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A proof of energy gap for Yang–Mills connections

Une preuve du gap d'énergie pour les connexions de Yang–Mills

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A R T I C L E I N F O

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A B S T R A C T

In this note, we prove an $L^{\frac{n}{2}}$-energy gap result for Yang–Mills connections on a principal G-bundle over a compact manifold without using the Lojasiewicz–Simon gradient inequality ([2] Theorem 1.1).

R É S U M É

Dans cette note, nous démontrons un résultat concernant le gap d'énergie $L^{\frac{n}{2}}$ pour les connexions de Yang–Mills sur un fibré principal de groupe structural G sur une variété compacte, sans utiliser l'inégalité du gradient de Lojasiewicz–Simon.

1. Introduction

Let X be a compact n-dimensional Riemannian manifold endowed with a smooth Riemannian metric g, $P \to X$ a principal G-bundle over X, where G is a compact Lie group. We define the Yang–Mills functional by

$$YM(A) = \int_X |F_A|^2 \mathrm{d}vol_g,$$

where A is a C^∞-connection on P and F_A is the curvature of A.

A connection A on P is called a Yang–Mills connection if it is a critical point of YM, i.e. it obeys the Yang–Mills equation with respect to the metric g:

$$d_A^* F_A = 0. \tag{1.1}$$

In [2], Feehan proved an $L^{\frac{n}{2}}$-energy gap result for Yang–Mills connections on the principal G-bundle P over an arbitrary closed smooth Riemannian manifold with dimension $n \geq 2$ ([2] Theorem 1.1). Feehan applied the Lojasiewicz–Simon gradient

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inequality ([2] Theorem 3.2) to remove a positivity hypothesis on the Riemannian curvature tensors in a previous $L^{\frac{n}{2}}$-energy gap result due to Gerhardt [3] (Theorem 1.2).

In this note, we give another proof of this $L^{\frac{n}{2}}$-energy gap result of Yang–Mills connection without using the Lojasiewicz–Simon gradient inequality.

Theorem 1.1. *([2] Theorem 1.1) Let X be a compact Riemannian manifold without boundary of dimension $n \geq 2$ endowed with a smooth Riemannian metric g, P be a G-bundle over X. Then, either any smooth Yang–Mills connection A over X with compact Lie group G satisfies*

$$\int_X |F_A|^{\frac{n}{2}} dvol_g \geq C_0$$

for a constant $C_0 > 0$ depending only on X, n, G, or the connection A is flat.

2. Preliminaries and basic estimates

We shall generally adhere to the now standard gauge-theory conventions and notation of Donaldson and Kronheimer [1] and Feehan [2]. Throughout our article, G denotes a compact Lie group and P a smooth principal G-bundle over a compact Riemannian manifold X of dimension $n \geq 2$ endowed with a Riemannian metric g, $\mathfrak{g}_P$ denote the adjoint bundle of P, endowed with a G-invariant inner product and $\Omega^p(X, \mathfrak{g}_P)$ denote the smooth p-forms with values in $\mathfrak{g}_P$. Given a connection on P, we denote by ∇_A the corresponding covariant derivative on $\Omega^*(X, \mathfrak{g}_P)$ induced by A and the Levi-Civita connection of X. Let d_A denote the exterior derivative associated with ∇_A.

For $u \in L^p(X, \mathfrak{g}_P)$, where $1 \leq p < \infty$ and k is an integer, we denote

$$\|u\|_{L^p_{k,A}(X)} := \Big(\sum_{j=0}^{k} \int_X |\nabla_A^j u|^p dvol_g\Big)^{1/p},$$

where $\nabla_A^j := \nabla_A \circ \ldots \circ \nabla_A$ (repeated j times for $j \geq 0$). For $p = \infty$, we denote

$$\|u\|_{L^\infty_{k,A}(X)} := \sum_{j=0}^{k} ess\sup_X |\nabla_A^j u|.$$

At first, we review a key result due to Uhlenbeck for the connections with L^p-small curvature ($2p > n$) [5], which provides the existence of a flat connection Γ on P, of a global gauge transformation u of A to Coulomb gauge with respect to Γ, and of a Sobolev norm estimate for the distance between Γ and A.

Theorem 2.1. *([5] Corollary 4.3 and [2] Theorem 5.1) Let X be a closed, smooth manifold of dimension $n \geq 2$ endowed with a Riemannian metric, g, and G be a compact Lie group, and $2p > n$. Then there are constants, $\varepsilon = \varepsilon(n, g, G, p) \in (0, 1]$ and $C = C(n, g, G, p) \in [1, \infty)$, with the following property. Let A be a L^p_1 connection on a principal G-bundle P over X. If the curvature F_A obeys*

$$\|F_A\|_{L^p(X)} \leq \varepsilon,$$

then there exist a flat connection, $|\Gamma|$, on P, and a gauge transformation $u \in L^p_2(X)$ such that

(1) $d_\Gamma^*\big(u^*(A) - \Gamma\big) = 0$ *on* X,
(2) $\|u^*(A) - \Gamma\|_{L^p_{1,\Gamma}} \leq C\|F_A\|_{L^p(X)}$ *and*
(3) $\|u^*(A) - \Gamma\|_{L^{\frac{n}{2}}_{1,\Gamma}} \leq C\|F_A\|_{L^{\frac{n}{2}}(X)}$.

Next, we also review another key result due to Uhlenbeck concerning an a priori estimate for the curvature of a Yang–Mills connection over a closed Riemannian manifold.

Theorem 2.2. *([4] Theorem 3.5 and [2] Corollary 4.6) Let X be a compact manifold of dimension $n \geq 2$ endowed with a Riemannian metric g, let A be a smooth Yang–Mills connection with respect to the metric g on a smooth G-bundle P over X. Then there exist constants $\varepsilon = \varepsilon(X, n, g) > 0$ and $C = C(X, n, g)$ with the following property. If the curvature F_A obeys*

$$\|F_A\|_{L^{\frac{n}{2}}(X)} \leq \varepsilon,$$

then

$$\|F_A\|_{L^\infty(X)} \leq C\|F_A\|_{L^2(X)}.$$

3. Proof of Theorem 1.1

For any $p \geq 1$, the estimate in Theorem 2.2 yields

$$\|F_A\|_{L^p(X)} \leq C\|F_A\|_{L^\infty(X)} \leq C\|F_A\|_{L^2(X)}, \tag{3.1}$$

for $C = C(g, n)$.

If $n \geq 4$, using Hölder inequality, we have

$$\|F_A\|_{L^2(X)} \leq C\|F_A\|_{L^{\frac{n}{2}}(X)}. \tag{3.2}$$

If $n = 2, 3$, the L^p interpolation implies that

$$\begin{aligned}\|F_A\|_{L^2(X)} &\leq C\|F_A\|^{\frac{n}{4}}_{L^{\frac{n}{2}}(X)}\|F_A\|^{1-\frac{n}{4}}_{L^\infty(X)}\\ &\leq C\|F_A\|^{\frac{n}{4}}_{\frac{n}{2}(X)}\|F_A\|^{1-\frac{n}{4}}_{L^2(X)}\end{aligned}$$

and thus

$$\|F_A\|_{L^2(X)} \leq C\|F_A\|_{L^{\frac{n}{2}}(X)}. \tag{3.3}$$

Therefore, by combining (3.1)–(3.3), we obtain

$$\|F_A\|_{L^p(X)} \leq C\|F_A\|_{L^{\frac{n}{2}}(X)},\ \forall 2p \geq n \text{ and } n \geq 2.$$

Hence, if we suppose $\|F_A\|_{L^{\frac{n}{2}}(X)}$ sufficiently small so that $\|F_A\|_{L^q(X)}$ ($2q > n$ and $n \geq 2$) satisfies the hypothesis of Theorem 2.1, then Theorem 2.1 provides a flat connection Γ on P, a gauge transformation $u \in \mathcal{G}_P$, and the estimate

$$\|u^*(A) - \Gamma\|_{L^q_1(X)} \leq C(q)\|F_A\|_{L^q(X)},$$

and

$$d^*_\Gamma(u^*(A) - \Gamma) = 0.$$

We denote $\tilde{A} := u^*(A)$ and $a := u^*(A) - \Gamma$, then the curvature of $\tilde{A}$ is

$$F_{\tilde{A}} = d_\Gamma a + a \wedge a.$$

The connection $\tilde{A}$ also satisfies Yang–Mills equation

$$0 = d^*_{\tilde{A}} F_{\tilde{A}}. \tag{3.4}$$

Hence, taking the L^2-inner product of (3.4) with a, we obtain

$$\begin{aligned}0 &= (d^*_{\tilde{A}} F_{\tilde{A}}, a)_{L^2(X)}\\ &= (F_{\tilde{A}}, d_{\tilde{A}} a)_{L^2(X)}\\ &= (F_{\tilde{A}}, d_\Gamma a + 2a \wedge a)_{L^2(X)}\\ &= (F_{\tilde{A}}, F_{\tilde{A}} + a \wedge a)_{L^2(X)}.\end{aligned}$$

Then we get

$$\begin{aligned}\|F_A\|^2_{L^2(X)} &= \|F_{\tilde{A}}\|^2_{L^2(X)}\\ &= -(F_{\tilde{A}}, a \wedge a)_{L^2(X)}\\ &\leq \|F_{\tilde{A}}\|_{L^2(X)}\|a \wedge a\|_{L^2(X)}\\ &= \|F_A\|_{L^2(X)}\|a \wedge a\|_{L^2(X)}\end{aligned}$$

here we use the fact $|F_{u^*(A)}| = |F_A|$ since $F_{u^*(A)} = u \circ F_A \circ u^{-1}$.

If $n \geq 4$:

$$\begin{aligned}\|a \wedge a\|_{L^2(X)} &\leq C\|a\|^2_{L^4(X)} \\ &\leq C\|a\|^2_{L^n(X)} \\ &\leq C\|a\|^2_{L_1^{\frac{n}{2}}(X)} \\ &\leq C\|F_A\|^2_{L^{\frac{n}{2}}(X)} \\ &\leq C\|F_A\|^2_{L^\infty(X)} \\ &\leq C\|F_A\|^2_{L^2(X)},\end{aligned}$$

where we apply the Sobolev embedding $L_1^{\frac{n}{2}} \hookrightarrow L^n$.

If $n = 2, 3$,

$$\begin{aligned}\|a \wedge a\|_{L^2(X)} &\leq C\|a\|^2_{L^4(X)} \\ &\leq C\|a\|^2_{L_1^2(X)} \\ &\leq C\|F_A\|^2_{L^2(X)},\end{aligned}$$

where we apply the Sobolev embedding $L_1^2 \hookrightarrow L^4$.

Combining the preceding inequalities, we have

$$\|F_A\|^2_{L^2(X)} \leq C\|F_A\|^3_{L^2(X)}.$$

We can choose $\|F_A\|_{L^2(X)}$ sufficiently small so that $C\|F_A\|_{L^2(X)} < 1$, hence $\|F_A\|_{L^2(X)} \equiv 0$ and thus A must be a flat connection. This completes the proof.

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