \mathcal{P}(\Omega) \mapsto \pi(A) := \{[u]_A : u \in U\}, \tag{8}$$

and we denote by $\Pi_{\text{ind}}(\mathcal{I})$ the image of the operator π, i.e.

$$\Pi_{\text{ind}}(\mathcal{I}) := \{\pi(A) : A \in \mathcal{P}(\Omega)\}. \tag{9}$$

We also set

$$\mathbb{P}_{\text{ind}}(\mathcal{I}) := (\Pi_{\text{ind}}(\mathcal{I}), \preceq), \tag{10}$$

so that $\mathbb{P}_{\text{ind}}(\mathcal{I})$ is a sub-poset of the partition lattice $\mathbb{P}(U)$.

Let us note that if $A' \in \mathcal{P}(\Omega)$, then

$$A \subseteq A' \Longrightarrow \pi(A') \preceq \pi(A). \tag{11}$$

Furthermore, the properties given in (11) can be equivalently expressed by saying that π is an order-preserving map between the lattices $\mathbb{B}^*(\Omega)$ and $\mathbb{P}(U)$.

We introduce now a binary relation $\approx$ on $\mathcal{P}(\Omega)$ that will play a basic role in this paper.
If $A, A' \in \mathcal{P}(\Omega)$, we put

$$A \approx A' :\Longleftrightarrow \pi(A) = \pi(A'),$$

which is equivalent to the following

$$u \equiv_A u' \Longleftrightarrow u \equiv_{A'} u', \tag{12}$$

for all $u, u' \in U$.

Then $\approx$ is an equivalence relation on $\mathcal{P}(\Omega)$ having the following basic property.

Proposition 3.2. *Let $A \approx B$ and $D \in \mathcal{P}(\Omega)$. Then*

$$A \cup D \approx B \cup D. \tag{13}$$

Proof. It is an immediate consequence of (7) and (12). □

Definition 3.3. We call $\approx$ the *indistinguishability relation* of $\mathcal{I}$, and we denote by $[A]_\approx$ the equivalence class of A with respect to $\approx$.

Another basic property of the indistinguishability relation is given in the following result.

Proposition 3.4. *The indistinguishability class $[A]_\approx$ is a union-closed family.*

Proof. Straightforward. □

We set now

$$M(A) := \bigcup [A]_\approx. \tag{14}$$

By Proposition 3.4 we obtain the following immediate consequence.

Corollary 3.5. *$M(A)$ is the maximum of the family $[A]_\approx$.*

Proposition 3.6. *We have that*

$$M(A) = \{b \in \Omega : A \cup \{b\} \approx A\}. \tag{15}$$

Proof. Let $b \in M(A)$. Then there exists $A' \in [A]_\approx$ such that $b \in A'$. Let now $u, u' \in U$ such that $u \equiv_{A \cup \{b\}} u'$, then $u \equiv_A u'$ because A is a subset of $A \cup \{b\}$. On the other hand, let assume that $u \equiv_A u'$. Since $A' \in [A]_\approx$, we have $u \equiv_{A'} u'$, and therefore $F(u, b) = F(u', b)$ because $b \in A'$. Hence $u \equiv_{A \cup \{b\}} u'$, and by (12) we deduce that $A \approx A \cup \{b\}$.

Let now $b \in \Omega$ such that $A \approx A \cup \{b\}$. Then $b \in A \cup \{b\} \in [A]_\approx$ and we deduce that $b \in M(A)$. □

The conditions (15) can be also respectively expressed in the following equivalent way:

$$b \in M(A) \Longleftrightarrow (\forall u, u' \in U : u \equiv_A u' \Longrightarrow F(u, b) = F(u', b)). \tag{16}$$

We consider now the following set operator:

$$M : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$$

such that

$$M : A \in \mathcal{P}(\Omega) \mapsto M(A) \in \mathcal{P}(\Omega).$$

Then we obtain the following result.

Proposition 3.7. *M is a closure operator.*

Proof. It is obvious that $A \subseteq M(A)$. Let now $A' \in \mathcal{P}(\Omega)$ such that $A \subseteq A'$. Let $b \in M(A)$. Then $A \approx A \cup \{b\}$. Let $u, u' \in U$ such that $u \equiv_{A'} u'$. Since A is a subset of A', we have that $u \equiv_A u'$, and this implies $F(u,b) = F(u',b)$ because $A \approx A \cup \{b\}$. This shows that $u \equiv_{A' \cup \{b\}} u'$, i.e. $b \in M(A')$. Hence $M(A) \subseteq M(A')$. Finally, we must prove that $M(M(A)) = M(A)$; for this, it is sufficient to show that $M(M(A)) \subseteq M(A)$. Let then $b \in M(M(A))$ and $u, u' \in U$ such that $u \equiv_A u'$. Since $A \approx M(A)$, it follows that $u \equiv_{M(A)} u'$, therefore $F(u,b) = F(u',b)$ by (16), because b is an element of $M(M(A))$. Hence $b \in M(A)$ again by (16). $\square$

We set now

$$MAXP(\mathcal{I}) := \{M(A) : A \in \mathcal{P}(\Omega)\} = \{B \in \mathcal{P}(\Omega) : M(B) = B\}, \tag{17}$$

and

$$\mathbb{M}(\mathcal{I}) := (MAXP(\mathcal{I}), \subseteq^*). \tag{18}$$

By Proposition 3.7, we immediately obtain the following result.

Theorem 3.8. *The following hold:*

(*i*) *$MAXP(\mathcal{I})$ is a closure system.*
(*ii*) *$\mathbb{M}(\mathcal{I})$ is a complete lattice.*

Moreover, the join and the meet in the lattice $\mathbb{M}(\mathcal{I})$ are characterized as follows.
If $\{A_j : j \in J\} \subseteq MAXP(\mathcal{I})$ then:

(*iii*) *the join of $\{A_j : j \in J\}$ in $\mathbb{M}(\mathcal{I})$ is $\bigcap_{j \in J} A_j$;*
(*iv*) *the meet of $\{A_j : j \in J\}$ in $\mathbb{M}(\mathcal{I})$ is $M(\bigcup_{j \in J} A_j)$.*

Proof. (*i*) By (17), it follows that $MAXP(\mathcal{I})$ is the family of all closed sets for the closure operators M. Therefore the result follows by Theorem 2.6.

(*ii*) By Theorem 2.6 we deduce that $(MAXP(\mathcal{I}), \subseteq)$ is a complete lattice, therefore also its corresponding dual lattice $\mathbb{M}(\mathcal{I})$ is a complete lattice.

(*iii*) The result is a direct consequence of the duality and Theorem 2.6.

(*iv*) Let $\{A_j : j \in J\} \subseteq MAXP(\mathcal{I})$. Since $A_j \subseteq \bigcup_{j \in J} A_j$ and M is a closure operator, $M(\bigcup_{j \in J} A_j) \subseteq^* M(A_j) = A_j$, so $M(\bigcup_{j \in J} A_j)$ is a lower bound of $\{A_j : j \in J\}$ in $\mathbb{M}(\mathcal{I})$. Let now $C \in MAXP(\mathcal{I})$ such that, for all $j \in J$, $C \subseteq^* A_j$. Then $C \subseteq^* \bigcup_{j \in J} A_j$, therefore $C = M(C) \subseteq^* M(\bigcup_{j \in J} A_j)$. This proves that the meet of $\{A_j : j \in J\}$ in $\mathbb{M}(\mathcal{I})$ is $M(\bigcup_{j \in J} A_j)$. $\square$

We use the following terminology.

Definition 3.9. We call the elements of $MAXP(\mathcal{I})$ the *maximum partitioners* of $\mathcal{I}$.

Proposition 3.10. *Let $A' \in \mathcal{P}(\Omega)$. Then the followings hold:*

$$\pi(A) \preceq \pi(A') \iff M(A') \subseteq M(A). \tag{19}$$

Proof. Let $M(A') \subseteq M(A)$. Then $\pi(A) = \pi(M(A)) \preceq \pi(M(A')) = \pi(A')$ by virtue of (11). We assume now that $\pi(A) \preceq \pi(A')$. Let $a \in M(A')$ and suppose by contradiction that $a \notin M(A)$, then $M(A) \subsetneq M(A) \cup \{a\}$. Therefore, by (15), we deduce that $\pi(M(A)) \neq \pi(M(A) \cup \{a\})$. Then there exist two objects $u, u' \in U$ such that $u \equiv_{M(A)} u'$ and $u \not\equiv_{M(A) \cup \{a\}} u'$, and this is possible only if $F(u,a) \neq F(u',a)$. Now, since $\pi(M(A)) \preceq \pi(M(A'))$ and $u \equiv_{M(A)} u'$, we obtain $u \equiv_{M(A')} u'$, therefore $F(u,a) = F(u',a)$ because $a \in M(A')$. This shows the contradiction and we are done. $\square$

By Proposition 3.10, we immediately obtain the following interesting order isomorphism.

Corollary 3.11. *The map $\pi_{\mathcal{I}}$ induces an order isomorphism between the posets $\mathbb{M}(\mathcal{I})$ and $\mathbb{P}_{\mathrm{ind}}(\mathcal{I})$. Moreover, since $\mathbb{M}(\mathcal{I})$ is a complete lattice, the same holds also for $\mathbb{P}_{\mathrm{ind}}(\mathcal{I})$.*

Definition 3.12. We call $\mathbb{M}(\mathcal{I})$ the *maximum partitioner lattice* of $\mathcal{I}$ and $\mathbb{P}_{\mathrm{ind}}(\mathcal{I})$ the *indiscernibility partition lattice* of $\mathcal{I}$.

We now give another characterization of $MAXP(\mathcal{I})$.

Proposition 3.13. *Let $A \in \mathcal{P}(\Omega)$. Then:*

$$M(A) = \bigcap \{B : B \in MAXP(\mathcal{I}),\ A \subseteq B\}. \tag{20}$$

Proof. Since M is a closure operator, $M(A) \subseteq B$ for any $B \in MAXP(\mathcal{I})$ such that $A \subseteq B$. Vice versa, since $A \subseteq M(A)$, the intersection in the right side of (20) is obviously contained in $M(A)$. Hence (20) holds. □

We set now

$$\Gamma(W) := \{a \in \Omega\ :\ \forall\, u, u' \in W, F(u,a) = F(u',a)\}. \tag{21}$$

Let us observe that $\Gamma : \mathcal{P}(U) \to \mathcal{P}(\Omega)$. Moreover, if $W' \in \mathcal{P}(U)$ it is clear that

$$W \subseteq W' \Longrightarrow \Gamma(W) \supseteq \Gamma(W'). \tag{22}$$

The following result is immediate.

Proposition 3.14. $\Gamma(W) = A$ *if and only if the following hold:*

(i) $W \preccurlyeq \pi(A)$;
(ii) *if* $A' \in \mathcal{P}(\Omega)$ *and* $W \preccurlyeq \pi(A')$, *then* $A' \subseteq A$.

Definition 3.15. We call the set operator Γ the *indiscernibility operator* of $\mathcal{I}$.

We set now

$$\Pi_{\mathrm{ind}}(\mathcal{I}|W) := \{\pi \in \Pi_{\mathrm{ind}}(\mathcal{I}) : W \preccurlyeq \pi\} \text{ and } \mathbb{P}_{\mathrm{ind}}(\mathcal{I}|W) := (\Pi_{\mathrm{ind}}(\mathcal{I}|W), \preceq), \tag{23}$$

$$MAXP(\mathcal{I}|W) := \{B \in MAXP(\mathcal{I}) : W \preccurlyeq \pi(B)\} \text{ and } \mathbb{M}(\mathcal{I}|W) := (MAXP(\mathcal{I}|W), \subseteq^*). \tag{24}$$

In the next result we show that the generalized indiscernibility relation is also a specific type of maximum partitioner having an important role in the lattice structure of $\mathbb{M}(\mathcal{I})$.

Theorem 3.16. *We have that* $\Gamma(W) \in MAXP(\mathcal{I}|W)$ *and the poset* $\mathbb{M}(\mathcal{I}|W)$ *is a complete lattice that coincides with the upset of* $\Gamma(W)$ *in the lattice* $\mathbb{M}(\mathcal{I})$.

Proof. Let $C := \Gamma(W)$ and $D := M(C)$. If $u, u' \in W$, by definition of $\Gamma(W)$, we have $u \equiv_C u'$, therefore we also obtain $u \equiv_D u'$, because $\pi(D) = \pi(C)$. Hence $W \preccurlyeq \pi(D)$. By (ii) of Proposition 3.14 we deduce then that $D \subseteq C$, i.e. $C = M(C)$. This shows that $C \in MAXP(\mathcal{I})$. By part (i) of Proposition 3.14 we also have $W \preccurlyeq \pi(C)$, so that $C = \Gamma(W) \in MAXP(\mathcal{I}|W)$.

We show that $MAXP(\mathcal{I}|W)$ is an upset generated by $\Gamma(W)$ in the poset $\mathbb{M}(\mathcal{I})$, which is equivalent to the following condition:

$$A \in MAXP(\mathcal{I}|W) \iff A \in MAXP(\mathcal{I}) \text{ and } A \subseteq \Gamma(W). \tag{25}$$

Let $A \in MAXP(\mathcal{I}|W)$. Then $A \in MAXP(\mathcal{I})$ by definition of $MAXP(\mathcal{I}|W)$. If $u, u' \in W$ we have $u \equiv_A u'$ because $W \preccurlyeq \pi(A)$. Therefore, by (ii) of Proposition 3.14, it follows that $A \subseteq \Gamma(W)$.

Conversely, let now $A \in MAXP(\mathcal{I})$ such that $A \subseteq \Gamma(W)$. Let us prove that $W \preccurlyeq \pi(A)$. For this, let $u, u' \in W$. Then $\forall a \in A$, since $a \in \Gamma(W)$, $F(u, a) = F(u', a)$. Then $u \equiv_A u'$ and thus $W \preccurlyeq \pi(A)$. So $A \in MAXP(\mathcal{I}|W)$. This proves (25).

Now, we assume that $\{A_j : j \in J\} \subseteq MAXP(\mathcal{I}|W)$. Then, by (25), $A_j \in MAXP(\mathcal{I})$ and $A_j \subseteq \Gamma(W)$ for any $j \in J$. By (iii) of Theorem 3.8, the join of $\{A_j : j \in J\}$ is $\bigcap_{j \in J} A_j$, which is again an element of $MAXP(\mathcal{I})$ contained in $\Gamma(W)$. Therefore, by (25) it also belongs to $MAXP(\mathcal{I}|W)$.

On the other hand, by (iv) of Theorem 3.8 the meet of $\{A_j : j \in J\}$ is $M(\bigcup_{j \in J} A_j)$. Since M is a closure operator, we deduce that

$$M(\bigcup_{j \in J} A_j) \subseteq M(\Gamma(W)) = \Gamma(W).$$

Again by (25), $M(\bigcup_{j \in J} A_j) \in MAXP(\mathcal{I}|W)$. Thus the thesis follows. □

Proposition 3.17. *The poset* $\mathbb{P}_{\mathrm{ind}}(\mathcal{I}|W)$ *is a complete lattice isomorphic to the lattice* $\mathbb{M}(\mathcal{I}|W)$.

Proof. It is sufficient to observe that the map $\pi_{\mathcal{J}}$ induces an order isomorphism between $\mathbb{M}(\mathcal{J})$ and $\mathbb{P}_{\text{ind}}(\mathcal{J})$ whose image is exactly $\Pi_{\text{ind}}(\mathcal{J}|W)$. Therefore, the thesis follows directly by Theorem 3.16. □

By virtue of the previous results, we give the following definition.

Definition 3.18. We call the lattice $\mathbb{P}_{\text{ind}}(\mathcal{J}|W)$ the *indiscernibility partition lattice* of $\mathcal{J}$ *conditioned* to W.

If $\pi = \{B_i : i \in I\} \in \Pi(U)$, we set

$$\text{Max}(\pi) := \bigcap_{i \in I} \Gamma(B_i). \tag{26}$$

The relevance of the previous set operator Γ is due to the following result.

Theorem 3.19. *Let $\pi = \{B_i : i \in I\} \in \Pi(U)$ and $A = \text{Max}(\pi)$. Then:*

(i) $A \in MAXP(\mathcal{J})$;
(ii) $\pi \preceq \pi(A)$;
(iii) $\pi \in \Pi_{\text{ind}}(\mathcal{J}) \iff \pi = \pi(A)$;
(iv) *Let $A' \in \mathcal{P}(\Omega)$ and $\pi(A') = \{C_l : l \in I'\}$. Then*

$$M(A') = \bigcap_{l \in I'} \Gamma(C_l).$$

Proof. (i) Since $MAXP(\mathcal{J})$ is a closure system and $\Gamma(B_i) \in MAXP(\mathcal{J})$, the thesis holds.

(ii) Let $u, u' \in U$ be two elements belonging to a same block B_k of π. We must show that $u \equiv_A u'$. Therefore, let $a \in A$. Then $a \in \Gamma(B_k)$, so $F(u,a) = F(u'a)$ by definition of $\Gamma(B_k)$.

(iii) The implication $\Longleftarrow$ is obvious. We assume therefore that $\pi \in \Pi_{\text{ind}}(\mathcal{J})$. Let $C = \text{Max}(\pi)$. Then we have $\pi = \pi(C)$, and this implies that $F(u,c) = F(u',c)$ for all $u, u' \in B_i$, for each $i \in I$ and each $c \in C$. By definition of $\Gamma(B_i)$, we deduce then that $C \subseteq \Gamma(B_i)$ for every $i \in I$. Hence $C \subseteq A$.

Let now $a \in A$. By definition of A, we have

$$F(u,a) = F(u',a), \quad \forall u, u' \in B_i, \quad \text{for each } i \in I. \tag{27}$$

Let us suppose now by contradiction that $a \notin C$. Then, since $C \in MAXP(\mathcal{J})$ by part (i), it follows that $\pi(C \cup \{a\}) \prec \pi(C) = \pi$. Therefore, there exists at least an index $r \in I$ and two distinct objects $u, u' \in B_r$ such that $F(u,a) \neq F(u',a)$, and this in contrast with (27). This shows that $A \subseteq C$. Thus $A = C$ and $\pi = \pi(C) = \pi(A)$. This concludes the proof of (iii).

(iv): We prove only the statement for A'. Let $\pi := \pi(A')$. If we set $A := \bigcap_{l \in I'} \Gamma(C_l)$, we deduce that $\pi(A') = \pi = \pi(A)$ by part (iii). Hence, $A \approx A'$. But $A \in MAXP(\mathcal{J})$ by part (i), thus $M(A') = A$. □

4. Separators and related notions

In this section we assume a topological point of view for our investigation of the maximum partitioner family. We set

$$S(A) := \Omega \setminus M(A). \tag{28}$$

Definition 4.1. We call any element $s \in S(A)$ a *separator* of A and we set

$$SEP(\mathcal{J}) := \{S(A) : A \in \mathcal{P}(\Omega)\}.$$

By (16) the following characterization is immediate.

Proposition 4.2. *Let $s \in \Omega$. Then s is a separator of A if and only if there exist two points $u, u' \in U$ such that:*

(i) $F(u,a) = F(u',a)\ \forall a \in A$;
(ii) $F(u,s) \neq F(u',s)$.

If $\mathcal{A} = \{A_i : i \in I\} \subseteq \mathcal{P}(\Omega)$, we set

$$S(\mathcal{A}) := \bigcap_{i \in I} S(A_i). \tag{29}$$

Proposition 4.3. $S(\cup\mathcal{A}) \subseteq S(\mathcal{A})$.

Proof. Let $s \in S(\mathcal{A})$, then there exist $u, u' \in U$ with the property that for any $a \in \bigcup_{i\in I} A_i$, $F(u,a) = F(u',a)$ and $F(u,s) \neq F(u',s)$. In other words, for any fixed $\tilde{i} \in I$ and for any choice of $a \in A_{\tilde{i}} \subseteq \bigcup_{i\in I} A_i$, it follows that $F(u,a) = F(u',a)$ and $F(u,s) \neq F(u',s)$, i.e. $s \in S(A_{\tilde{i}})$. It readily follows that $s \in S(A_i)$ for any $i \in I$ and hence $s \in S(\mathcal{A})$. □

Definition 4.4. We say that a chain $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ is *union preserving separated* if $S(\mathcal{A}) \subseteq S(\cup\mathcal{A})$.

Let us consider now the following two very natural infinite function systems.

Definition 4.5. We call the Boolean symmetric function system $\mathcal{R} = \langle \mathbb{R}, E_U, \{0,1\}\rangle$, where $E_U(x,y) := 1$ if $d(x,y) = 1$ and $E_U(x,y) := 0$ otherwise, the *unitary Euclidean real function system*. Similarly, we call the Boolean symmetric function system $\mathcal{N} = \langle \mathbb{N}, E_U, \{0,1\}\rangle$, where E_U is defined as before, the *unitary Euclidean natural function system*.

Example 4.6. Let $\mathcal{A} = \{A_n : n \geq 1\}$ where $A_n := [0, (1+\frac{1}{n})^n]$. Since each A_n is an interval, by Proposition 4.13 we have that $A_n \in MAXP(\mathcal{R})$, for any $n \geq 1$, and

$$S(A_n) = \mathbb{R} \setminus A_n =]-\infty, 0[\cup \left]\left(1+\frac{1}{n}\right)^n, +\infty\right[.$$

It follows immediately that

$$S(\mathcal{A}) := \bigcap_{n\geq 1} S(A_n) =]-\infty, 0[\cup]\mathrm{e}, +\infty[.$$

On the other hand, by the fact that

$$\bigcup_{n\geq 1} A_n = [0, \mathrm{e}],$$

again by Proposition 4.13, we deduce that

$$S(\cup\mathcal{A}) = S([0,\mathrm{e}]) = \mathbb{R} \setminus [0,\mathrm{e}].$$

In other terms, we have that $S(\mathcal{A}) = S(\cup\mathcal{A})$, so $\mathcal{A}$ is a union preserving separated chain.

In what follows, we define a new class of function systems, i.e. those for which any chain is union preserving.

Definition 4.7. We say that the function system $\mathcal{I}$ is *union chain preserving*, abbreviated *UCP*, if any chain $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ is union preserving separated.

Remark 4.8. Any finite function system is UCP.

We now characterize the UCP function systems.

Theorem 4.9. *The following three conditions are equivalent:*

(*i*) *$\mathcal{I}$ is UCP;*
(*ii*) *$MAXP(\mathcal{I})$ is a chain union closed family;*
(*iii*) *For any $A \in \mathcal{P}(\Omega)$, we have*

$$M(A) = \bigcup\{M(F) : F \subseteq_f A\}. \tag{30}$$

Proof. $(i) \Longrightarrow (ii)$: Let $\mathcal{I}$ be a UCP function system and let $\mathcal{B} = \{B_i : i \in I\}$ be a chain of maximum partitioners of Ω. Let $B := \cup\mathcal{B}$ and $s \in \Omega \setminus B$. Then $s \in \Omega \setminus B_i = \Omega \setminus M(B_i) = S(B_i)$ for all $i \in I$, because any B_i is a maximum partitioner and therefore $M(B_i) = B_i$. Hence $s \in \bigcap_{i\in I} S(B_i) := S(\mathcal{B})$, and $S(\mathcal{B}) \subseteq S(\cup\mathcal{B})$ by hypothesis, so that $s \in S(B)$. This shows that $\Omega \setminus B \subseteq S(B) = \Omega \setminus M(B)$, i.e. $M(B) \subseteq B$, and hence $B \in MAXP(\mathcal{I})$. Therefore $MAXP(\mathcal{I})$ is a chain union closed family.

$(ii) \Longrightarrow (i)$: Let $MAXP(\mathcal{I})$ be a chain union closed family and let $\mathcal{A} = \{A_i : i \in I\} \subseteq \mathcal{P}(\Omega)$ be a chain. Since $A_i \subseteq A_j$ implies that $M(A_i) \subseteq M(A_j)$, it follows that $\mathcal{A}_M := \{M(A_i) : i \in I\}$ is a chain of maximum partitioners. Then, by hypothesis we have that $K := \bigcup_{i\in I} M(A_i) \in MAXP(\mathcal{I})$. Let us note now that

$$S(\mathcal{A}) := \bigcap_{i\in I} S(A_i) = \bigcap_{i\in I} \Omega \setminus M(A_i) = \Omega \setminus K. \tag{31}$$

Then, since $K = M(K)$, by (31) we deduce that

$$S(\mathcal{A}) = \Omega \setminus M(K) = S(K). \tag{32}$$

Now, since $A_i \subseteq M(A_i)$ for all $i \in I$, we have $\cup\mathcal{A} := \bigcup_{i\in I} A_i \subseteq K$, therefore

$$S(K) \subseteq S(\cup\mathcal{A}). \tag{33}$$

Hence, by (32) and (33) we deduce that $S(\mathcal{A}) \subseteq S(\cup\mathcal{A})$. This shows that $\mathcal{I}$ is an UCP.

$(ii) \Longrightarrow (iii)$: If $\mathcal{I}$ is a function system on a finite set Ω, then the thesis is obvious. Let now $\mathcal{I}$ be a function system on an infinite set Ω and let $A \subseteq \Omega$. We endow A with a well-order $<$. Set

$$C(a) := \{b \in A : b < a\}$$

for any $a \in A$. Let us assume that the thesis holds for sets whose cardinality is less than $|A|$ and, furthermore, that for any $a \in A$, the set $C(a)$ has cardinality strictly less than $|A|$. In particular, the thesis holds for $C(a)$, for any $a \in A$. Let $\mathfrak{Q} := \{M(C(a)) : a \in A\}$. It is clearly a chain in $MAXP(\mathcal{I})$. Since $MAXP(\mathcal{I})$ is a chain union closed family, we have that the union

$$D := \bigcup \mathfrak{Q}$$

belongs to $\mathfrak{Q}$. Let $K := \bigcup_{a\in A} C(a)$. Hence, we have that $M(C(a)) \subseteq M(K)$ for any $a \in A$, so $D \subseteq M(K)$. On the other hand, it is obvious that $M(K) \subseteq D$, thus $D = M(K)$. By transfinite inductive hypothesis, the thesis holds for $C(a)$, for any $a \in A$. We observe that for any $F \subseteq_f A$, there exists $a \in A$ such that $F \subseteq_f C(a)$. This shows the claim.

$(iii) \Longrightarrow (ii)$: Let $\mathcal{C}$ be a non-empty chain of maximum partitioners in $MAXP(\mathcal{I})$ and let $B := \bigcup \mathcal{C}$. We will show that $B \in MAXP(\mathcal{I})$. Suppose by contradiction that there exists $F \subseteq_f \bigcup B$ such that $F \not\subseteq_f C$ for any $C \subseteq B$. Let us fix $a \in F$, then there exists $C_1 \in B$ such that $x \in C_1$ and, in a similar way, there exists $C_2 \in B$ such that $F \setminus \{a\} \subseteq C_2$. Therefore, if $C_1 \subseteq C_2$, we would have $F \subseteq C_2$, which is absurd and, analogously, if $C_2 \subseteq C_1$, then $F \subseteq C_1$, once again a contradiction. Thus F must be contained in some $C \subseteq B$. Furthermore, we must have $M(F) \subseteq C \subseteq B$ so, by our hypothesis, the claim has been showed. □

In the next result, we provide some sufficient condition in order to verify the existence of a maximal element for any non-empty subfamily $\mathcal{A}$ of $MAXP(\mathcal{I})$.

Theorem 4.10. *If $MAXP(\mathcal{I})$ is a chain union closed family and for any $C \in MAXP(\mathcal{I})$, there exists $F \subseteq_f C$ such that $F \approx C$, then any non-empty subfamily $\mathcal{A}$ of $MAXP(\mathcal{I})$ admits a maximal element.*

Proof. Let $\mathcal{A}$ be a non-empty subfamily of $MAXP(\mathcal{I})$. Since $MAXP(\mathcal{I})$ is a chain union closed family, then $\bigcup\mathcal{K} \in MAXP(\mathcal{I})$ for any $\mathcal{K}$ chain in $MAXP(\mathcal{I})$. Therefore, let us take a chain $\mathcal{K}$ in $\mathcal{A}$, so $C := \bigcup\mathcal{K} \in MAXP(\mathcal{I})$. By our assumption, there exists $F \subseteq_f C$ such that $F \approx C$. Let us define the map $\phi : a \in F \mapsto A_a \in \mathcal{K}$, where A_a is a maximum partitioner in $\mathcal{K}$ containing a. Then, we obtain a finite chain $\{A_a : a \in F\}$ in $\mathcal{A}$ clearly having a maximal element, which we call A_b, for some $b \in F$. Thus, $A_b \in MAXP(\mathcal{I})$. Since $F \subseteq A_b \subseteq C$, we deduce that $M(A_b) = A_b = C$. But since $\mathcal{K}$ is a maximal chain, we conclude that A_b is a maximal element in $\mathcal{A}$ and this completes the proof. □

Let now $\mathbb{K}$ be equal to $\mathbb{N}$ or $\mathbb{R}$, $\mathcal{K} := \langle \mathbb{K}, E_U, \{0, 1\}\rangle$ and let A be a given subset of $\mathbb{K}$. In order to give a concise description of the A-indiscernibility partition, we set:

$$A^* := (A-1) \cup (A+1), \quad A_* := (A-1) \cap (A+1)^c \cap (A-3)^c. \tag{34}$$

In the next result, we provide the general form for an A-indiscernibility partition of $\mathcal{K}$.

Proposition 4.11. *We have that*

$$\pi(A) = \mathbb{K} \setminus A^* | \{x, x+2\}_{x\in A_*} | \{y\}_{y\in A^*\setminus(A_*\cup A_*+2)}.$$

Proof. Let $x, y \in \mathbb{K}$ such that $x < y$. It is easy to prove that $x \equiv_A y$ in $\mathcal{K}$ if and only if $x, y \in \mathbb{R} \setminus A^*$ or $x \in A_*$ and $y = x + 2$. Moreover, in this second case, since $A_* = \{x \in \mathbb{K} : x - 1 \notin A, x + 3 \notin A, x + 1 \in A\}$, there exist no other real number equivalent to x beside y. □

In the next result, we characterize the maximum partitioners of $\mathcal{K}$.

Proposition 4.12. *$A \in MAXP(\mathcal{K})$ if and only if for all $a \notin A$*

(1′) *($a - 2 \notin A$ or $a + 2 \notin A$) and the subset $\mathbb{R} \setminus A^*$ has at least two elements;*
(2′) *($a + 2 \in A$ and $a + 4 \notin A$) or ($a - 2 \in A$ and $a - 4 \notin A$).*

Proof. A subset $A \subseteq \mathbb{K}$ is a maximum partitioner in $\mathcal{K}$ if and only if, for all $a \notin A$, there exist $x, y \in \mathbb{K}$ such that $x \equiv_A y$ but $x \not\equiv_{A\cup\{a\}} y$. By (4.11), it is equivalent to say that, for each choice of $a \notin A$, it holds at least one of the followings:

(1) $\exists x, y \in \mathbb{K} \setminus A^*$ such that $x < y$ and $\{x, y\} \cap (A \cup \{a\})^* \neq \emptyset$;
(2) $\exists x \in A_* \setminus (A \cup \{a\})_*$.

These conditions are equivalent, respectively, to $1'$ and $2'$. $\square$

We now show that any interval of the real line is a maximum partitioner.

Proposition 4.13. *Let A be any interval of the real line, then $A \in MAXP(\mathcal{R})$.*

Proof. Let A be an interval of the real line. Let us consider $A = [\alpha, \beta)$; the other cases are similar. Let $a \notin A$. If $a \in (-\infty, \alpha+2)$ or $a \in [\beta-2, +\infty)$, then $a-2 \notin A$ or $a+2 \notin A$, so condition $(1')$ is satisfied. If $a \in (-\infty, \alpha+2) \cup [\beta-2, +\infty)$, then $\beta-\alpha < 2$, $\{a-2, a+2\} \cap A \neq \emptyset$ and thus if $a-2 \in A$, then $a-4 \notin A$ and if $a+2 \in A$, then $a+4 \notin A$. It follows the thesis. $\square$

We prove now that both $\mathcal{R}$ and $\mathcal{N}$ are UCP.

Theorem 4.14. *$MAXP(\mathcal{K})$ is a chain union closed family.*

Proof. Let $\mathcal{A} = \{A_i : i \in I\}$ be a chain of maximum partitioners of $\mathcal{K}$, and let $A := \bigcup_{i\in I} A_i$. Let $a \notin A$ and suppose $a-2, a+2, a+4 \in A$. Since $\mathcal{A}$ is a chain, there exists $i \in I$ such that $a-2$, $a+2$ and $a+4$ are all in A_i. It follows that $a-4 \notin A_i$, and thus, for all $j \in I$ such that $A_j \subseteq A_i$, $a-4 \notin A_j$. Moreover, if $j \in I$ is such that $A_i \subseteq A_j$, then $a-2, a+2, a+4 \in A_j$, thus, since $A_j \in MAXP(\mathcal{K})$, $a-4 \notin A_j$. We conclude that $a-4 \notin A$ and $A \in MAXP(\mathcal{K})$.

Let us suppose now $|\mathbb{K} \setminus A^*| \leq 1$ and that $a+2 \notin A$ and $a-2 \in A$. We have to prove that $a-4 \notin A$. If $a-4 \in A$ then for some $i \in I$, $|\mathbb{K} \setminus A_i^*| \leq 1$, $a+2 \notin A_i$, $a-2 \in A_i$ and $a-4 \in A_i$, and this contradicts the fact that A_i is a maximum partitioner. Moreover, it can not hold simultaneously that both $a+2$ and $a-2$ do not belong to A, because in this case, since $a \notin A$, it would hold that $|\mathbb{K} \setminus A^*| \geq 1$, contradicting our assumption.

Suppose $a+4 \in A$. If $a-2 \notin A$, then $a+2 \in A$ and there exists $i \in I$ such that $|\mathbb{K} \setminus A_i^*| \leq 1$, $a+2 \in A_i$, $a-2 \notin A_i$ and $a+4 \in A_i$, but this is impossible because $A_i \in MAXP(\mathcal{K})$. Finally suppose that both $a-4$ and $a+4$ are in A. Then, in this case for some $i \in I$, $|\mathbb{K} \setminus A_i^*| \leq 1$, $a-4 \in A_i$ and $a+4 \in A_i$, so A_i is not a maximum partitioner by Proposition 4.12, which is a contradiction.

It follows that $A \in MAXP(\mathcal{K})$. $\square$

5. Indiscernibility hypergraphic structures of a function system

In this section, we introduce three new hypergraphs that we use to investigate the closure system $MAXP(\mathcal{I})$ in relation to an associated simplicial complex $MINP(\mathcal{I})$ that we will introduce in Section 6.

Let $\mathcal{I} = \langle U, \Omega, F, \Lambda\rangle$ be a function system.
For $A \in \mathcal{P}(\Omega)$ we set $\Delta_A : U \times U \to \mathcal{P}(A)$ defined by

$$\Delta_A(u, u') := \{a \in A : F(u, a) \neq F(u', a)\}, \tag{35}$$

for any $u, u' \in U$.

In particular, we set $\Delta := \Delta_{U,\Omega}$, so that $\Delta_A(u, u') = \Delta(u, u') \cap A$. We call the hypergraph having vertex set A and hyperedge set the following subset family of A:

$$DISC(A) := \{\Delta_A(u, u') : u, u' \in U \text{ and } \Delta_A(u, u') \neq \emptyset\} \tag{36}$$

the *A-discernibility hypergraph*. In particular, we set $DISC(\mathcal{I}) := DISC(\Omega)$ and we call $\mathfrak{D}(\mathcal{I}) := (\Omega, DISC(\mathcal{I}))$ the *discernibility hypergraph* of $\mathcal{I}$.

The following result relates the subsets $\Delta_A(u, u')$ to the indiscernibility relation.

Proposition 5.1. *Let $D \subseteq A$ and $u, u' \in U$. Then:*

(i) $D = \Delta_A(u, u') \Longrightarrow u \equiv_{A\setminus D} u'$;
(ii) $u \equiv_{A\setminus D} u' \Longrightarrow \Delta_A(u, u') \subseteq D$;
(iii) *Let $C \subseteq A$. Then* $\Delta_A(u, u') \cap C = \emptyset \Longleftrightarrow u \equiv_C u'$.

Proof. Straightforward. $\square$

We now introduce some new notions that we will use in what follows.
If $A \in \mathcal{P}(\Omega)$, we set

$$CORE(A) := \{a \in A : \pi(A) \neq \pi(A \setminus \{a\})\},$$
$$\mathfrak{N}(A) := \{B \subseteq A : CORE(B) = \emptyset\},$$
$$MAXP(A) := \{B \cap A : B \in MAXP(\mathfrak{I})\}.$$

We call $CORE(A)$ the *A-core* of $\mathfrak{I}$ and we can use the notion of A-core to establish the following characterization.

Theorem 5.2. *Let $A \in \mathcal{P}(\Omega)$. Then:*

(i) $CORE(A) = \{a \in A : A \setminus \{a\} \in MAXP(A)\}$.
(ii) $CORE(M(A)) \subseteq A$.
(iii) $CORE(A) = \{a \in A : a \notin M(A \setminus \{a\})\}$.
(iv) $CORE(A) = \emptyset$ *if and only if for any* $B \in SEP(\mathfrak{I})$ *and for any* $a \in A \cap B$*, it results*

$$A \setminus \{a\} \cap B \neq \emptyset. \tag{37}$$

(v) $\mathfrak{N}(A)$ *is a union-closed family.*

Proof. (i): Let $a \in CORE(A)$. We must show that $A \setminus \{a\} \in MAXP(A)$, i.e. there exists $B \in MAXP(\mathfrak{I})$ such that $A \setminus \{a\} = A \cap B$. Let us assume by absurd that $A \setminus \{a\} \neq A \cap B$ for any $B \in MAXP(\mathfrak{I})$. Since $\pi(A) \neq \pi(A \setminus \{a\})$, then $A \setminus \{a\} \not\approx A$. Let $B = M(A \setminus \{a\})$. Clearly, $A \setminus \{a\} \subseteq B$, and we have $A \setminus \{a\} \subseteq A \cap B$; therefore, by our assumption, it follows that $A \setminus \{a\} \subsetneq A \cap B$. Hence $A \cap B = A$, i.e. $A \subseteq B$. It follows that $\pi(B) = \pi(A \setminus \{a\}) \preceq \pi(A) \preceq \pi(A \setminus \{a\}) = \pi(B)$, i.e. $\pi(A) = \pi(B)$, which is equivalent to say that $A \approx B \approx A \setminus \{a\}$, which is absurd. Thus $A \setminus \{a\} \in MAXP(A)$.

Conversely, let $a \in A$ such that $A \setminus \{a\} \in MAXP(A)$. Hence, there exists $B \in MAXP(\mathfrak{I})$ such that $A \setminus \{a\} = B \cap A$. If $B = A \setminus \{a\}$, then $A \setminus \{a\} \in MAXP(\mathfrak{I})$. In this case, by maximality of $A \setminus \{a\}$, then $A \notin [A \setminus \{a\}]_{\approx}$, so $\pi(A) \neq \pi(A \setminus \{a\})$, therefore $a \in MAXP(A)\} \subseteq CORE(A)$, and the claim is proved. Otherwise, let $A \setminus \{a\} \subsetneq B$. Clearly, it results that $a \notin B$. Furthermore, it results $\pi(B) \preceq \pi(A \setminus \{a\})$. Suppose by contradiction that $\pi(A) = \pi(A \setminus \{a\})$. Then, we have $\pi(B) \preceq \pi(A)$, so

$$u \equiv_B u' \Longrightarrow u \equiv_A u' \Longrightarrow F(u, a) = F(u', a) \Longrightarrow u \equiv_{B \cup \{a\}} u'.$$

In other terms, we have shown that $\pi(B) \preceq \pi(B \cup \{a\})$. Nevertheless, we also have $\pi(B \cup \{a\}) \preceq \pi(B)$, i.e. $\pi(B) = \pi(B \cup \{a\})$. This contradicts the maximality of B, which is absurd. Thus, $a \in CORE(A)$.

(ii): Let $a \in CORE(M(A))$ and suppose, by contradiction, that $a \notin A$. Therefore, we have $A \subseteq M(A) \setminus \{a\} \subseteq M(A)$ and, hence, that $\pi(M(A)) \preceq \pi(M(A) \setminus \{a\}) \preceq \pi(A)$. Since $\pi(M(A)) = \pi(A)$, we conclude that $\pi(M(A) \setminus \{a\}) = \pi(M(A))$, contradicting the fact that $a \in CORE(A)$.

(iii): Let $a \in CORE(A)$, then $\pi(A) \neq \pi(A \setminus \{a\})$. Therefore, $M(A \setminus \{a\}) \subsetneq M(A)$. We must have $a \notin M(A \setminus \{a\})$, otherwise $M(A) \subseteq M(A \setminus \{a\})$, which is a contradiction. Hence $CORE(A) \subseteq \{a \in A : a \notin M(A \setminus \{a\})\}$. Conversely, if $a \notin M(A \setminus \{a\})$, it is clear that $\pi(A) \neq \pi(A \setminus \{a\})$, i.e. $a \in CORE(A)$.

(iv): Let $CORE(A) = \emptyset$. Then for any $a \in A$ it results $M(A) = M(A \setminus \{a\})$. Let us fix $a \in A$. Suppose by contradiction that there exists $B \in SEP(\mathfrak{I})$ containing a and such that

$$A \setminus \{a\} \cap B = \emptyset.$$

Thus $A \setminus \{a\} \subseteq B^c = C$, where $C \in MAXP(\mathfrak{I})$. In particular, it follows that

$$M(A \setminus \{a\}) = M(A) \subseteq C$$

but this means that $a \in C$, which is absurd.

Conversely, suppose by contradiction that $CORE(A) \neq \emptyset$, then there exists $a \in A$ such that $M(A \setminus \{a\}) \subsetneq M(A)$. Hence, if we set $B = M(A \setminus \{a\})^c$, we have just found $B \in SEP(\mathfrak{I})$ such that $a \in A \cap B$ and $A \setminus \{a\} \cap B = \emptyset$, contradicting our assumption.

(v): It follows immediately by (iv). $\square$

By (v) of Theorem 5.2, we deduce that $\mathfrak{N}(A)$ has a unique maximum element. We set therefore

$$C(A) := \bigcup_{B \in \mathfrak{N}(A)} B. \tag{38}$$

We now provide the notion of *A-reduct*.

Definition 5.3. Let $B \subseteq A$. We say that B is an *A-reduct* of $\mathfrak{I}$ if:

$(R1)$ $\pi(A) = \pi(B)$;
$(R2)$ $\pi(A) \neq \pi(B')$ for all $B' \subsetneq B$.

We denote by $RED(A)$ the family of all A-reducts of $\mathfrak{I}$ and we set $RED(\mathfrak{I}) := RED(\Omega)$. We call the members of $RED(\mathfrak{I})$ the *reducts* of $\mathfrak{I}$, the hypergraph $\mathfrak{R}(A) := (A, RED(A))$ the *A-reduct hypergraph*, and the hypergraph $\mathfrak{R}(\mathfrak{I}) := \mathfrak{R}(\Omega)$ the *reduct hypergraph* of $\mathfrak{I}$.

The A-reduct family is related to the A-core in the following natural way.

Proposition 5.4. $CORE(A) := \bigcap\{C : C \in RED(A)\}$.

Proof. Let $a \notin CORE(A)$. Then $\pi(A) = \pi(A \setminus \{a\})$. This means that there exists $B \subseteq A \setminus \{a\}$ such that $\pi(A) = \pi(B)$, i.e. $B \in RED(A)$ and $a \notin B$. In other terms, we have shown that $\bigcap\{C : C \in RED(A)\} \subseteq CORE(A)$. On the other hand, let $B \in RED(A)$ and $a \in A \setminus B$. Then we have $\pi(B) \preceq \pi(A \setminus \{a\}) \preceq \pi(A)$ but, since $\pi(B) = \pi(A)$, we deduce that $\pi(A \setminus \{a\}) = \pi(A)$, i.e. $a \notin CORE(A)$. Hence, $CORE(A) \subseteq \bigcap\{C : C \in RED(A)\}$ and the thesis follows. □

We now give the notion of *A-essential*, which provides a generalization to the notion of A-core.

Definition 5.5. Let $B \subseteq A$. We say that B is *A-essential* if:

$(E1)$ $\pi(A \setminus B) \neq \pi(A)$;
$(E2)$ $\pi(A \setminus B') = \pi(A)$ for all $B' \subsetneq B$.

We denote by $ESS(A)$ the family of all A-essential subsets of A and we set $ESS(\mathfrak{I}) := ESS(\Omega)$. We call the members of $ESS(\mathfrak{I})$ the *essentials* of $\mathfrak{I}$, the hypergraph $\mathfrak{E}(A) := (A, ESS(A))$ the *A-essential hypergraph* of A and the hypergraph $\mathfrak{E}(\mathfrak{I}) := \mathfrak{E}(\Omega)$ the *essential hypergraph* of $\mathfrak{I}$.

The next result tells us that the A-essentials are exactly the minimal elements of the A-discernibility hypergraph.

Theorem 5.6. *We have that* $ESS(A) = \min(DISC(A))$.

Proof. Let $B \in ESS(A)$. By Definition 5.5 it results that $\pi(A \setminus B) \neq \pi(A)$. Hence, there exist two distinct elements $v, w \in U$ such that $v \equiv_{A \setminus B} w$ and $v \not\equiv_B w$. Equivalently, we can express the previous condition by saying that $A \setminus B \subseteq A \setminus \Delta_A(v, w)$, i.e. $\Delta_A(v, w) \subseteq B$. This shows that any A-essential contains some hyperedge of the A-discernibility hypergraph. We now claim that $\Delta_A(v, w) = B$. Indeed, if $b \in B$ and $B' := B \setminus \{b\} \subsetneq B$, we deduce that $v \not\equiv_{A \setminus B'} w$ by Definition 5.5. Therefore $b \in \Delta_A(v, w)$. By the arbitrariness of $b \in B$, it follows that $\Delta_A(v, w) = B$. This proves that $B \in DISC(A)$. Moreover, we proved that whenever two elements $v, w \in U$ satisfy the relation $\Delta_A(v, w) \subseteq B$, then $\Delta_A(v, w) = B$. This means that B is minimal in $DISC(A)$ with respect to the set-theoretical inclusion.

Let now $B = \Delta_A(v, w) \neq \emptyset$ minimal in the poset $(DISC(A), \subseteq)$, for some $v, w \in U$. Since B is not empty, by (iii) of Proposition 5.1, it follows that $v \not\equiv_A w$. Moreover, by (i) of Proposition 5.1 we also obtain $v \equiv_{A \setminus B} w$. Then we have $\pi(A \setminus B) \neq \pi(A)$, and thus B satisfies (i) of Definition 5.5. Let now $B' \subsetneq B$. B minimal in $DISC(A)$ implies that, for all $u, u' \in U$ such that $\Delta_A(u, u') \neq \emptyset$, $\Delta_A(u, u') \nsubseteq B'$. We claim that $\pi(A \setminus B') = \pi(A)$. It is obvious that $u \equiv_A u'$ implies $u \equiv_{A \setminus B'} u'$; furthermore suppose that $u \equiv_{A \setminus B'} u'$ and assume by contradiction that $u \not\equiv_A u'$. Then we have $\Delta_A(u, u') \subseteq B'$, which is absurd. Hence $u \equiv_A u' \iff u \equiv_{A \setminus B'} u'$, so $\pi(A \setminus B') = \pi(A)$. In this way, we have shown that B satisfies also condition (ii) of Definition 5.5. Then $B \in ESS(A)$ and the theorem is proved. □

Remark 5.7. By Theorem 5.6, it is clear that $\mathrm{Tr}(DISC(A)) = \mathrm{Tr}(ESS(A))$.

The next result shows that the A)-reducts are exactly the minimal transversals of the family $ESS(A)$.

Theorem 5.8. *Let* $B \subseteq A$. *Then* $RED(A) = \mathrm{Tr}(DISC(A)) = \mathrm{Tr}(ESS(A))$.

Proof. We firstly claim that $\pi(B) = \pi(A)$ if and only if $B \in \mathrm{Tr}(DISC(A))$. To prove this, let $B \subseteq A$. Let us note that the equality $\pi(B) = \pi(A)$ is obviously equivalent to the identity $\equiv_B = \equiv_A$. Therefore, we assume first that $\equiv_B = \equiv_A$; we must show that B is a transversal of $DISC(A)$. Let then $D \in DISC(A)$. By definition of $DISC(A)$, it results that D is not empty and that there exist two distinct elements $u, u' \in U$ such that $D = \{a \in A : F(u, a) \neq F(u', a)\}$. Since D contains at least one element, we deduce that $u \not\equiv_A u'$, and this also implies $u \not\equiv_B u'$ from the hypothesis $\equiv_B = \equiv_A$. Hence, by definition of $\equiv_B$, we find an element $b \in B$ such that $F(u, b) \neq F(u', b)$, i.e. $b \in B \cap D$. This shows that $B \cap D \neq \emptyset$; therefore, B is a transversal of $DISC(A)$.

We suppose now that B is a transversal of $DISC(A)$ and let u, u' be two any distinct elements in U. If $u \equiv_A u'$, it is obvious that we also have $u \equiv_B u'$. We can assume therefore that $u \not\equiv_A u'$. By definition of $\equiv_A$ and by (35), it follows then that the $D := \Delta_A(u, u')$ is not empty, so that $D \in DISC(A)$. Since B is a transversal of $DISC(A)$ we have that $B \cap D \neq \emptyset$. Let $b \in B \cap D$. Then we obtain a function $b \in B$ such that $F(u, b) \neq F(u', b)$, and this implies that $u \not\equiv_B u'$. Hence $\equiv_B = \equiv_A$. This proves that $\pi(B) = \pi(A)$ if and only if $B \in \mathrm{Tr}(DISC(A))$.

Let $B \in RED(A)$. By Definition 5.3, we have then $\pi(B) = \pi(A)$, and by part (i) this implies that B is a transversal of $DISC(A)$. Now, if $b \in B$, again by Definition 5.3 we have that $\pi(B \setminus \{b\}) \neq \pi(A)$, therefore by (i) it follows that $B \setminus \{b\}$ is not a transversal of $DISC(A)$. Hence B is a minimal transversal of $DISC(A)$. On the other hand, let B be a minimal transversal of $DISC(A)$, then by (i) it follows that $\pi(B) = \pi(A)$. Now, if $b \in B$ the subset $B \setminus \{b\}$ is not a transversal of $DISC(A)$ by virtue of the minimality of B, therefore, again by (i) we obtain $\pi(B \setminus \{b\}) \neq \pi(A)$. Hence, $B \in RED(A)$ and the thesis follows. $\square$

By virtue of Theorems 5.6 and 5.8, we give the following terminology.

Definition 5.9. We call $\mathfrak{D}(\mathcal{I})$, $\mathfrak{R}(\mathcal{I})$ and $\mathfrak{E}(\mathcal{I})$ the *indiscernibility hypergraphic structures* of $\mathcal{I}$.

We now study the deep link between relative maximum partitioners and $RED(A)$.

Proposition 5.10. *Let $B \in MAXP(A)$ such that $B \subsetneq A$. Then $A \setminus B$ is a transversal of $RED(A)$.*

Proof. Suppose by contradiction that $A \setminus B$ is not a transversal of $RED(A)$, i.e. that there exists $C \in RED(A)$ such that $C \cap (A \setminus B) = \emptyset$. Thus, $C \subseteq B \subsetneq A$. In particular, it results that

$$M(C) \subseteq M(B) \subseteq M(A),$$

but $M(C) = M(A)$, so $M(A) = M(B)$. Since there exists $D \in MAXP(\mathcal{I})$ such that $B = A \cap D$, we observe that $M(B) \subseteq D$, hence

$$A \subseteq M(A) \subseteq D$$

or, equivalently, $B = A \cap D = A$, which is a contradiction. Therefore, $A \setminus B$ is a transversal of $RED(A)$. $\square$

We now prove two interesting properties of the indistinguishability classes.

If $B, C \in [A]_\approx$ we write $B \lessdot_A C$ if C covers B in the poset $([A]_\approx, \subseteq)$. We also set

$$\mathcal{C}_A(B) := \{C \in [A]_\approx : C \lessdot_A B\}, \tag{39}$$

and

$$\mathcal{S}(A) := \{B \in [A]_\approx : |\mathcal{C}_A(B)| \leq 1\}. \tag{40}$$

Let us note that

$$RED(A) \subseteq \mathcal{S}(A). \tag{41}$$

We have the following result.

Theorem 5.11. *Any element of $[A]_\approx$ is a union of elements of $\mathcal{S}(A)$.*

Proof. Let $B \approx A$ such that $|\mathcal{C}_A(B)| \geq 2$. Firstly, we claim that if A_1, A_2, A_3 are three distinct non-empty function subsets such that $A_1, A_2 \approx A_3$ and $A_1, A_2 \lessdot A_3$, then $A_1 \cup A_2 = A_3$. It clearly results that $A_1 \cup A_2 \subseteq A_3$; suppose therefore that $A_1 \cup A_2 \subsetneq A_3$. Hence, $A_1 \subsetneq A_1 \cup A_2 \subsetneq A_3$, so A_3 does not cover A_1, contradicting our assumption. Thus, if $\mathcal{C}_A(B) = \{D, E\}$, we have

$$B = D \cup E. \tag{42}$$

If D (or E) covers more than one function subset, in (42) we can substitute D with the union of two of the subsets it covers; in this way, the thesis is proved. $\square$

Based on the result obtained in Theorem 5.11 we introduce the following terminology.

Definition 5.12. We call $\mathcal{S}(A)$ the *spanning family* of $[A]_\approx$.

In the next result, we find some conditions so that $RED(A)$ coincides with the spanning family of $[A]_\approx$.

	1	2	3	4	5
u_1	1	0	0	1	0
u_2	0	1	0	0	1
u_3	0	0	0	0	1
u_4	1	0	0	1	0
u_5	1	0	0	0	1

Fig. 1. The functional table $\mathcal{J}$.

Proposition 5.13. *Let $B, C \in [A]_{\approx}$ and $K = M(A)$. If:*

(i) $B \cap C = \emptyset$,
(ii) $B, C \lessdot_A K$,

then $\mathcal{C}_A(K) = \{B, C\} = RED(K)$ and $[A]_{\approx} = \{K, B, C\}$.

Proof. By (39), we have $\{B, C\} \subseteq \mathcal{C}_A(K)$. Suppose by contradiction that $\{B, C\} \subsetneqq \mathcal{C}_A(K)$ and let $D \in \mathcal{C}_A(K) \setminus \{B, C\}$. By the proof of Theorem 5.11, we have that the union of any pair of elements of $\mathcal{C}_A(K)$ is exactly K, hence it results $B = K \setminus C \subseteq D$ and $C = K \setminus B \subseteq D$. In this way, we conclude that $D = K$, contradicting our assumption. Therefore, $\mathcal{C}_A(K) = \{B, C\}$. On the other hand, in order to prove that $RED(K) = \{B, C\}$, by (41) we must only prove that $B, C \in RED(K)$. Suppose that $B \notin RED(K)$, thus there exists $D \subsetneqq B$ such that $D \in RED(K)$. Hence, it results

$$C \subsetneqq C \cup D \subsetneqq K,$$

contradicting the fact that $C \in \mathcal{C}_A(K)$. Similarly if $C \notin RED(K)$. Therefore $RED(K) = \{B, C\} = \mathcal{C}_A(K)$ and $[K]_{\approx} = \{K, B, C\}$. $\square$

6. Local indistinguishability posets and minimal partitioners

In this section, we introduce an abstract simplicial complex $MINP(\mathcal{I})$ that can be investigated as a potential type of matroid having $MAXP(\mathcal{I})$ as its closed set family.

Let $\mathcal{I}$ be a finite function system and $MAXP(\mathcal{I}) = \{C_1, \ldots, C_k\}$. We set now

$$IND_{\approx}(\mathcal{I}) = \{[C_1]_{\approx}, \ldots, [C_k]_{\approx}\}. \tag{43}$$

We introduce the following partial order $\sqsubseteq$ on $IND_{\approx}(\mathcal{I})$. If $[C_l]_{\approx}, [C_j]_{\approx} \in IND_{\approx}(\mathcal{I})$ we set

$$C_l \subseteq^* C_j \iff : [C_l]_{\approx} \sqsubseteq [C_j]_{\approx} \tag{44}$$

and

$$\mathbb{I}(\mathcal{I}) := (IND_{\approx}(\mathcal{I}), \sqsubseteq). \tag{45}$$

We have then the following immediate result

Theorem 6.1. *$\mathbb{I}(\mathcal{I})$ is a lattice that is order isomorphic to the maximum partitioner lattice $\mathbb{M}(\mathcal{I})$.*

Proof. The thesis follows immediately by the definition of $IND_{\approx}(\mathcal{I})$ and by (44). $\square$

Definition 6.2. We call $\mathbb{I}(\mathcal{I})$ the *indistinguishability lattice* of $\mathcal{I}$.

Example 6.3. Let us consider the function system $\mathcal{J}$ given in Fig. 1.

We represent the diagram of $\mathbb{M}(\mathcal{J})$ in Fig. 2 and the diagram of $\mathbb{I}(\mathcal{J})$ in Fig. 3.

Definition 6.4. We say that $([C_1]_{\approx}, \subseteq), \ldots, ([C_k]_{\approx}, \subseteq)$ are the *local indistinguishability posets* of $\mathcal{I}$.

Proposition 6.5. *Let $C_i \in MAXP(\mathcal{I})$. Then $([C_i]_{\approx} \cup \{\emptyset\}, \subseteq)$ is a lattice $([C_i]_{\approx} \cup \{\emptyset\}, \wedge, \vee)$ where*

$$A \wedge B = \bigcup \{C \in [C_i]_{\approx} \cup \{\emptyset\} : C \subseteq A \cap B\} \tag{46}$$

and

$$A \vee B = A \cup B \tag{47}$$

for any $A, B \in [C_i]_{\approx} \cup \{\emptyset\}$.

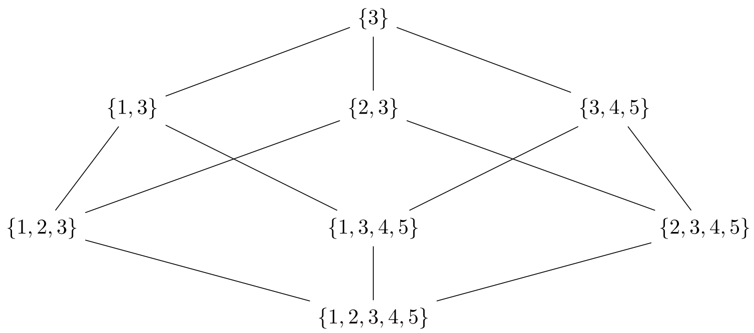

Fig. 2. The lattice $\mathbb{M}(\mathcal{J})$.

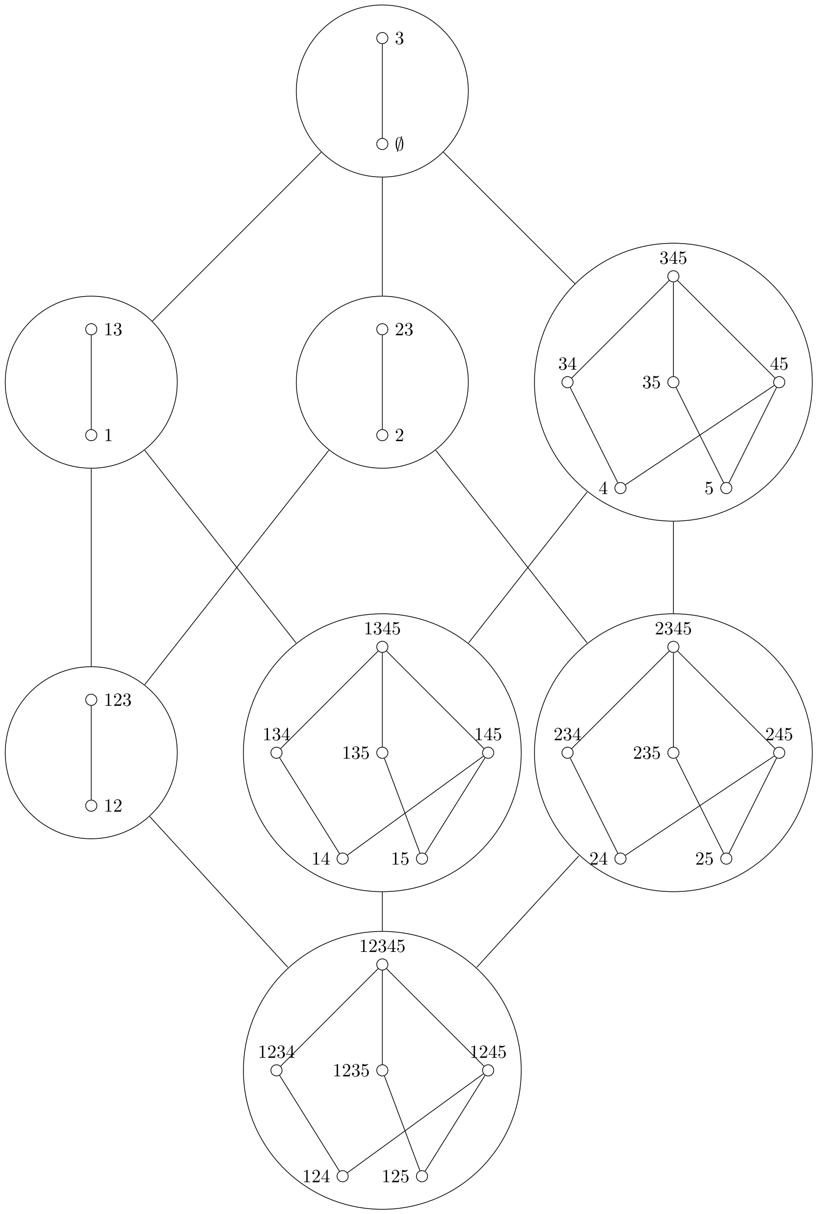

Fig. 3. The lattice $\mathbb{I}(\mathcal{J})$.

Proof. By Proposition 3.4, $[C_i]_{\approx} \cup \{\emptyset\}$ is a join-semilattice with a least element. So the thesis follows. □

We set now

$$K_{\mathfrak{J}}(A) := \{a \in A : \exists B \in [A]_{\approx} : B \setminus \{a\} \notin [A]_{\approx}\} \tag{48}$$

and $K^c_{\mathfrak{J}}(A) := A \setminus K_{\mathfrak{J}}(A)$.

Example 6.6. By referring to the function system given in Fig. 1, let $A = \{1,3,4,5\}$, we have that $K_{\mathfrak{J}}(A) = \{1,4,5\}$ and $K^c_{\mathfrak{J}}(A) = \{3\}$.

The next set family has a relevant role in this paper. We set

$$MINP(\mathfrak{J}) := \bigcup\{\min([A]_{\approx}) : A \in MAXP(\mathfrak{J})\}. \tag{49}$$

It is clear that $MINP(\mathfrak{J})$ can be considered as a dual version of $MAXP(\mathfrak{J})$.

In the next result, we establish some basic properties of the set families above introduced.

Theorem 6.7. *The following conditions hold:*

(i) *Let $A \in MAXP(\mathfrak{J})$. Then $a \in K_{\mathfrak{J}}(A)$ if and only if there exists $C \in MAXP(A)$ such that $a \notin C$ and $M(C \cup \{a\}) = A$.*
(ii) *Let $A \in MAXP(\mathfrak{J})$. Then $K^c_{\mathfrak{J}}(A) = \bigcap\{B : B \in [A|\mathbb{M}(\mathfrak{J})\uparrow]\}$.*
(iii) *If $A \in \mathcal{P}(\Omega)$, then $CORE(A) \subseteq K_{\mathfrak{J}}(A)$.*
(iv) *If $A \subseteq A'$ and $A \approx A'$, then $K_{\mathfrak{J}}(A) \subseteq K_{\mathfrak{J}}(A')$.*
(v) *If $A \in MINP(\mathfrak{J})$, then $K_{\mathfrak{J}}(A) = A$.*
(vi) *If $A \in \mathcal{P}(\Omega)$, then $A \approx K_{\mathfrak{J}}(M(A))$.*
(vii) *If $A \in \mathcal{P}(\Omega)$, then $M(A) = M(K_{\mathfrak{J}}(M(A)))$.*
(viii) *If $K_{\mathfrak{J}}(A) = \emptyset$ then for any $B \in SEP(\mathfrak{J})$ and for any $a \in A \cap B$, it results*

$$A \setminus \{a\} \cap B \neq \emptyset. \tag{50}$$

Proof. (i): Let $a \in K_{\mathfrak{J}}(A)$. By definition of $K_{\mathfrak{J}}(A)$, there exists $B \in [A]_{\approx}$ such that

$$B \setminus \{a\} \not\approx A. \tag{51}$$

Since $B \approx A$, by (51) we have that $a \in B$. Let $C := M(B \setminus \{a\})$. We first show that $a \notin C$. In fact, let us assume by absurd that $a \in C$. In this case, $B \subseteq C$ because $B \setminus \{a\} \subseteq M(B \setminus \{a\}) = C$. Therefore, $A = M(B) \subseteq M(C) = C$ because $A, C \in MAXP(\mathfrak{J})$ and $B \in [A]_{\approx}$, and this implies that

$$\pi(C) \preceq \pi(A). \tag{52}$$

On the other hand, we also have

$$\pi(A) = \pi(B) \preceq \pi(B \setminus \{a\}) = \pi(C). \tag{53}$$

Then, by (52) and (53) we deduce $\pi(A) = \pi(C)$, i.e. $A \approx C$, and this implies (by definition of C) $A \approx B \setminus \{a\}$, which is in contrast with (51). Hence $a \notin C$. Let us observe now that

$$C = M(B \setminus \{a\}) \subseteq M(B) = M(A) = A, \tag{54}$$

because $A \in MAXP(\mathfrak{J})$. By (54) we have then $C \in MAXP(A)$. Finally, since $B \setminus \{a\} \subseteq C$, we have $B \subseteq C \cup \{a\} \subseteq A$, so that

$$\pi(A) \preceq \pi(C \cup \{a\}) \preceq \pi(B),$$

and this implies $\pi(C \cup \{a\}) = \pi(A)$ because $\pi(A) = \pi(B)$. Hence $M(C \cup \{a\}) = M(A) = A$ because $A \in MAXP(\mathfrak{J})$. This proves the first implication.

For the other implication, we assume that there exists $C \in MAXP(A)$ such that $a \notin C$ and $M(C \cup \{a\}) = A$. Then $a \in A$ and we set $B := C \cup \{a\}$. Therefore, $B \in [A]_{\approx}$ and $B \setminus \{a\} = C$. Let us note that $C \notin [A]_{\approx}$. In fact, if $C \approx A$ then $C = A$ because $C \in MAXP(A)$ and $A \in MAXP(\mathfrak{J})$, contradicting $a \in A \setminus C$. Hence, we obtain a function subset $B \in [A]_{\approx}$ such that $B \setminus \{a\} \notin [A]_{\approx}$ and $a \in A$, i.e. $a \in K_{\mathfrak{J}}(A)$.

(ii) Let $a \in \bigcap\{B : B \in [A|\mathbb{M}(\mathfrak{J})\uparrow]\}$ and suppose by contradiction that $a \in K_{\mathfrak{J}}(A)$. By (i), there exists $B \in MAXP(A)$ such that $a \notin B$ and $M(B \cup \{a\}) = A$. We clearly have that $B \notin [A|\mathbb{M}(\mathfrak{J})\uparrow]$. Therefore, there exists $C \in MAXP(A)$ such that $B \subsetneqq C \subsetneqq A$. Hence

$$\pi(A) \preceq \pi(C). \tag{55}$$

We claim that $a \notin C$. In fact, if $a \in C$, we would have $B \cup \{a\} \subseteq C$, hence

$$\pi(C) \preceq \pi(B \cup \{a\}) = \pi(A). \tag{56}$$

Thus, by (55) and (56), $\pi(A) = \pi(C)$ and, so, $C = A$, contradicting our assumption on C. Proceeding in this way, we will find a function subset $D \in [A|\mathbb{M}(\mathcal{I}) \uparrow]$ not containing a, which is absurd. Thus, $a \in K_{\mathcal{I}}^c(A)$.

Conversely, let $a \in K_{\mathcal{I}}^c(A)$. Suppose by contradiction that there exists $B \in [A|\mathbb{M}(\mathcal{I}) \uparrow]$ such that $a \notin B$. Hence $C := B \cup \{a\} \approx A$, so we have found an element $C \in [A]_{\approx}$ such that $C \setminus \{a\} \not\approx A$, i.e. $a \in K_{\mathcal{I}}(A)$, which is absurd.

(iii) Let $a \in CORE(A)$. Hence $\pi(A) \neq \pi(A \setminus \{a\})$, i.e. $a \in K_{\mathcal{I}}(A)$.

(iv) Let $a \in K_{\mathcal{I}}(A)$. Hence, for any $B \approx A$, it results that $B \setminus \{a\} \approx A$. Therefore, $a \in K_{\mathcal{I}}(A')$.

(v) Since $A \in MINP(\mathcal{I})$, we have that $\pi(A) \neq \pi(A \setminus \{a\})$ for any $a \in A$, hence $K_{\mathcal{I}}(A) = A$.

(vi) Set $A' = K_{\mathcal{I}}(M(A)))$. We have that $A' \subseteq M(A)$. Let $B \in \min([A]_{\approx})$, then $B \subseteq M(A)$; hence, by part (iv), we have that $K_{\mathcal{I}}(B) \subseteq A'$. Since $B \in MINP(\mathcal{I})$, by part (v), $K_{\mathcal{I}}(B) = B$, so $B \subseteq A'$. We conclude that

$$B \subseteq A' \subseteq M(A).$$

Therefore, we have

$$\pi(M(A)) = \pi(A) \preceq \pi(A') \preceq \pi(B) = \pi(A),$$

so $\pi(A) = \pi(A')$ and $A \approx A' = K_{\mathcal{I}}(M(A))$.

(vii) It follows directly by part (vi).

$(viii)$ It follows by part (iii) and by (37). □

In the next result, we provide a first basic link between relative reducts and $MINP(\mathcal{I})$.

Proposition 6.8. *Let $A \in \mathcal{P}(\Omega)$. Then*

$$RED(A) = \{X : X \in \min([A]_{\approx}),\ X \subseteq A\}. \tag{57}$$

Proof. Let $C \in RED(A)$. By (i) of Definition 5.3 we have then $\pi(C) = \pi(A)$, therefore $C \in [A]_{\approx}$ and, by (ii) of the same definition, there is no $D \in [A]_{\approx}$ is such that $D \subsetneq C$. Hence C is necessarily minimal in the subset family $[A]_{\approx}$. Hence $RED(A) \subseteq \{X : X \in \min([A]_{\approx}),\ X \subseteq A\}$.

Conversely, let $C \in \min([A]_{\approx})$. Then $\pi(C) = \pi(A)$. Let now $C' \subsetneq C$. Since C is minimal in $[A]_{\approx}$, it follows that $C' \notin [A]_{\approx}$, so that $\pi(C') \neq \pi(A)$. This shows that C satisfies both the conditions of Definition 5.3, i.e. $C \in RED(A)$. □

For the maximum partitioners, we have the following stronger results.

Theorem 6.9. *Let $A \in MAXP(\mathcal{I})$. Then $RED(A) = \min([A]_{\approx})$.*

Proof. Since A is a maximum partitioner, the subset family $\{X : X \in \min([A]_{\approx}),\ X \subseteq A\}$ in (57) coincides with $\min([A]_{\approx})$. The thesis follows then by Proposition 6.8. □

Corollary 6.10. $RED(\mathcal{I}) = \min([\Omega]_{\approx})$.

It is now immediate to see that

$$MINP(\mathcal{I}) := \bigcup_{A \in MAXP(\mathcal{I})} \min([A]_{\approx}) = \bigcup_{A \in MAXP(\mathcal{I})} RED(A). \tag{58}$$

Moreover, we also set

$$MINP^c(\mathcal{I}) := \mathcal{P}(\Omega) \setminus MINP(\mathcal{I}), \tag{59}$$

$$MXMN(\mathcal{I}) := \mathrm{Max}(MINP(\mathcal{I})), \tag{60}$$

$$MNMN^c(\mathcal{I}) := \min(MINP^c(\mathcal{I})). \tag{61}$$

We call any member of $MINP(\mathcal{I})$ a *minimal partitioner* of $\mathcal{I}$ and the hypergraph $\mathfrak{m}(\mathcal{I}) := (\Omega, MINP(\mathcal{I}))$ the *minimal partitioner hypergraph* of $\mathcal{I}$.

Definition 6.11. We call the hypergraphs $\mathfrak{M}(\mathcal{I})$ and $\mathfrak{m}(\mathcal{I})$ the *indistinguishability hypergraphic structures* of $\mathcal{I}$.

Example 6.12. Again in reference to the finite function system $\mathcal{J}$ introduced in Example 6.3, it results that (we write the members of $MINP(\mathcal{J})$ and $MINP^c(\mathcal{J})$ as strings)

$$MINP(\mathcal{J}) = \{\emptyset, 1, 2, 4, 5, 12, 14, 15, 24, 25, 124, 125\}, \tag{62}$$

$$MINP^c(\mathcal{J}) = \{3, 13, 23, 34, 35, 45, 123, 134, 135, 234, 245, 345, 1234, 1235, 1245, 2345, 12345\}, \tag{63}$$

so that $MXMN(\mathcal{J}) = \{124, 125\}$ and $MNMN^c(\mathcal{J}) = \{3\}$.

Let us note that for $\mathcal{J}$ we also have $RED(\mathcal{J}) = \{124, 125\}$.

In general, the reducts do not coincide with the max-min partitioners, although any reduct is always a max-min partitioner, as the next result shows.

Proposition 6.13. *Let $MAXP(\mathfrak{I}) = \{C_1, \dots, C_k\}$. The following conditions hold:*

(i) Let $C_i \subsetneqq C_j$. Then, if $X \in [C_i]_\approx$ and $Y \in [C_j]_\approx$ we have $Y \nsubseteq X$.
(ii) $RED(\mathfrak{I}) \subseteq MXMN(\mathfrak{I})$.

Proof. (i) Since C_i and C_j are two distinct maximum partitioners such that $C_i \subsetneqq C_j$, we have $\pi(C_j) \prec \pi(C_i)$. Now, by absurd, let $X \in [C_i]_\approx$ and $Y \in [C_j]_\approx$ such that $Y \subseteq X$. Then

$$\pi(C_i) = \pi(X) \preceq \pi(Y) = \pi(C_j),$$

which is a contradiction.

(ii) It is a direct consequence of the previous (i), because $RED(\mathfrak{I}) = \min([\Omega]_\approx)$, $\Omega \in MAXP(\mathfrak{I})$ and $C_i \subsetneqq \Omega$ for any maximum partitioner $C_i \neq \Omega$. $\square$

In the following result, we establish a deepest link between minimal partitioners and reducts.

Theorem 6.14. *The following conditions hold:*

(i) Let $A \subseteq \Omega$ and $B \in RED(A)$ and $C \subseteq B$. Then $\pi(C \setminus \{x\}) \neq \pi(C)$ for any $x \in C$.
(ii) We have that

$$MINP(\mathfrak{I}) \supseteq \bigcup_{B \in RED(A)} \mathcal{P}(B), \ \forall A \subseteq \Omega. \tag{64}$$

Proof. (i) Let us suppose by absurd that there exists an element $x \in C$ such that $\pi(C \setminus \{x\}) = \pi(C)$. Then, since $C \setminus \{x\} \subseteq B \setminus \{x\}$, we have

$$\pi(B \setminus \{x\}) \preceq \pi(C \setminus \{x\}) = \pi(C). \tag{65}$$

Therefore, if $u, u' \in U$ by (65) it follows that

$$u \equiv_{B \setminus \{x\}} u' \Longrightarrow u \equiv_C u' \Longrightarrow F(u, x) = F(u', x) \Longrightarrow u \equiv_B u',$$

therefore

$$\pi(B \setminus \{x\}) \preceq \pi(B). \tag{66}$$

On the other hand, since we also have $\pi(B) \preceq \pi(B \setminus \{x\})$, by (66) we deduce that $\pi(B \setminus \{x\}) = \pi(B)$, which is in contrast with the hypothesis that $B \in RED(A)$. This concludes the proof of part (i).

(ii): Let $C \in \bigcup_{B \in RED(A)} \mathcal{P}(B)$. Then there is some reduct $B \in RED(A)$ such that $C \subseteq B$. Let $j \in \{1, \dots, k\}$ such that $C \in [C_j]_\approx$. Let us assume, by absurd, that $C \notin \min([C_j]_\approx)$. Then there exists some $C' \in [C_j]_\approx$ such that $C' \subsetneqq C$. Let $x \in C \setminus C'$. Then

$$C' \subseteq C \setminus \{x\} \subseteq C,$$

and therefore

$$\pi(C) \preceq \pi(C \setminus \{x\}) \preceq \pi(C'). \tag{67}$$

Since $C, C' \in [C_j]_\approx$, we have $\pi(C) = \pi(C')$; therefore, by (67), we deduce that

$$\pi(C) = \pi(C \setminus \{x\}). \tag{68}$$

But the identity in (68) is in contrast with part (i), because $C \subseteq B$ and $B \in RED(A)$. Hence, $C \in \min([C_j]_\approx) \subseteq MINP(\mathfrak{I})$. This concludes the proof of part (ii). $\square$

Corollary 6.15. *We have that:*

$$MINP(\mathcal{I}) \supseteq \bigcup_{A \in RED(\mathcal{I})} \mathcal{P}(A) = \{X \in \mathcal{P}(\Omega) : X \subseteq A,\ A \in RED(\mathcal{I})\}. \tag{69}$$

In the following result, we show that $MINP(\mathcal{I})$ is an abstract simplicial complex.

Theorem 6.16. *If $C \in MINP(\mathcal{I})$ and $K \subseteq C$, then $K \in MINP(\mathcal{I})$.*

Proof. Let $C \in MINP(\mathcal{I})$. Then there exists $B \in MAXP(\mathcal{I})$ such that $C \in \min([B]_{\approx})$. Let $K \subsetneqq C$, then there exists $B' \in MAXP(\mathcal{I})$ such that $K \in [B']_{\approx}$. Suppose by contradiction that $K \notin \min([B']_{\approx})$, then there exists $K' \subsetneqq K$ such that $K' \in \min([B']_{\approx})$. Hence, there is $x \in K \setminus K'$. We deduce that

$$K' \subseteq K \setminus \{x\} \subseteq K$$

i.e.

$$\pi(K) \preceq \pi(K \setminus \{x\}) \preceq \pi(K').$$

But since $\pi(K) = \pi(K') = \pi(B)$, we conclude that $\pi(K) = \pi(K \setminus \{x\})$, contradicting (i) of Theorem 6.14. Hence $K \in MINP(\mathcal{I})$. □

If $A \in \mathcal{P}(\Omega)$ we set

$$MINP(A) := \{X : X \subseteq A,\ X \in MINP(\mathcal{I})\} \tag{70}$$

and

$$MXMN(A) := \mathrm{Max}(MINP(A)). \tag{71}$$

In particular, we have $MINP(\Omega) = MINP(\mathcal{I})$ and $MXMN(\Omega) = MXMN(\mathcal{I})$.

In the next proposition we establish the basic links between relative reducts and relative minimum partitioners.

Proposition 6.17. *Let $A \in \mathcal{P}(\Omega)$ and $W \in \mathcal{P}(U)$. Then:*

$$RED(A) \subseteq MXMN(A). \tag{72}$$

Proof. Let $X \in RED(A)$, then $X \subseteq A$ and $X \in \min([A]_{\approx}) \subseteq MINP(\mathcal{I})$ by Proposition 6.8. This proves that $X \in MINP(A)$. Suppose by contradiction that there exists $Y \in MINP(A)$ such that $X \subsetneqq Y$. Then $\pi(Y) \preceq \pi(X) = \pi(A)$. Moreover, we have that $Y \in MINP(\mathcal{I})$, hence there exists $B \in MAXP(\mathcal{I})$ such that $Y \in \min([B]_{\approx})$. Thus $\pi(Y) = \pi(B)$ and, since $Y \subseteq A$, it results that $\pi(A) \preceq \pi(Y) = \pi(B)$. Hence $\pi(A) = \pi(B)$, i.e. $A = B$. Therefore, $Y \in \min([A]_{\approx})$ and, by (57), we conclude that $Y \in RED(A)$, absurd since it contains X. So $X \in MXMN(A)$. □

By means of Proposition 6.17, we obtain the following characterization of $MINP(\mathcal{I})$.

Theorem 6.18. *We have that*

$$MINP(\mathcal{I}) = \{A \in \mathcal{P}(\Omega) : RED(A) = \{A\}\} \tag{73}$$

Proof. Let $A \in MINP(\mathcal{I})$, then $A \in \min([A]_{\approx})$. Moreover, the unique subset satisfying both conditions of Definition 5.3 is exactly A, i.e. $RED(A) = \{A\}$.

Vice versa, suppose that $RED(A) = \{A\}$. Then, by (72), we have that $A \in MXMN(A)$, so by (70) we have that $A \in MINP(\mathcal{I})$. □

Remark 6.19. It is clear by Definition 5.3 that the identity in (73) can be restated equivalently as follows:

$$MINP(\mathcal{I}) = \{A \in \mathcal{P}(\Omega) : \forall B \subsetneqq A\ \ \pi(A) \prec \pi(B)\}. \tag{74}$$

7. Some properties of matroidal type for $MINP(\mathcal{I})$

In this section, we discuss some basic cases in which the hypergraph $\mathfrak{m}(\mathcal{I})$ is a matroid. To this purpose, we introduce the following numerical function on $\mathcal{P}(\Omega)$. For any $A \in \mathcal{P}(\Omega)$, we set

$$\psi(A) := \max\{|X| : X \in MINP(A)\} \tag{75}$$

so that we obtain the map

$$\psi : A \in \mathcal{P}(\Omega) \mapsto \psi(A) \in \mathbb{N} \cup \{0\}.$$

We call ψ the *max-min function* of $\mathcal{I}$.

Remark 7.1. The function ψ satisfies the following properties:

(i) $\psi(\emptyset) = 0$;
(ii) If $A \subseteq B$, then $\psi(A) \le \psi(B)$;
(iii) $0 \le \psi(A) \le |A|$ for any $A \in \mathcal{P}(\Omega)$;
(iv) $\psi(A) \le \psi(A \cup \{x\}) \le \psi(A) + 1$.

Definition 7.2. Let $C \in \mathcal{P}(\Omega)$. We say that C is a ψ*-incremental subset* if

$$\psi(C \cup \{x\}) := \psi(C) + 1,$$

for all $x \in \Omega \setminus C$.

We denote by $INCR(\psi)$ the family of all ψ-incremental subsets. Moreover, if $A \in \mathcal{P}(\Omega)$, we denote by $INCR(A|\psi)$ the family of all ψ-incremental subsets containing A.

In the next result, we give three new different characterizations for $MINP(\mathcal{I})$.

Theorem 7.3. *The following conditions hold:*

(i) $MINP(\mathcal{I}) = \{A \in \mathcal{P}(\Omega) : C \nsubseteq A \ \forall C \in MNMN^c(\mathcal{I})\}$;
(ii) $MINP(\mathcal{I}) = \{A \in \mathcal{P}(\Omega) : \psi(A) = |A|\}$;
(iii) $MINP(\mathcal{I}) = \{A \in \mathcal{P}(\Omega) : a \notin M(A \setminus \{a\}) \ \forall a \in A\}$.

Proof. (i): Let $A \in MINP(\mathcal{I})$. Then, by Theorem 6.16, any subset of A belongs to $MINP(\mathcal{I})$ hence A cannot contain any element of $MNMN^c(\mathcal{I})$. So $MINP(\mathcal{I}) \subseteq \{A \in \mathcal{P}(\Omega) : C \nsubseteq A \ \forall C \in MNMN^c(\mathcal{I})\}$. On the other hand, let $A \in \mathcal{P}(\Omega)$ such that $C \nsubseteq A$ for any $C \in MNMN^c(\mathcal{I})$. Hence $A \notin MINP^c(\mathcal{I})$, otherwise or $A \in MNMN^c(\mathcal{I})$ or it contains a function subset $C \in MNMN^c(\mathcal{I})$. In both cases, we are contradicting our assumption on A, hence $A \in MINP(\mathcal{I})$ and (i) has been shown.

(ii): Let $A \in MINP(\mathcal{I})$. Since $\psi(A) = \max\{|X| : X \subseteq A, X \in MINP(\mathcal{I})\}$, it is clear that $\psi(A) = |A|$, thus $MINP(\mathcal{I}) \subseteq \{A \in \mathcal{P}(\Omega) : \psi(A) = |A|\}$. Conversely, let $A \in \mathcal{P}(\Omega)$ such that $\psi(A) = |A|$ and suppose by contradiction that $A \in MINP^c(\mathcal{I})$. This is equivalent to require the existence of $B \in MINP(\mathcal{I})$ such that $B \subsetneqq A$ and $\pi(B) = \pi(A)$. Hence, we should have $|B| = \psi(A) = |A|$, contradicting our assumption on B. So (ii) has been shown.

(iii): Let $A \in MINP(\mathcal{I})$ and $a \in A$. Then $A \setminus \{a\} \notin [A]_{\approx}$, so $M(A \setminus \{a\}) \notin [A]_{\approx}$. Since $A \setminus \{a\} \subsetneqq A$, we have $\pi(A) \preceq \pi(A \setminus \{a\})$. Moreover, suppose by contradiction that $a \in M(A \setminus \{a\})$. Then $A \subseteq M(A \setminus \{a\})$ and, by (20), $M(A) \subseteq M(A \setminus \{a\})$. By (19), we have $\pi(A \setminus \{a\}) \preceq \pi(A)$, so $\pi(A \setminus \{a\}) = \pi(A)$ and $A \setminus \{a\} \in [A]_{\approx}$, which is absurd. So $a \notin M(A \setminus \{a\})$ and $MINP(\mathcal{I}) \subseteq \{A \in \mathcal{P}(\Omega) : a \notin M(A \setminus \{a\}) \ \forall a \in A\}$. On the other hand, let $A \in \mathcal{P}(\Omega)$ such that $a \notin M(A \setminus \{a\})$ for any $a \in A$. Suppose by contradiction that $A \in MINP^c(\mathcal{I})$, i.e. there exists $B \subsetneqq A$ such that $\pi(A) = \pi(B)$. Then there exists $a \in A$ such that $B \subseteq A \setminus \{a\} \subseteq A$. This implies that $\pi(A) \preceq \pi(A \setminus \{a\}) \preceq \pi(B)$, i.e. $\pi(A \setminus \{a\}) = \pi(A)$. In other terms, we have $M(A \setminus \{a\}) = M(A) \supseteq A$, which is a contradiction. So (iii) follows. □

Corollary 7.4. *Let $A \in MINP(\mathcal{I})$. Then $C(A) = \emptyset$.*

Proof. Let $A \in MINP(\mathcal{I})$. By (iii) of Theorem 7.3, we have that $a \notin M(A \setminus \{a\})$ for any $a \in A$. Hence, by (iii) of Theorem 5.2, it follows that $CORE(A) = A$. But Theorem 6.16 asserts that any subset of A belongs to $MINP(\mathcal{I})$; therefore, we conclude that $CORE(B) = B$ for any $B \subseteq A$. So $C(A) = \emptyset$. □

At this point, we introduce a new class of function subsets that will be useful in the sequel.

Definition 7.5. Let $A \in \mathcal{P}(\Omega)$. We say that

- A is *reduct uniform* if $RED(A)$ has uniform cardinality.

	1	2	3	4	5
u_1	0	0	1	2	1
u_2	1	0	2	2	0
u_3	2	1	0	1	1
u_4	1	1	2	0	2
u_5	1	0	0	0	2
u_6	0	1	2	2	0

Fig. 4. The function system $\mathcal{I}$.

- A is *maxp-reduct uniform* if $RED(M(A))$ has uniform cardinality.

Example 7.6. Let $A = \{1, 2, 3, 4\}$. We observe that $RED(A) = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}\}$, hence A is reduct uniform. Nevertheless, as we can observe in Fig. 5, $M(A) = \Omega$ but $RED(\mathcal{I}) = RED(\Omega)$ has not uniform cardinality, hence A is not maxp-reduct uniform.

We introduce now an operator $\hat{\psi} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ as follows.
If $A \in \mathcal{P}(\Omega)$ and $c \in \Omega$ we set

$$c \vdash_{\psi} A :\Longleftrightarrow \psi(A \cup \{c\}) = \psi(A), \tag{76}$$

and

$$\hat{\psi}(A) := \{x \in \Omega : x \vdash_{\psi} A\}. \tag{77}$$

Remark 7.7. If $\mathfrak{m}(\mathcal{I})$ is a matroid, then $\hat{\psi}$ coincides with the closure operator of $\mathfrak{m}(\mathcal{I})$.

Example 7.8. By referring to the function system $\mathcal{J}$ of the Example 6.3, it is easy to verify that we have $\hat{\psi}(A) = M(A)$ for any function subset A. We also observe that in this case $\mathfrak{m}(\mathcal{I})$ is a matroid.

However, the identity $\hat{\psi}(A) = M(A)$ does not hold in general, as we can see in the next example.

Example 7.9. Let $\mathcal{I}$ be the function system given in Fig. 4 and let $A = \{1, 2\}$. It is immediate to see that $\psi(\{1, 2\}) = \psi(\{1, 2, 5\}) = 2$ and that $5 \notin M(\{1, 2\}) = \{1, 2\}$. Hence, $M(A) \neq \hat{\psi}(A)$.

It is simple to find examples of function systems such that $\mathfrak{m}(\mathcal{I})$ is a matroid, but $\hat{\psi} \neq M$.

Example 7.10. Let us consider the function system $\mathcal{T}$ whose functional table is given in Fig. 6.

It is easy to see that $\mathfrak{m}(\mathcal{T})$ is a matroid. In fact, for the empty set, Property $(M3)$ holds trivially. Moreover, if we take A, B such that $|A| = 1$ and $|B| = 2$, two cases occur: if $A \subsetneq B$, $(M3)$ is trivially verified; otherwise, we clearly have that $A \cap B = \emptyset$. Let us consider, for example $A = \{a_1\}$ and $B = \{a_3, a_4\}$. Since both $\{a_1, a_3\}$ and $\{a_1, a_4\}$ belong to $MINP(\mathcal{T})$, we conclude that $(M3)$ is satisfied for these two particular sets. A similar argument holds for any pair of subsets as A and B. Therefore, $\mathfrak{m}(\mathcal{T})$ is a matroid. Nevertheless, it is easy to see that, we have that $\hat{\psi}(\{a_1, a_4\}) = \{a_1, a_2, a_3, a_4\} \neq M(\{a_1, a_4\}) = \{a_1, a_4\}$.

The next result is a consequence of (71) and a well-known basic theorem of matroid theory (see [54]).

Proposition 7.11. *$\mathfrak{m}(\mathcal{I})$ is a matroid if and only if $MXMN(A)$ has uniform cardinality for any $A \subseteq \Omega$.*

We can establish now two basic links between some properties of the reducts and the matroidal properties of the hypergraph $\mathfrak{m}(\mathcal{I})$.

Corollary 7.12. *If $\mathfrak{m}(\mathcal{I})$ is a matroid, then A is reduct uniform for any $A \subseteq \Omega$.*

Corollary 7.13. *If $A, B \in RED(\mathcal{I})$ are such that $|A| \neq |B|$, then $\mathfrak{m}(\mathcal{I})$ is not a matroid.*

We can easily provide a sufficient condition in order to ensure that $\mathfrak{m}(\mathcal{I})$ is a matroid.

Proposition 7.14. *If M is a MLS closure operator on Ω then $\mathfrak{m}(\mathcal{I})$ is a matroid.*

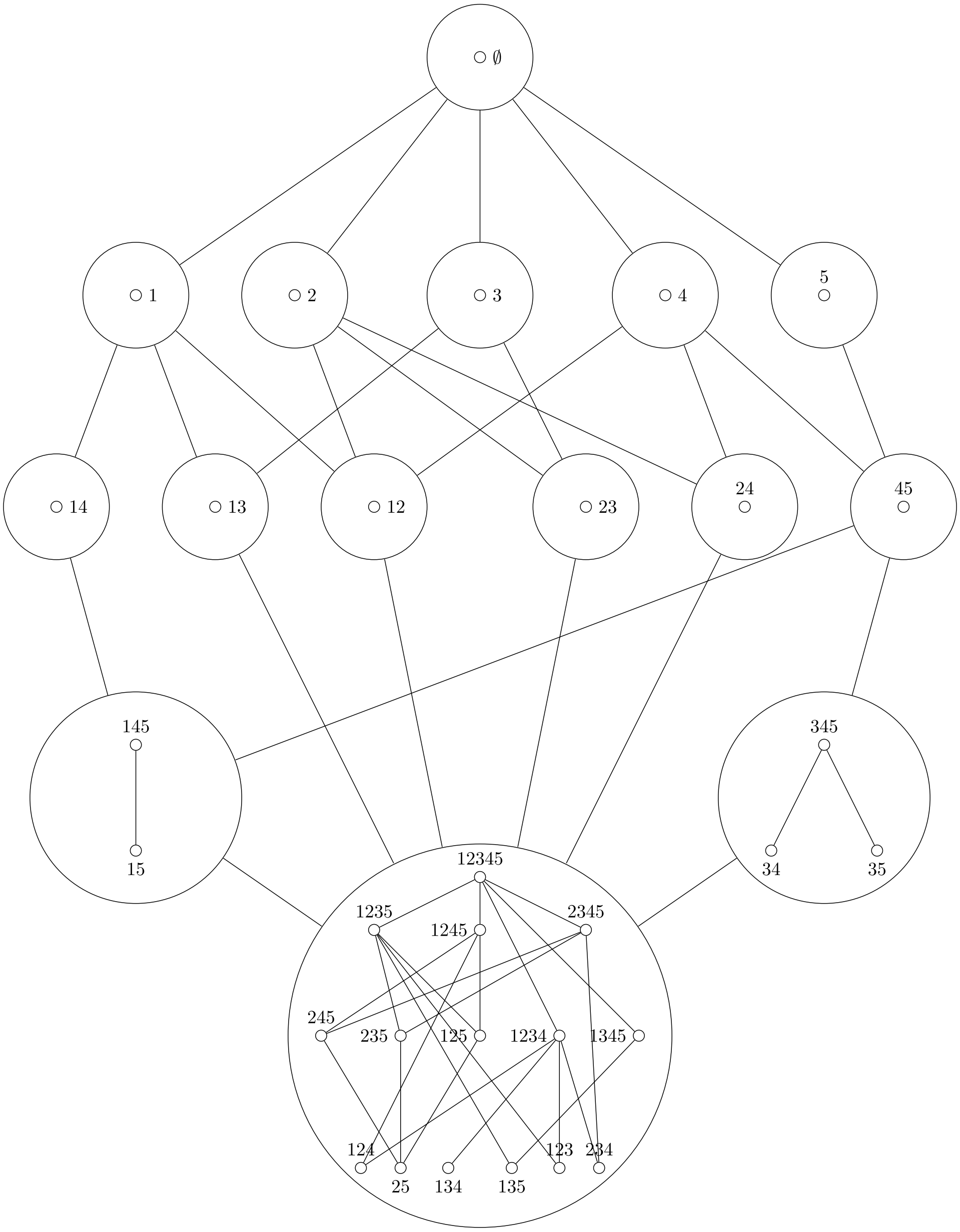

Fig. 5. The lattice $\mathbb{I}(\mathcal{I})$.

$\mathcal{T}$	a_1	a_2	a_4	a_5
u_1	0	0	0	0
u_2	1	1	1	1
u_3	0	0	0	2
u_4	1	2	2	1
u_5	2	1	1	2

Fig. 6. The functional table $\mathcal{T}$.

Proof. Since M is a MLS closure operator on Ω, the family

$$\mathcal{F} := \{A \in \mathcal{P}(\Omega) : a \notin M(A \setminus \{a\}) \ \forall a \in A\}$$

coincides with the independent set family of a matroid on Ω by Theorem 2.8. By virtue of (iii) of Theorem 7.3, we deduce that $\mathcal{F} = MINP(\mathcal{I})$, hence the claim has been showed. $\square$

We can now characterize the property of $\mathfrak{m}(\mathcal{I})$ of being a matroid in terms of MLS closure operators.

Corollary 7.15. *Let $\mathcal{I}$ be a function system such that $\hat{\psi} = M$. Then $\mathfrak{m}(\mathcal{I})$ is a matroid if and only if M is a MLS closure operator.*

Proof. Suppose that $\mathfrak{m}(\mathcal{I})$ is a matroid. By Remark 7.7, $\hat{\psi}$ is the closure operator associated with $\mathfrak{m}(\mathcal{I})$, hence it is a MLS closure operator by Theorem 2.8. But, since $\hat{\psi} = M$, we deduce that M is a MLS closure operator. The converse is given in Proposition 7.14. $\square$

However, the particular cases described in Example 7.6 and in Example 7.8 induce us to investigate in more detail the link between the dependency operator and the maximum partitioner operator. In the next definition, we introduce a first class of function systems for which the identity $\hat{\psi}(A) = M(A)$ holds for any $A \subseteq \Omega$.

Definition 7.16. We say that a function system $\mathcal{I}$ is *maxp-reduct uniform* if the following two conditions are satisfied:

(i) any $A \in \mathcal{P}(\Omega)$ is maxp-reduct uniform;
(ii) if $A, B \in \mathcal{P}(\Omega)$, $A \subsetneqq B$ and $A \not\approx B$, then $||RED(M(A))|| < ||RED(M(B))||$.

Example 7.17. It is easy to verify that the function system $\mathcal{J}$ of Example 6.3 is maxp-reduct uniform.

Theorem 7.18. *Let $\mathcal{I}$ be a maxp-reduct uniform function system. Then:*

(i) *$\hat{\psi}(A) = M(A)$ for any $A \in \mathcal{P}(\Omega)$;*
(ii) *$MAXP(\mathcal{I}) = INCR(\psi)$.*

Proof. (i) Let $A \subseteq \Omega$ and $a \in \hat{\psi}(A)$. Then $\psi(A) = \psi(A \cup \{a\})$. Suppose by contradiction that $a \notin M(A)$, then $A \cup \{a\} \notin [A]_{\approx}$. By (ii) of Definition 7.16, we have that $||RED(A)|| < ||RED(A \cup \{a\})||$, so $\psi(A \cup \{a\}) > \psi(A)$, which is absurd. On the other hand, let $a \in M(A)$. Then, by (ii) of Remark 7.1, we have that

$$\psi(A) \le \psi(A \cup \{a\}) \le \psi(M(A)).$$

But, since $||RED(A)|| = ||RED(M(A))||$ and $A \cup \{a\} \in [A]_{\approx}$, we have that $\psi(A) = \psi(A \cup \{a\})$, so $a \in \hat{\psi}(A)$.

(ii) Let $A \in MAXP(\mathcal{I})$, hence by (ii) of Definition 7.16, for any $a \in \Omega \setminus A$, we have that $\psi(A \cup \{a\}) = \psi(A) + 1$, so $A \in INCR(\psi)$. Conversely, let $A \in INCR(\psi)$. Then $\psi(A \cup \{a\}) = \psi(A) + 1$ for any $a \in \Omega \setminus A$, hence $a \notin \hat{\psi}(A) = M(A)$. Thus $A \in MAXP(\mathcal{I})$ and the thesis follows. $\square$

Hence, by Corollary 7.15 and Theorem 7.18 we obtain an immediate consequence.

Corollary 7.19. *If $\mathcal{I}$ is a maxp-reduct uniform function system then $\mathfrak{m}(\mathcal{I})$ is a matroid.*

8. Graphs as Boolean function systems

In this section, we deal exclusively with finite undirected simple graphs and we refer the reader to [31] for any general notion concerning graph theory. Here we recall only some basic definitions, and we fix some notations that we will use in the sequel. We always denote by $G = (V(G), E(G))$ a finite simple (i.e. no loops and no multiple edges are allowed) undirected graph, with vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$. In this case, we also use the term *n-graph*. If $v, v' \in V(G)$, we will write $v \sim v'$ if $\{v, v'\} \in E(G)$ and $v \nsim v'$ otherwise.

Definition 8.1. Let $v \in V(G)$. We call the set $N_G(v) := \{w \in V(G) : v \sim w\}$ the *neighborhood* of v in G. In particular, if $A \subseteq V(G)$, we call the set

$$N_G(A) := \bigcup_{v \in A} N_G(v) \tag{78}$$

neighborhood of A in G.

We now associate a function system with any n-graph G.

Definition 8.2. We call the Boolean function system

$$\mathfrak{I}[G] := \langle V(G), V(G), \{0,1\}, F_G \rangle,$$

where

$$F_G(u,v) := \begin{cases} 1 & \text{if } u \sim v \\ 0 & \text{otherwise} \end{cases}$$

adjacency function system of the n-graph G.

Hereafter, we write G instead of $\mathfrak{I}[G]$.

Let $A \subseteq V(G)$ a vertex subset. We can define the A-indiscernibility relation $\equiv_A$ for the function system of the graph G as follows:

$$v \equiv_A v' :\Longleftrightarrow F_G(v,a) = F_G(v',a), \forall a \in A. \tag{79}$$

We denote by $\pi_G(A)$ the set partition of $V(G)$ induced from the equivalence relation $\equiv_A$. If $v \in V(G)$, we denote by $[v]_A$ the equivalence class of the vertex v with respect to $\equiv_A$.

For the graphs, we have the following interpretation of the indiscernibility relation.

Proposition 8.3. *If $A \subseteq V(G)$ and $v, v' \in V(G)$, then*

$$v \equiv_A v' :\Longleftrightarrow N_G(v) \cap A = N_G(v') \cap A. \tag{80}$$

Proof. Straightforward. □

We recall that two vertices $v_i, v_j \in V(G)$ are called *twin* if $N_G(v_i) = N_G(v_j)$ and a graph without twin vertices is called a *twin-free graph*.

By means of this classical terminology, we can establish the following immediate characterization for the graphs G such that $\pi_G(V(G)) = v_1|\ldots|v_n$.

Proposition 8.4. *Let G be a n-graph. Then $\pi_G(V(G)) = v_1|\ldots|v_n$ if and only if G has at most a single isolated vertices and it is twin-free.*

Proof. Immediate. □

In what follows, we analyze the case of *complete multipartite graph*.

Definition 8.5. A n-graph $G = (V(G), E(G))$ is said *complete multipartite* if there exist s non-empty subsets $B_1, \ldots, B_s$ of $V(G)$ such that

(i) $|B_i| = r_i$;
(ii) $B_i \cap B_j = \emptyset$ if $i \neq j$;
(iii) $\bigcup_{i=1}^{s} B_i = V(G)$;
(iv) $E(G) = \{\{x,y\} : x \in B_i,\ y \in B_j,\ i \neq j\}$.

We denote a complete multipartite graph by $K_{r_1,\ldots,r_s}$ or $(B_1|\ldots|B_s)$.

In the next result we provide the A-indiscernibility partition of $K_{r_1,\ldots,r_s}$. To this regard, we introduce the following notation: let $G := K_{r_1,\ldots,r_s} = (B_1|\ldots|B_s)$ and $A \subseteq V(G)$, then we set

$$Q_G(A) := \{B_i :\ A \cap B_i \neq \emptyset\}.$$

Proposition 8.6. *Let $G = K_{r_1,\ldots,r_s} = (B_1|\ldots|B_s)$ and $A \subseteq V(G)$. Then:*

$$\pi_G(A) = B_{i_1}|\ldots|B_{i_r}|Q_G(A)^c \tag{81}$$

where $r := |Q_G(A)| = 1, \ldots, s$.

Proof. Let $A \subseteq V(G)$ and let $Q_G(A) = \{B_{i_1}, \dots, B_{i_r}\}$. Let us fix and index i_j, where $1 \le j \le r \le n$. Then any vertex of B_{i_j} is linked to all vertices of $A \setminus B_{i_j}$ while, all vertices of $Q_G(A)^c$ are linked to any vertex of A. Therefore we conclude that

$$\pi_G(A) = B_{i_1}|\dots|B_{i_r}|Q_G(A)^c. \qquad \square$$

We can therefore express $MAXP(K_{r_1,\dots,r_s})$. We first introduce the following notation: if $n \ge k$ we set

$$\mathcal{V}(n,k) := \{K \subseteq [n] : |K| \le n-k\} \cup \{[n]\}. \tag{82}$$

Proposition 8.7. *Let $G := K_{r_1,\dots,r_s} = (B_1|\dots|B_s)$. Then*

$$\mathbb{M}(G) \cong (\mathcal{V}(n,2), \subseteq^*). \tag{83}$$

Proof. It follows immediately by (81) that

$$MAXP(K_{r_1,\dots,r_s}) = \{A \subseteq V(G) : A = \bigcup_{j=1}^{|Q_G(A)|} B_{i_j},\ |Q_G(A)| = 1, \dots, n-2\} \cup \{V(G)\}. \tag{84}$$

Therefore, the map $\phi : \mathbb{M}(G) \to \mathcal{V}(n,2)$ that associates with a block B_i the integer i is an order isomorphism between $\mathbb{M}(G)$ and $\mathcal{V}(n,2)$, and the claim has been proved. $\square$

In the next result, we show that $\mathfrak{m}(G)$ is a matroid when G is a complete multipartite graph $K_{r_1,\dots,r_s}$, with $r_i \ge 2$ for any $i = 1, \dots, s$.

Theorem 8.8. *Let $G = K_{r_1,\dots,r_s}$ be a complete multipartite graph where $r_i \ge 2$ for any $i = 1, \dots, s$. Then $\mathfrak{m}(G)$ is a matroid.*

Proof. Firstly, we observe that

$$\min([A]_\approx) = \{B \subseteq V(G) : B = \{v_{i_1}, \dots, v_{i_l}\},\ v_{i_j} \in B_{i_j},\ B_{i_j} \in Q_G(A)\}. \tag{85}$$

We must prove that $\hat{\psi}(A) = M(A)$ for any vertex subset $A \subseteq V(G)$. Fix $A \subseteq V(G)$ and let $B_i \in Q_G(A)$. Let $v \in B_i$, then $A \cup \{v\} \approx A$, so by (85), $\psi(A \cup \{v\}) = \psi(A)$. On the other hand, if $v \in B_j$ for some $B_j \in Q_G(A)^c$, then $A \cup \{v\} \notin [A]_\approx$, so by (85), $\psi(A \cup \{v\}) \ne \psi(A)$. Thus $v \vdash_\psi A$ if and only if $v \in M(A)$. Finally, by Corollary 7.15, we have to show that M is an MLS closure operator. Let $A \subseteq V(G)$. Let $a, b \in V(G) \setminus M(A)$ such that $b \in M(A \cup \{a\}) \setminus M(A)$. Let $M(A) = \{B_{i_1}, \dots, B_{i_r}\}$ and $a \in B_l$. Hence, $b \in B_l$ too. Thus

$$M(A \cup \{a\}) = Q_G(A) \cup B_l = M(A \cup \{b\}),$$

so $a \in M(A \cup \{b\})$, as claimed. $\square$

As an immediate consequence, we show the next result.

Corollary 8.9. *Let $G = K_{r_1,\dots,r_s}$ be a complete multipartite graph where $r_i \ge 2$ for any $i = 1, \dots, s$. Then $RED(K_{r_1,\dots,r_s})$ has uniform cardinality and $||RED(K_{r_1,\dots,r_s})|| = n - 1$.*

Proof. Straightforward. $\square$

8.1. The case of C_n

We study now in detail the case $G = C_n$, i.e. the undirected cycle on n vertices.

Lemma 8.10. *Let $G = C_n$ and let $V = V(G) = \{v_1, \dots, v_n\}$. Fix a subset $A \subseteq V$, $A = \{v_{i_1}, \dots, v_{i_k}\}$, and let $v_i, v_j \in V$, with $i < j$. Then $v_i \equiv_A v_j$ if and only if $N_G(v_i) \cap A = N_G(v_j) \cap A = \emptyset$ or $N_G(v_i) \cap A = N_G(v_j) \cap A = \{v_{i+1}\} = \{v_{j-1}\}$.*

Proof. We firstly observe that, for each $i \in \{1, \dots, n\}$, $N_{C_n}(v_i) = \{v_{i-1}, v_{i+1}\}$, where the index sums are all taken mod(n). The proof follows easily by observing that, since $v_i \ne v_j$, then $|N_G(v_i) \cap N_G(v_j)| \le 1$, and the equality holds if and only if $j = i + 2$. It follows that, if $N_G(v_i) \cap A = N_G(v_j) \cap A$, then $N_G(v_i) \cap A = (N_G(v_i) \cap N_G(v_j)) \cap A \subseteq N_G(v_i) \cap N_G(v_j)$. By (80), $v_i \equiv_A v_j$ if and only if $N_G(v_i) \cap A = N_G(v_j) \cap A$. Thus, $|N_G(v_i) \cap A| = |N_G(v_j) \cap A| \le 1$ and the equality holds if and only if $j = i + 2$ and $v_{i+1} = v_{j-1} \in A$. This proves the thesis. $\square$

We give now a complete description of the indiscernibility partition for C_n.

Proposition 8.11. *Let G and A as in Lemma 8.10. We set $B_G(A) := (N_G(A))^c$ and*

$$C_G(A) := \{v_i \in A : v_{i-2} \notin A \wedge v_{i+2} \notin A\}.$$

Then, if $v_{s_1}, \dots, v_{s_l}$ are the vertices in $V(G) \setminus [B_G(A) \cup N_G(C_G(A))]$ and $v_{j_1}, \dots, v_{j_h}$ are the vertices in $C_G(A)$, we have

$$\pi_G(A) = B_G(A)|v_{j_1-1}v_{j_1+1}|\cdots|v_{j_h-1}v_{j_h+1}|v_{s_1}|\cdots|v_{s_l}.$$

Proof. The proof follows directly by Lemma 8.10. In fact, let $v_i, v_j \in V(G)$, with $i < j$ and $v_i \equiv_A v_j$. By the previous lemma, then either (a) $N_G(v_i) \cap A = N_G(v_j) \cap A = \emptyset$ or (b) $N_G(v_i) \cap A = N_G(v_j) \cap A = v_{i+1} = v_{j-1}$. But (a) is equivalent to say that $v_i, v_j \in B_A$, (b) that $\{v_i, v_j\} = N_G(v)$, for some $v \in C_A$. The proposition is thus proved. □

Example 8.12. Let $n = 7$ and let $G = C_7$ the 7-cycle and the 7-path on the set $V = \{v_1, \dots, v_7\}$. Let $A = \{v_3, v_4, v_7\}$. Then $B_G(A) = \{v_7\}$, $C_G(A) = \{v_3, v_4, v_7\}$ and $V(G) \setminus [B_G(A) \cup N_G(C_G(A))] = \emptyset$. Thus,

$$\pi_G(A) = v_1v_6|v_2v_4|v_3v_5|v_7.$$

In the next results, we provide a characterization of the maximum partitioners and the minimum partitioners of C_n.

Theorem 8.13. *Let $G = C_n$ and $A \subseteq V(G)$. Then $A \in MAXP(G)$ if and only if for any $v_i \notin A$ it holds one of the following conditions:*

(i) *$(v_{i-2} \notin A$ or $v_{i+2} \notin A)$ and $(B_G(A) \neq \{v_{i-1}, v_{i+1}\})$;*
(ii) *$(v_{i-2} \in A$ and $v_{i-4} \notin A)$ or $(v_{i+2} \in A$ and $v_{i+4} \notin A)$.*

Proof. If $A \subseteq V(G)$ is a maximum partitioner, then, for all $v_i \notin A$ there exist $v_j, v_k \in V(G)$ such that $v_j \equiv_A v_k$ and $F_G(v_i, v_j) \neq F_G(v_i, v_k)$, which is equivalent to require that one of the two conditions holds: (1) v_j, v_k are both in $B_G(A)$ and $|\{v_j, v_k\} \cap B_G(A \cup \{a\})| = 1$, or (2) $|k - j| = 2$, $v_{j+1} \in C_G(A)$ and either $v_i = v_{j-1}$ or $v_i = v_{k+1}$. Case (1) is equivalent to say that $v_{i-2} \notin A$ or $v_{i+2} \notin A$ and if both v_{i-2} and v_{i+2} are not in A, there exists another vertex $v_l \in B_G(A) \setminus \{v_{i-1}, v_{i+1}\}$. In this way, we obtain (i) of the Theorem. Condition (2) holds if and only if $v_{j+1} \in C_G(A) \setminus C_G(A \cup \{a\})$. This is equivalent to say that $v_{j+1} \in A$, $v_{j-1} \notin A$, $v_{j+3} = v_{k+1} \notin A$ and $v_i \in \{v_{j-1}, v_{k+1}\}$. This means that $v_{i+2} \in A$ and $v_{i+4} \notin A$ or $v_{i-2} \in A$ and $v_{i-4} \notin A$, which is condition (ii). The theorem is thus proved. □

Theorem 8.14. *Let $G = C_n$ and $A \subseteq V(C_n)$. Then $A \in MINP(G)$ if and only if for any $v_i \in A$ it holds one of the following conditions:*

(i) *$(v_{i-2} \notin A$ or $v_{i+2} \notin A)$ and $(B_G(A) \neq \emptyset)$;*
(ii) *$(v_{i-2} \in A$ and $v_{i-4} \notin A)$ or $(v_{i+2} \in A$ and $v_{i+4} \notin A)$.*

Proof. Let $A \subseteq V(G)$ and suppose there exists $v_i \in A$ such that both (1) and (2) are false. If $B_G(A) = \emptyset$, then

$$B_G(A \setminus \{v_i\}) = \begin{cases} \{v_{i-1}\} & \text{if } v_{i-2} \notin A \text{ and } v_{i+2} \in A, \\ \{v_{i+1}\} & \text{if } v_{i-2} \in A \text{ and } v_{i+2} \notin A, \\ \{v_{i-1}, v_{i+1}\} & \text{if } v_{i-2} \notin A \text{ and } v_{i+2} \notin A, \\ \emptyset & \text{otherwise.} \end{cases}$$

Moreover, since $v_{i-2} \in A \Longrightarrow v_{i-4} \in A$ and $v_{i+2} \in A \Longrightarrow v_{i+4} \in A$, then

$$C_G(A \setminus \{v_i\}) = \begin{cases} C_G(A) \setminus \{v_i\} & \text{if } v_{i-2} \notin A \text{ and } v_{i+2} \notin A, \\ C_G(A) & \text{otherwise.} \end{cases}$$

Thus, we conclude that $A \approx A \setminus \{v_i\}$ and, accordingly, $A \notin MINP(G)$.

On the other hand, let now $A \notin MINP(G)$ and $v_i \in A$ such that $A \approx A \setminus \{v_i\}$. Then there exist $v_j, v_k \in V(G)$ such that $v_j \equiv_{A \setminus \{v_i\}} v_k$ and $v_j \not\equiv_A v_k$. This happens if and only if both the following assertions are false:

(1) $v_j, v_k \in B_G(A \setminus \{v_i\})$, $\{v_j, v_k\} \nsubseteq B_G(A)$ and $B_G(A) \neq \emptyset$;
(2) $v_{i-2} \in C_G(A)$ or $v_{i+2} \in C_G(A)$.

These conditions are equivalent to conditions (i) and (ii) respectively. This proves the theorem. □

We now provide the following characterization for the reducts of C_n.

Theorem 8.15. *Let $A \subseteq V(C_n)$. Then A is a reduct of C_n if and only if $C_{C_n}(A) = \emptyset$ and for all $v_i \in A$ one of the following conditions holds:*

(1) $|B_{C_n}(A)| = 1 \wedge (v_{i-2} \notin A \vee v_{i+2} \notin A)$;
(2) $|B_{C_n}(A)| \leq 1 \wedge ((v_{i-4} \notin A \wedge v_{i-2} \in A) \vee (v_{i+2} \in A \wedge v_{i+4} \notin A))$.

Proof. Let $V = V(C_n)$. By Definition 5.3, it is clear that a vertex subset $A \subseteq V(C_n)$ is a reduct if and only if $|B_{C_n}(A)| \leq 1$ and $C_{C_n}(A) = \emptyset$. We claim that A is minimal with respect to (i) of Definition 5.3 if and only if (1) or (2) holds. Let A be a reduct of C_n and $v_i \in A$ such that $|B_{C_n}(A \setminus \{v_i\})| \geq 2$. If $|B_{C_n}(A)| = \emptyset$, we would have $\{v_{i-2}, v_{i+2}\} \cap A = \emptyset$ or, equivalently, $C_{C_n}(A) \neq \emptyset$, which contradicts our assumptions. Thus $|B_{C_n}(A)| = 1$ and one of the two vertices v_{i-2} or v_{i+2} is not in A, which is condition (1). On the other hand, suppose that A is a reduct of C_n and $v_i \in A$ such that $C_{C_n}(A \setminus \{v_i\}) \neq \emptyset$. Hence $v_{i-2} \in C_{C_n}(A \setminus \{v_i\})$ or $v_{i+2} \in C_{C_n}(A \setminus \{v_i\})$. This is equivalent to condition (2). Conversely, suppose that $A \subseteq V$ is a vertex subset satisfying conditions (1) or (2) and such that $C_{C_n}(A) = \emptyset$. It is obvious that A satisfies (i) of Definition 5.3. Let $v_i \in A$ and let us consider the vertex subset $A \setminus \{v_i\}$. Suppose $B_{C_n}(A) = \{v_j\}$. Hence $j \neq i \pm 1$. If $v_{i-2} \notin A$, then $v_{i-1} \equiv_{A \setminus \{v_i\}} v_j$; similarly if $v_{i+2} \notin A$, then $v_{i+1} \equiv_{A \setminus \{v_i\}} v_j$. On the other hand, let $v_i \in A$ such that $v_{i-4} \notin A \wedge v_{i-2} \in A$. In this case, $v_{i-3} \equiv_{A \setminus \{v_i\}} v_{i-1}$; similarly in the other case. In both cases, $\pi(A \setminus \{v_i\}) \neq \pi(V)$. This completes the proof. □

Acknowledgements

The authors are extremely grateful to an anonymous referee for his help in the improvement of the structure of this paper.

References

[1] G.E. Andrews, Euler's "De Partitio numerorum", Bull. Amer. Math. Soc. 44 (4) (2007) 561–573.
[2] N. Apollonio, M. Caramia, Recognizing Helly edge path tree graphs and their clique graphs, Discrete Appl. Math. 159 (2011) 1166–1175.
[3] N. Apollonio, B. Simeone, Improved approximation of maximum vertex coverage problem on bipartite graphs, SIAM J. Discrete Math. 28 (3) (2014) 1137–1151.
[4] N. Apollonio, B. Simeone, The maximum vertex coverage problem on bipartite graphs, Discrete Appl. Math. 165 (2014) 37–48.
[5] N. Apollonio, M. Caramia, P.G. Franciosa, On the Galois lattice of bipartite distance hereditary graphs, Discrete Appl. Math. 190 (2015) 13–23.
[6] R.A. Bayley, Orthogonal partitions in designed experiments, Des. Codes Cryptogr. 8 (3) (1996) 45–77.
[7] R.A. Bayley, Association Schemes: Designed Experiments, Algebra and Combinatorics, Cambridge University Press, Cambridge, UK, 2004, 387 p.
[8] C. Berge, Hypergraphs: Combinatorics of Finite Sets, Elsevier, Amsterdam, 1984.
[9] L. Biacino, Generated envelopes, J. Math. Anal. Appl. 172 (1993) 179–190.
[10] L. Biacino, G. Gerla, An extension principle for closure operators, J. Math. Anal. Appl. 198 (1996) 1–24.
[11] G. Birkhoff, Lattice Theory, 3rd edition, American Mathematical Society, Providence, Rhode Island, 1967.
[12] C. Bisi, G. Chiaselotti, A class of lattices and boolean functions related to the Manickam–Miklös–Singhi conjecture, Adv. Geom. 13 (1) (2013) 1–27.
[13] C. Bisi, G. Chiaselotti, G. Marino, P.A. Oliverio, A natural extension of the Young partition lattice, Adv. Geom. 15 (3) (2015) 263–280.
[14] C. Bisi, G. Chiaselotti, D. Ciucci, T. Gentile, F. Infusino, Micro and macro models of granular computing induced by the indiscernibility relation, Inf. Sci. 388–389 (2017) 247–273.
[15] C. Bisi, G. Chiaselotti, T. Gentile, P.A. Oliverio, Dominance order on signed partitions, Adv. Geom. 17 (1) (2017) 5–29.
[16] T. Boykett, Orderly algorithm to enumerate central groupoids and their graphs, Acta Math. Sin. Engl. Ser. 23 (2) (2007) 249–264.
[17] T. Boykett, Rectangular groupoids and related structures, Discrete Math. 313 (2013) 1409–1418.
[18] G. Cattaneo, G. Chiaselotti, D. Ciucci, T. Gentile, On the connection of hypergraph theory with formal concept analysis and rough set theory, Inf. Sci. 330 (2016) 342–357.
[19] G. Cattaneo, G. Chiaselotti, P.A. Oliverio, F. Stumbo, A new discrete dynamical system of signed integer partitions, Eur. J. Comb. 55 (2016) 119–143.
[20] B. Chen, M. Yan, Eulerian stratification of polyhedra, Adv. Appl. Math. 21 (1998) 22–57.
[21] B. Chen, M. Yan, The geometric cone relations for simplicial and cubical complexes, Discrete Math. 183 (1998) 39–46.
[22] B. Chen, S.T. Yau, Y.N. Yeh, Graph homotopy and Graham homotopy, Discrete Math. 241 (2001) 153–170.
[23] G. Chiaselotti, T. Gentile, Intersection properties of maximal directed cuts in digraphs, Discrete Math. 340 (2017) 3171–3175.
[24] G. Chiaselotti, G. Infante, G. Marino, New results related to a conjecture of Manickam and Singhi, Eur. J. Comb. 29 (2) (2008) 361–368.
[25] G. Chiaselotti, T. Gentile, P.A. Oliverio, Parallel and sequential dynamics of two discrete models of signed integer partitions, Appl. Math. Comput. 232 (2014) 1249–1261.
[26] G. Chiaselotti, D. Ciucci, T. Gentile, Simple graphs in granular computing, Inf. Sci. 340–341 (2016) 279–304.
[27] G. Chiaselotti, T. Gentile, F. Infusino, Dependency structures for decision tables, Int. J. Approx. Reason. 88 (2017) 333–370.
[28] G. Chiaselotti, T. Gentile, F. Infusino, Knowledge pairing systems in granular computing, Knowl.-Based Syst. 124 (2017) 144–163.
[29] G. Chiaselotti, T. Gentile, F. Infusino, P.A. Oliverio, The adjacency matrix of a graph as a data table. A geometric perspective, Ann. Mat. Pura Appl. 196 (3) (2017) 1073–1112.
[30] G. Chiaselotti, T. Gentile, F. Infusino, P. Oliverio, Local dissymmetry on graphs and related algebraic structures, preprint.
[31] R. Diestel, Graph Theory, 4th edition, Grad. Texts Math., Springer, 2010.
[32] T. Eiter, G. Gottlob, Identifying the minimal transversals of a hypergraph and related problems, SIAM J. Comput. 24 (1995) 1278–1304.
[33] K. Elbassioni, On the complexity of monotone dualization and generating minimal hypergraph transversals, Discrete Appl. Math. 32 (2) (2008) 171–187.
[34] P. Erdös, A. Rényi, Asymmetric graphs, Acta Math. Hung. 14 (3–4) (1963) 295–315.
[35] B. Ganter, R. Wille, Formal Concept Analysis. Mathematical Foundations, Springer-Verlag, 1999.
[36] L. Guo, F. Huang, Q. Lia, G. Zhang, Power contexts and their concept lattices, Discrete Math. 311 (2011) 2049–2063.
[37] A. Gyárfás, J. Lehel, Hypergraph families with bounded edge cover or transversal number, Combinatorica 3 (3–4) (1983) 351–358.
[38] M. Hagen, Lower bounds for three algorithms for transversal hypergraph generation, Discrete Appl. Math. 157 (2009) 1460–1469.
[39] G. Hahn, G. Sabidussi (Eds.), Graph Symmetry. Algebraic Methods and Applications, NATO ASI Ser., vol. 497, Springer, 1997.
[40] P.W. Harley III, Metric and symmetric spaces, Proc. Amer. Math. Soc. 43 (2) (1974) 428–430.
[41] A. Huang, H. Zhao, W. Zhu, Nullity-based matroid of rough sets and its application to attribute reduction, Inf. Sci. 263 (2014) 153–165.
[42] W.J. Keith, A bijective toolkit for signed partitions, Ann. Comb. 15 (2011) 95–117.
[43] A. Kelarev, C.E. Praeger, On transitive Cayley graphs of groups and semigroups, Eur. J. Comb. 24 (1) (2003) 59–72.
[44] A. Kelarev, S.J. Quinn, Directed graphs and combinatorial properties of semigroups, J. Algebra 251 (1) (2002) 16–26.

[45] A. Kelarev, J. Ryan, J. Yearwood, Cayley graphs as classifiers for data mining: the influence of asymmetries, Discrete Math. 309 (2009) 5360–5369.
[46] H.W. Martin, Metrization of symmetric spaces and regular maps, Proc. Amer. Math. Soc. 35 (1972) 269–274.
[47] L. Molnár, Orthogonality preserving transformations on indefinite inner product spaces: generalization of Uhlhorn's version of Wigner's theorem, J. Funct. Anal. 194 (2002) 248–262.
[48] P. Pagliani, M.K. Chakraborty, A Geometry of Approximation. Rough Set Theory: Logic, Algebra and Topology of Conceptual Patterns, Springer, 2008.
[49] B. Poonen, Union-closed families, J. Comb. Theory, Ser. A 59 (1992) 253–268.
[50] C.M. Reidys, Sequential dynamical systems over words, Ann. Comb. 10 (4) (2006) 481–498.
[51] C.M. Reidys, Combinatorics of sequential dynamical systems, Discrete Math. 308 (4) (2008) 514–528.
[52] J. Tanga, K. Shea, F. Min, W. Zhu, A matroidal approach to rough set theory, Theor. Comput. Sci. 471 (2013) 1–11.
[53] P.M. Van den Broek, Symmetry transformations in indefinite metric spaces: a generalization of Wigner's theorem, Physica A 127 (3) (1984) 599–612.
[54] D.J.A. Welsh, Matroid Theory, Academic Press, 1976.
[55] W. Zhu, S. Wang, Rough matroids based on relations, Inf. Sci. 232 (2013) 241–252.