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Dynamical systems

A notion of Denjoy sub-system

Une notion de sous-système de Denjoy

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ABSTRACT

We introduce a notion of Denjoy sub-system that generalizes that of the Aubry–Mather set. For such systems, we prove a result similar to Denjoy theorem (non-existence of C^2 Denjoy sub-systems), and study their Lyapunov exponents.

RÉSUMÉ

Nous introduisons une notion de sous-système de Denjoy qui généralise celle d'ensemble d'Aubry–Mather. Pour ces systèmes, nous montrons un analogue du théorème de Denjoy (la non-existence de sous-systèmes de Denjoy de classe C^2) et étudions leurs exposants de Lyapunov.

Version française abrégée

Nous introduisons une notion de sous-système de Denjoy pour un difféomorphisme d'une variété, notion qui généralise celle d'ensemble d'Aubry–Mather.

Définition 0.1. Soit $f : M \to M$ un difféomorphisme de classe C^k d'une variété M. Un C^k *(resp. Lipschitz) sous-système de Denjoy pour* f est un triplet (K, γ, h) où

- $\gamma : \mathbb{T} \to M$ est un plongement de classe C^k (resp. biLipschitz) ;
- $h : \mathbb{T} \to \mathbb{T}$ est un contre-exemple de Denjoy d'ensemble minimal $K \subset \mathbb{T}$;
- $f(\gamma(K)) = \gamma(K)$;
- $\gamma \circ h_{|K} = f \circ \gamma_{|K}$.

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Remarque 1.

(i) Pour les difféomorphismes exact symplectiques de l'anneau qui dévient la verticale, la théorie d'Aubry–Mather fournit de nombreux sous-systèmes de Denjoy (voir [7]).
(ii) D'autres ensembles de Cantor qui ne sont pas des sous-systèmes de Denjoy apparaissent de manière naturelle en dynamique. Par exemple, un fer à cheval a des points périodiques et un odomètre a des sous-ensembles fermés non triviaux périodiques.

Notre premier théorème énonce qu'une dynamique de Denjoy ne peut pas être insérée dans une courbe (non nécessairement invariante) de classe C^2.

Théorème 0.2. *Il n'existe pas de sous-système de Denjoy de classe C^2.*

Corollaire 0.3. *Soit $f : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ un difféomorphisme de classe C^2 qui dévie la verticale (et n'est pas forcément symplectique). Si $E \subset \mathbb{T} \times \mathbb{R}$ est un ensemble d'Aubry–Mather de nombre de rotation irrationnel contenu dans le graphe d'une fonction de classe C^2 $\psi : \mathbb{T} \to \mathbb{R}$, alors E coïncide avec ce graphe.*

L'entropie topologique et métrique d'un sous-système de Denjoy est nulle. Le résultat suivant exprime que l'unique mesure de probabilité définie par un sous-système de Denjoy de classe C^1 a au moins un exposant de Lyapunov nul.

Théorème 0.4. *Soit f un difféomorphisme de classe C^1 d'une variété M. Si (K, γ, h) est un sous-système de Denjoy de classe C^1 pour f, alors l'unique mesure de probabilité borélienne f invariante à support dans $\gamma(K)$ a au moins un exposant de Lyapunov nul.*

Corollaire 0.5. *Soit f un difféomorphisme de classe C^1 qui préserve l'aire d'une surface M. Si (K, γ, h) est un sous-système de Denjoy de classe C^1 pour f, alors l'unique mesure de probabilité borélienne f invariante à suppport dans $\gamma(K)$ a ses deux exposants de Lyapunov nuls.*

Remarque 2.

(i) Le second auteur a montré dans [6] qu'un difféomorphisme générique symplectique déviant la verticale a beaucoup d'ensembles d'Aubry–Mather qui sont uniformément hyperboliques. Parmi ceux-ci, ceux qui ont un nombre de rotation irrationnel définissent alors des sous-systèmes de Denjoy qui sont Lipschitz, mais ne sont pas de classe C^1.
(ii) Dans [2], il est montré que l'unique mesure de probabilité invariante portée par un lacet invariant d'un difféomorphisme de classe $C^{1+\alpha}$ ne contenant pas de point périodique a au moins un exposant de Lyapunov nul. La première remarque montre que ce résultat ne peut pas être étendu aux sous-systèmes de Denjoy Lipschitz définis à l'aide d'une courbe non invariante.
(iii) Le premier auteur a montré dans [1] une sorte de réciproque au corollaire 0.5 : si les exposants de Lyapunov de la mesure à support dans un ensemble d'Aubry–Mather de nombre de rotation irrationnel d'une application symplectique déviant la verticale sont nuls, alors le support de la mesure est en un certain sens de classe C^1 presque partout. La proposition suivante montre que ce résultat est faux si on enlève l'hypothèse de déviation de la verticale.

Proposition 0.6. *Il existe un difféomorphisme de classe C^1 F de $\mathbb{T} \times \mathbb{R}$ qui admet un sous-système Lipschitz de Denjoy (K, γ, h) tel que*

- *les exposants de Lyapunov de l'unique mesure de probabilité à support dans $\gamma(\mathbb{T})$ sont nuls ;*
- *il n'existe pas de sous-système de Denjoy de classe C^1 (K_0, γ_0, h_0) pour un difféomorphisme de classe C^1 f de $\mathbb{T} \times \mathbb{R}$ tel que $\gamma_0(K_0) = \gamma(K)$.*

Remarque 3. L'exemple de sous-système de Denjoy construit dans la proposition 0.6 admet en fait un feuilletage Lipschitz de courbes invariantes dont chacune porte un contre-exemple de Denjoy.

Cette notion de sous-système de Denjoy s'étend pour les flots.

Définition 0.7. Soit $(\varphi_t)_{t\in\mathbb{R}}$ un flot de classe C^k d'une variété M. Un *sous-flot de Denjoy de classe C^k (resp. Lipschitz) pour $(\varphi_t)_{t\in\mathbb{R}}$* est un triplet $(K, j, (\psi_t)_{t\in\mathbb{R}})$ où

- $j : \mathbb{T}^2 \to M$ est un plongement de classe C^k (resp. Lipschitz) ;
- $(\psi_t)_{t\in\mathbb{R}}$ est un flot de classe C^1 sur $\mathbb{T}^2$ sans point périodique ni orbite dense ;

– si K est l'unique ensemble fermé minimal invariant de $(\psi_t)_{t\in\mathbb{R}}$, alors $j(K)$ est invariant par $(\varphi_t)_{t\in\mathbb{R}}$;
– pour tout $x \in K$ et tout $t \in \mathbb{R}$, on a $j \circ \psi_t(x) = \varphi_t \circ j(x)$.

Théorème 0.8. *Il n'existe pas de sous-flot de Denjoy de classe C^2.*

Un résultat similaire pour les graphes est prouvé dans [4] dans le cadre de la théorie K.A.M. faible.

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1. Introduction

In 1932, Arnaud Denjoy proved in [3] that if a C^2 vector-field of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ has a rotation vector that does not belong to $\mathbb{R}.\mathbb{Z}^2$, then every orbit is dense in $\mathbb{T}^2$. A similar result for diffeomorphisms of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ can be stated in the following way: if an orientation-preserving C^2 diffeomorphism of the circle has an irrational rotation number, then it is C^0 conjugated to a rotation. There exist examples of orientation-preserving C^1 diffeomorphisms of $\mathbb{T}$ that have an irrational rotation number and are not C^0 conjugated with a rotation, or equivalently are not minimal (see [5] for example). Such examples in any regularity (i.e. C^0 or C^1) are now called *Denjoy counter-examples*, and they have a unique minimal invariant closed set that is a Cantor set (i.e. compact, totally disconnected and with no isolated point).

This implies of course that if a C^2 diffeomorphism f of a manifold has an invariant loop Γ such that $f_{|\Gamma}$ has an irrational rotation number and is not minimal, then Γ is not C^2. Here we address the question when we do not require the curve to be invariant: can a Denjoy dynamics be contained in some non-invariant C^2 curve? In other words, can the set that supports this dynamics be C^2 in some sense? To explain in detail this question, we have to introduce some notions.

Definition 1.1. Let $f : M \to M$ be a C^k diffeomorphism of a manifold M. A *C^k (resp. Lipschitz) Denjoy sub-system for f* is a triplet (K, γ, h) where

– $\gamma : \mathbb{T} \to M$ is a C^k (resp. biLipschitz) embedding;
– $h : \mathbb{T} \to \mathbb{T}$ is a Denjoy counter-example with invariant minimal set $K \subset \mathbb{T}$;
– $f(\gamma(K)) = \gamma(K)$;
– $\gamma \circ h_{|K} = f \circ \gamma_{|K}$.

Remark 1.

(i) For area-preserving twist maps of the annulus, Aubry–Mather theory tells us that there are many Lipschitz Denjoy sub-systems (see [7]).
(ii) Other kinds of invariant Cantor sets appear naturally in dynamical systems that are not Denjoy sub-systems. For example, a horseshoe has periodic points, an odometer has periodic non-trivial closed subsets.

Our first result asserts that a Denjoy sub-system is never C^2.

Theorem 1.2. *There exist no C^2 Denjoy sub-system.*

We immediately get:

Corollary 1.3. *Let $f : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ be a C^2 twist map (non necessarily area preserving). If $E \subset \mathbb{T} \times \mathbb{R}$ is an Aubry–Mather set of irrational rotation number that is contained in the graph of a C^2 function $\psi : \mathbb{T} \to \mathbb{R}$, then E coincides with this graph.*

The topological entropy of a circle homeomorphism is equal to zero. Moreover, such a homeomorphism is uniquely ergodic in case its rotation number is irrational. Consequently the restriction of a diffeomorphism of a manifold to a Denjoy sub-system has zero entropy and is uniquely ergodic. The next result asserts that the unique invariant probability measure has one zero Lyapunov in case the sub-system is C^1.

Theorem 1.4. *Let f be a C^1 diffeomorphism of a manifold M. If (K, γ, h) is a C^1 Denjoy sub-system for f, then the unique f invariant Borel probability measure that is supported on $\gamma(K)$ has at least one vanishing Lyapunov exponent.*

We easily deduce the following.

Corollary 1.5. *Let f be an area-preserving C^1 diffeomorphism of a surface M. If (K, γ, h) is a C^1 Denjoy sub-system for f, then the unique f invariant Borel probability measure supported on $\gamma(K)$ has its two Lyapunov exponents equal to zero.*

Remark 2.

(i) The second author proved in [6] that, for a generic area-preserving twist map, many Aubry–Mather sets are uniformly hyperbolic. Such an Aubry–Mather set cannot be a loop. Among them, those that have an irrational rotation number are Lipschitz Denjoy sub-systems, but not C^1 Denjoy sub-systems.
(ii) It was proved in [2] that any continuous invariant loop by a $C^{1+\alpha}$ diffeomorphism of a surface that contains no periodic point carries a unique invariant measure that has at least one zero Lyapunov exponent. The first remark shows that this result cannot be extended to Lipschitz Denjoy sub-systems that are not defined via an invariant curve.
(iii) The first author proved in [1] a kind of reverse result of Corollary 1.5: if the Lyapunov exponents of a measure that is supported on an Aubry–Mather set with irrational rotation number of a symplectic twist map are zero, then the support of the measure is, in some sense, C^1 regular almost everywhere. The next proposition gives a counter-example in the non-twist setting.

Proposition 1.6. *There exists a C^1 diffeomorphism F of $\mathbb{T}\times\mathbb{R}$ that admits a Lipschitz Denjoy sub-system (K,γ,h) such that*

- *the Lyapunov exponents of the unique invariant probability measure that is supported on $\gamma(\mathbb{T})$ are equal to zero;*
- *there exists no C^1 Denjoy sub-system (K_0,γ_0,h_0) for a C^1 diffeomorphism f of $\mathbb{T}\times\mathbb{R}$ such that $\gamma_0(K_0)=\gamma(K)$.*

Remark 3. The diffeomorphism built in Proposition 1.6 has a Lipschitz foliation into invariant Lipschitz graphs on which the dynamics is a Denjoy counter-example.

This notion of Denjoy sub-system can be extended to flows.

Definition 1.7. Let $(\varphi_t)_{t\in\mathbb{R}}$ be a C^k flow on a manifold M. A *C^k (resp. Lipschitz) Denjoy sub-flow for* $(\varphi_t)_{t\in\mathbb{R}}$ is a triplet $(K,j,(\psi_t)_{t\in\mathbb{R}})$ where

- $j:\mathbb{T}^2\to M$ is a C^k (resp. Lipschitz) embedding;
- $(\psi_t)_{t\in\mathbb{R}}$ is a C^1 flow of $\mathbb{T}^2$ with neither periodic orbit nor dense orbit;
- if K is the unique minimal invariant closed subset of $(\psi_t)_{t\in\mathbb{R}}$, then $j(K)$ is invariant by $(\varphi_t)_{t\in\mathbb{R}}$;
- for every $x\in K$ and every $t\in\mathbb{R}$, one has $j\circ\psi_t(x)=\varphi_t\circ j(x)$.

Theorem 1.8. *There exists no C^2 Denjoy sub-flow.*

In [4], a similar result for graphs is proved in the setting of weak K.A.M. theory.

2. Proof of Theorems 1.2 and 1.4

The key result, in the proofs of Theorems 1.2 and 1.4 is the following:

Proposition 2.1. *Let $k\geq 1$ be some integer. Let (K,γ,h) be a C^k Denjoy sub-system for some C^k-diffeomorphism $f:M\to M$. There exists an orientation-preserving C^k diffeomorphism $\varphi:\mathbb{T}\to\mathbb{T}$ such that $\varphi_{|K}=h_{|K}$.*

Proof. With the notations of Proposition 2.1, observe that the two loops $\gamma:\mathbb{T}\to M$ and $f\circ\gamma:\mathbb{T}\to M$ are tangent along the f invariant Cantor set $\gamma(K)$. Indeed, this set is included in both loops and has no isolated point.

Let $\mathcal{N}$ be a tubular neighborhood of $\gamma(\mathbb{T})$ in which we can define a C^k projection $p:\mathcal{N}\to\gamma(\mathbb{T})$. Observe that $p\circ f\circ\gamma$, coinciding with $\gamma\circ h$ on K, is a C^k embedding when restricted to a neighborhood of K. This implies that there exist two open neighborhoods U, U' of K in $\mathbb{T}$ such that $f\circ\gamma(U)\subset\mathcal{N}$ and such that $\gamma^{-1}\circ p\circ f\circ\gamma$ induces a C^k diffeomorphism between U and U'. The family of connected components of U is an open covering of K. By compactness of K, there are finitely many components that meet K. So, taking a subset of U if necessary, we can assume that the number of connected components of U is finite, larger than 1, and that each connected component meets K. We can index these components, getting a family $(U_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ ordered in the usual cyclic order along $\mathbb{T}$. Setting $U'_i=\gamma^{-1}\circ p\circ f\circ\gamma(U_i)$, one gets another family $(U'_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ ordered in the usual cyclic order along $\mathbb{T}$ because $\gamma^{-1}\circ p\circ f\circ\gamma_{|K}=h_{|K}$. We can find a covering $(V_i)_{i\in\mathbb{Z}/n\mathbb{Z}}$ of K by open and connected subsets of $\mathbb{T}$, satisfying $\overline{V_i}\subset U_i$ for every $i\in\mathbb{Z}/n\mathbb{Z}$. For every $j\in\mathbb{Z}/n\mathbb{Z}$, we define $V'_i=\gamma^{-1}\circ p\circ f\circ\gamma(V_i)$, and we denote by J_i the connected component of the complement of $\bigcup_{i\in\mathbb{Z}/n\mathbb{Z}} V_i$ that lies between V_i and V_{i+1} and J'_i the connected component of the complement of $\bigcup_{i\in\mathbb{Z}/n\mathbb{Z}} V'_i$ that lies between V'_i and V'_{i+1}. To extend $f_{|\bigcup_{1\leq i\leq n}\overline{V}_i}$ to a C^k diffeomorphism of $\mathbb{T}$, it is sufficient to find, for every $i\in\mathbb{Z}/n\mathbb{Z}$, an increasing C^k diffeomorphism

$\varphi_i : J_i \to J'_i$ with prescribed values of the derivatives $\varphi_i^{(l)}$, $1 \le l \le k$, at the two ends of J_i to ensure a global differentiability. There is no obstruction to such a construction. □

Denjoy's Theorem implies that under the assumption of Proposition 2.1, the integer k must be equal to 1, so Theorem 1.2 is proved. Moreover, when $k = 1$, then $\mathbb{R}.\gamma'$ is a Df invariant lines bundle along $\gamma(K)$ that satisfies $Df \cdot \gamma' = \varphi'(\gamma' \circ \varphi)$. So, for every integer $n \ge 0$, one has

$$Df^n \cdot \gamma' = (\varphi^n)'(\gamma' \circ \varphi^n).$$

We deduce that the unique f invariant probability measure supported on $\gamma(K)$ has a Lyapunov exponent that coincides with the Lyapounov exponent of the unique invariant measure of φ, a C^1 diffeomorphism of $\mathbb{T}$. This Lyapunov exponent being known to vanish, Theorem 1.4 is proved.

3. Proof of Proposition 1.6

As explained in [5], one can construct a C^1 Denjoy counter-example $g : \mathbb{T} \to \mathbb{T}$ such that:

- g has exactly two orbits of wandering intervals $(I_k)_{k\in\mathbb{Z}}$ and $(J_k)_{k\in\mathbb{Z}}$ where $\mathrm{Leb}(I_k) = \mathrm{Leb}(J_k) = \ell_k$;
- $\sum_{k\in\mathbb{Z}} \ell_k = \frac{1}{2}$;
- $\lim_{k\to\pm\infty} \frac{\ell_k}{\ell_{k+1}} = 1$;
- the derivative g' is equal to 1 on the minimal compact invariant set K.

In particular, one has $\mathbb{T} \setminus K = \bigcup_{k\in\mathbb{Z}} \mathrm{Int}(I_k) \cup \bigcup_{k\in\mathbb{Z}} \mathrm{Int}(J_k)$.

There exists a Lipschitz function $\psi : \mathbb{T} \to \mathbb{R}$, uniquely defined up to an additive constant, such that

$$\psi'_{|\bigcup_{k\in\mathbb{Z}} \mathrm{Int}(I_k)} = 1, \quad \psi'_{|\bigcup_{k\in\mathbb{Z}} \mathrm{Int}(J_k)} = -1.$$

Observe that ψ' is constant along every orbit of g that is contained in $\mathbb{T} \setminus K$.

The map $\eta : \mathbb{T} \to \mathbb{R}$ defined by $\eta = \psi \circ g - \psi$ is also Lipschitz and differentiable on $\mathbb{T} \setminus K$. Moreover, for every $\theta \in \mathbb{T} \setminus K$, one has

$$\eta'(\theta) = g'(\theta)\,\psi'(g(\theta)) - \psi'(\theta) = \psi'(\theta)\,(g'(\theta) - 1).$$

The fact that ψ' is bounded on $\mathbb{T} \setminus K$ and that $1 - g'$ is continuous on $\mathbb{T}$ and vanishes on K implies that η is C^1.

Finally, let us define a C^1 diffeomorphism F of $\mathbb{T} \times \mathbb{R}$ by the formula

$$F(\theta, r) = (g(\theta), r + \eta(\theta)).$$

The graph of ψ is invariant by F because

$$F(\theta, \psi(\theta)) = (g(\theta), \psi(\theta) + \eta(\theta)) = (g(\theta), \psi \circ g(\theta)).$$

Note that for every $\theta \in \mathbb{T}$ and every $k \in \mathbb{Z}$, one has

$$DF^k(\theta_0, \psi(\theta_0)) = \begin{pmatrix} (g^k)'(\theta_0) & 0 \\ b_k & 1 \end{pmatrix}.$$

Hence the two Lyapunov exponents of the unique invariant measure that is supported on the graph of ψ are zero.

Moreover, if $\theta \in K$, there exists two sequences $(I_{i_k})_{k\in\mathbb{N}}$ and $(J_{j_k})_{k\in\mathbb{N}}$ that converge to θ. The slope of ψ between the two ends of I_{i_k} is 1 and the slope of ψ between the two ends of J_{j_k} is -1. This implies that there exists no C^1 Denjoy sub-system (K_0, γ_0, h_0) of a diffeomorphism f of $\mathbb{T} \times \mathbb{R}$ such that $\gamma_0(K_0) = \mathrm{graph}(\psi_{|K})$.

4. Proof of Theorem 1.8

We assume that $(K_0, j, (\psi_t)_{t\in\mathbb{R}})$ is a C^2 Denjoy sub-flow for some C^2 flow $(\varphi_t)_{t\in\mathbb{R}}$ on a manifold M. We choose on $\mathbb{T}^2$ an essential C^2 embedding $\gamma : \mathbb{T} \to \mathbb{T}^2$ that is transverse to the flow direction $\dot\psi$ of $(\psi_t)_{t\in\mathbb{R}}$. The first return map $r : \gamma(\mathbb{T}) \to \gamma(\mathbb{T})$ is then well defined, of class C^1, and the map $h = \gamma^{-1} \circ r \circ \gamma : \mathbb{T} \to \mathbb{T}$ is a C^1 Denjoy counter-example. We denote K the minimal set of h and set $\mathcal{K} = j \circ \gamma(K)$ and $\mathcal{T} = j \circ \gamma(\mathbb{T})$.

Let $\mathcal{N}$ be a tubular neighborhood of $j(\mathbb{T}^2)$ in which we can define a C^2 projection $p : \mathcal{N} \to j(\mathbb{T}^2)$ and consider the C^2 hypersurface $\mathcal{H} = p^{-1}(\mathcal{T})$. The flow direction $\dot\varphi$ of $(\varphi_t)_{t\in\mathbb{R}}$ is transverse to $\mathcal{H}$ on $\mathcal{K}$. Moreover, the first return map of $(\varphi_t)_{t\in\mathbb{R}}$

on $\mathcal{H}$ is well defined on $\mathcal{K}$ and coincide with $j \circ \gamma \circ h \circ \gamma^{-1} \circ j^{-1}{}_{|\mathcal{K}}$. One deduces that there exist two neighborhoods $\mathcal{W}$ and $\mathcal{W}'$ of $\mathcal{K}$ in $\mathcal{H}$ such that the first return map of $(\varphi_t)_{t\in\mathbb{R}}$ on $\mathcal{H}$ is well defined on $\mathcal{W}$ and induces a C^2 diffeomorphism $\mathcal{R}$ between $\mathcal{W}$ and $\mathcal{W}'$. The map $\gamma^{-1} \circ j^{-1}{}_{|\mathcal{T}} \circ p \circ \mathcal{R} \circ j \circ \gamma$ is a C^2 map defined on a neighborhood of K with values in $\mathbb{T}$ that coincides with the C^1 diffeomorphism h on K. So it induces a C^2 diffeomorphism between two neighborhoods U, U' of K. As in the proof of Proposition 2.1, we may require that U and U' have finitely many connected components and we can construct a C^2 diffeomorphims of $\mathbb{T}$ that coincides with h in a neighborhood of K. This gives a contradiction.

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