Functional analysis

On a convergence theorem for semigroups of positive integral operators

Sur un théorème de convergence pour les semi-groupes d'opérateurs intégraux positifs

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A R T I C L E   I N F O

A B S T R A C T

We give a new and very short proof of a theorem of Greiner asserting that a positive and contractive C_0-semigroup on an L^p-space is strongly convergent in case it has a strictly positive fixed point and contains an integral operator. Our proof is a streamlined version of a much more general approach to the asymptotic theory of positive semigroups developed recently by the authors. Under the assumptions of Greiner's theorem, this approach becomes particularly elegant and simple. We also give an outlook on several generalisations of this result.

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R É S U M É

Nous présentons une nouvelle preuve très courte d'un théorème de Greiner qui dit qu'un semi-groupe de contractions positives sur un espace L^p converge fortement au cas où il contiendrait un opérateur intégral et posséderait un point fixe positif presque partout. Notre preuve est une version simplifiée d'une approche plus générale de la théorie asymptotique des semi-groupes positifs développée récemment par les auteurs. Dans la situation du théorème de Greiner, cette approche est particulièrement élégante et simple. Finalement, on présente un bref aperçu de plusieurs généralisations de ce résultat.

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Consider a positive and contractive C_0-semigroup $\mathscr{T} := (T_t)_{t \in [0,\infty)}$ on $L^p := L^p(\Omega, \mu)$, where (Ω, μ) is a σ-finite measure space and $p \in [1, \infty)$. By positivity, we mean that $f \geq 0$ implies $T_t f \geq 0$ for all $f \in L^p$ and all times $t \geq 0$. We are interested in studying the behaviour of T_t as $t \to \infty$.

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In applications it frequently occurs that $\mathscr{T}$ consists of so-called *integral operators* (or *kernel operators*). Here, a positive linear operator $T : L^p \to L^p$ is called an integral operators if for a measurable function $k : \Omega \times \Omega \to [0, \infty)$ and for all $f \in L^p$ the following holds: we have $k(\,\cdot\,, y)f(\,\cdot\,) \in L^1(\Omega, \mu)$ for almost all $y \in \Omega$ and $Tf = \int_\Omega k(x, \cdot\,)f(x)\,\mathrm{d}\mu(x)$. If at least one of the operators T_t is an integral operator and if the semigroup $\mathscr{T}$ has a fixed point that is strictly positive almost everywhere, then one automatically obtains strong convergence of T_t as time tends to infinity. This was first observed by Greiner [15, Kor 3.11], and his result reads as follows.

Theorem 1. *Let (Ω, μ) be a σ-finite measure space, let $p \in [1, \infty)$ and let $\mathscr{T} := (T_t)_{t \in [0, \infty)}$ be a positive and contractive C_0-semigroup on $L^p := L^p(\Omega, \mu)$. If $\mathscr{T}$ has a fixed point f_0 that fulfils $f_0 > 0$ almost everywhere and if T_{t_0} is an integral operator for at least one time $t_0 \geq 0$, then T_t converges strongly as $t \to \infty$.*

An application of this result to semigroups generated by elliptic operators on L^1 can, for instance, be found in [2]. Moreover, Theorem 1 can be used to derive a famous result of Doob about the convergence of Markov semigroups on spaces of measures, see [14] and [13, Sec 4]. Related results on ℓ^p-sequence spaces and, more generally, on atomic measure spaces can be found in [7,16,27].

One of the major drawbacks of Theorem 1 is its difficult proof. In fact, Greiner reduced the theorem to a 0–2-law whose proof is itself technically quite involved. Here, we present a proof of Theorem 1 that only uses the classical decomposition theorem by Jacobs, de Leeuw and Glicksberg and the observation that every positive integral operator on L^p maps order intervals to relatively compact sets. Indeed, a rather explicit proof of the latter fact is presented in the Appendix of [12]; however, the fact can also be deduced from abstract Banach lattice theory, see [26, Prop IV.9.8] and [20, Cor 3.7.3]. The methods used in the following proof are taken from a more general approach to the asymptotic theory of positive semigroup representations that was recently developed by the authors in [12]. In the setting discussed here, the arguments from this approach become particularly neat and yield a surprisingly simple proof of Greiner's theorem, so we find it worthwhile to devote the present short note to this special case.

Proof of Theorem 1. We first show that the orbits of the semigroup $\mathscr{T}$ are relatively compact. To this end, let $c > 0$ and consider a vector f in the order interval $[-cf_0, cf_0] := \{g \in L^p : -cf_0 \leq g \leq cf_0\}$. For every $t \geq t_0$, we have

$$T_t f \in T_{t_0} T_{t-t_0}[-cf_0, cf_0] \subseteq T_{t_0}[-cf_0, cf_0].$$

Since the latter set is relatively compact and independent of t, it follows that the orbit of f under $\mathscr{T}$ is relatively compact. Since $f_0 > 0$ almost everywhere, the so-called principal ideal $\bigcup_{c>0}[-cf_0, cf_0]$ is dense in L^p and, as the semigroup is bounded, it follows that the orbit of every vector in L^p is relatively compact [10, Lem V.2.13].

Hence, we can apply the decomposition theorem of Jacobs, de Leeuw and Glicksberg, which is, for instance, described in [17, Sec 2.4], [10, Sec V.2] and [9, Sec 16.3]. This theorem gives us a positive, contractive projection P on L^p that commutes with each operator T_t and which has the following properties: T_t converges strongly to 0 on $\ker P$ as t tends to ∞ and the restriction of $\mathscr{T}$ to the range $F := PE$ of P (which contains every fixed point of $\mathscr{T}$ and which is a sublattice of L^p as P is contractive) can be extended to a positive and contractive C_0-group $(S_t)_{t \in \mathbb{R}}$ on F.

As F is a closed sublattice of L^p, it is itself an L^p-space over some measure space. Let us show that F is actually isometrically lattice isomorphic to $\ell^p(I)$ for some index set I. It is known that this is the case if and only if every order interval in F is compact; this fact follows for instance from [1, Cor 21.13]. So let $f, g \in F$ with $f \leq g$. Then we have

$$[f, g]_F = T_{t_0} S_{-t_0}[f, g]_F \subseteq T_{t_0}[S_{-t_0}f, S_{-t_0}g]_E,$$

where we used the subscripts F and E to distinguish order intervals in the spaces F and E. The set $T_{t_0}[S_{-t_0}f, S_{-t_0}g]_E$ is relatively compact in E, so we conclude that $[f, g]_F$ is compact in F, and hence we have indeed $F \cong \ell^p(I)$.

Let $e_i \in \ell^p(I)$ be a canonical unit vector. Since each operator S_t is an isometric lattice isomorphism, $S_t e_i$ is also a canonical unit vector for each $t \in \mathbb{R}$ and $i \in I$. It now follows from the strong continuity of $(S_t)_{t \in \mathbb{R}}$ that $S_t e_i = e_i$ for all t sufficiently close to 0 and hence for all $t \in \mathbb{R}$ (cf. [27, Prop 2.3] for a more general observation). Thus, each operator S_t is the identity map on F, i.e. each operator T_t operates trivially on F. This proves the assertion. □

We point out that, combining the techniques presented here with results about positive group representations, one can derive considerable generalisations of Theorem 1. This is explained in detail in the author's recent paper [12]; we give a brief summary of those generalisations at the end of this note.

Now, we discuss a version of Theorem 1 that does not require the semigroup to contain an integral operator, but only to dominate a non-trivial integral operator. This result reads as follows.

Theorem 2. *Let (Ω, μ) be a σ-finite measure space, let $p \in [1, \infty)$ and let $\mathscr{T} := (T_t)_{t \in [0, \infty)}$ be a positive and contractive C_0-semigroup on $L^p := L^p(\Omega, \mu)$. Assume that $\mathscr{T}$ has a fixed point f_0 that fulfils $f_0 > 0$ almost everywhere and that the following assumption is fulfilled:*

(∗) *for every fixed point $0 \neq f \geq 0$ of $\mathscr{T}$, there exists a time $t \geq 0$ and a positive integral operator K on L^p such that $T_t \geq K$ and $Kf \neq 0$.*

Then T_t converges strongly as $t \to \infty$.

Note that the assumption (∗) is automatically fulfilled if $\mathscr{T}$ is irreducible, meaning that there exists no $\mathscr{T}$-invariant band in L^p except for 0 and L^p, and if we have in addition $T_{t_0} \geq K \geq 0$ for at least one time $t_0 \geq 0$ and a non-zero integral operator K. This follows since every positive non-zero fixed point of an irreducible semigroup is strictly positive almost everywhere according to [4, Prop C-III-3.5(a)].

For irreducible Markov semigroups on L^1-spaces, Theorem 2 was proved by Pichór and Rudnicki in [22, Thm 1]. This result has applications to various models from mathematical biology, see for instance [25,24,6,18,8,5,19]. Conditions similar to (∗) also occurred in the literature on several occasions, though in a more explicit form. We refer for instance to [23, pp. 308 and 309] and to the introduction of the recent article [21]. A version of Theorem 2 for irreducible semigroups on Banach lattices with order continuous norm was proved by the first author in [11, Thm 4.2].

We only give a sketch of the proof of Theorem 2. For details we refer to [12], where the theorem is shown in a considerably more general setting, but with a more complex and technically more involved proof.

Sketch of the proof of Theorem 2. The set of all integral operators that are regular (i.e. that can be written as the difference of two positive operators) is a band within the regular operators on L^p [20, Section 3.3]. Hence, for each $t \geq 0$, we find a maximal positive integral operator $K_t \leq T_t$ and define $R_t := T_t - K_t \geq 0$. As the product of a positive integral operator with a positive operator is again an integral operator [3, Prop 1.9(e)], it easily follows from the maximality of K_t and the semigroup law for $\mathscr{T}$ that $R_{t+s} \leq R_t R_s$ for all $s, t \geq 0$. Hence, $R_{t+s} f_0 \leq R_t R_s f_0 \leq R_t T_s f_0 = R_t f_0$ for all $s, t \geq 0$, so $(R_t f_0)_{t \geq 0}$ decreases in norm to a vector $0 \leq g \in L^p$. This vector fulfils $T_t g \geq R_t g = \lim_s R_t R_s f_0 \geq \lim_s R_{t+s} f_0 = g$ for each $t \geq 0$. Since each operator T_t is contractive, we conclude that actually $T_t g = R_t g = g$, and hence $K_t g = 0$ for all $t \geq 0$. Our condition (∗) now implies that $g = 0$, so we have shown that $R_t f_0 \searrow 0$ in norm as $t \to \infty$.

Now we can see that the orbit of each vector $f \in [-f_0, f_0]$ is relatively compact. Indeed, let $\varepsilon > 0$ and choose $t_0 \geq 0$ such that $\|R_{t_0} f_0\| < \varepsilon$. For each $t \geq 0$, we obtain

$$T_{t_0+t} f \in K_{t_0}[-f_0, f_0] + [-R_{t_0} f_0, R_{t_0} f_0]$$

and thus the orbit of f under $\mathscr{T}$ is contained in the set

$$\{T_t f : t \in [0, t_0]\} \cup \left(K_{t_0}[-f_0, f_0] + B_\varepsilon(0) \right),$$

where $B_\varepsilon(0)$ denotes the open ball with radius ε around 0. Hence, the orbit of f is totally bounded and thus relatively compact.

Since the principal ideal $\bigcup_{c>0}[-cf_0, cf_0]$ is dense in L^p, we conclude that the orbit of actually every vector $f \in L^p$ under $\mathscr{T}$ is relatively compact [10, Lem V.2.13], so we can apply the Jacobs–de Leeuw–Glicksberg decomposition theorem. Now one proceeds as in the proof of Theorem 1. The only difficulty in this situation is to see that each order interval $[f, g]_F$ in F is compact. To this end, one first observes that

$$[-f_0, f_0]_F \subseteq K_t S_{-t}[-f_0, f_0]_F + R_t S_{-t}[-f_0, f_0]_F$$
$$\subseteq K_t[-f_0, f_0]_E + [-R_t f_0, R_t f_0]_E$$

for each $t \geq 0$, where $(S_t)_{t \in \mathbb{R}}$ is given as in the proof of Theorem 1. Hence, the order interval $[-f_0, f_0]_F$ is totally bounded and thus compact. Now one can use that the principal ideal $\bigcup_{c>0} c[-f_0, f_0]_F$ is dense in F according to [26, Cor 2 to Thm II.6.3] in order to conclude that every order interval $[f, g]_F$ is compact in F. □

As mentioned above, Theorems 1 and 2 allow for considerable generalisations. First of all, Theorem 1 remains true if L^p is replaced with a Banach lattice E with order continuous norm and if the semigroup $\mathscr{T}$ is only assumed to be bounded instead of contractive. In this case, the proof clearly requires a bit more lattice theory. Moreover, the range F of the projection P needs no longer be a sublattice, but it is still a so-called lattice subspace of E, meaning that it is a lattice with respect to the order induced by E but not with respect to the same lattice operations. We point out that even for $E = L^p$, the space F is no longer an ℓ^p-space in this case; instead, it is an atomic Banach lattice with order continuous norm. For more details, we refer to [12].

Theorem 2 can be generalised to bounded positive semigroups on Banach lattices with order continuous norm, too. However, the first part of the proof shows that one needs an additional technical assumption in this case: we have to assume that every *super-fixed point* of the semigroup is a fixed point, meaning that $T_t g = g$ for all $t \geq 0$ whenever $T_t g \geq g \geq 0$ for all $t \geq 0$. Again, we refer to [12] for details.

The most significant generalisation of Theorems 1 and 2 refers however to the strong continuity assumption with respect to the time parameter. In the proof of Theorem 2, this assumption is first employed when one uses that a set of the form $\{T_t f : t \in [0, t_0]\}$ is compact, but this step of the argument can easily be circumvented by using a bit more information

about the Jacobs–de Leeuw–Glicksberg decomposition. Much more importantly, the proofs of Theorems 1 and 2 both use the strong continuity to deduce that the positive and contractive group $(S_t)_{t\in\mathbb{R}}$ acts trivially on $\ell^p(I)$. Yet, it turns out that this can also be deduced without strong continuity by only using the algebraic properties of the additive group $\mathbb{R}$. Hence, if one is willing to invest more work in the proofs, one can show that Theorems 1 and 2 and their counterparts on Banach lattices remain true for semigroup representations $(T_t)_{t\in[0,\infty)}$ without any continuity or measurability assumptions with respect to t. For detailed results and proofs, we refer again to [12], where it is also demonstrated that the same methods can be used to obtain convergence results for positive representations of more general semigroups.

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