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The flowbox theorem for divergence-free Lipschitz vector fields

Le théorème du flot tubulaire pour les champs vectoriels Lipschitz à divergence nulle

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A R T I C L E   I N F O

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A B S T R A C T

In this note, we prove the flowbox theorem for divergence-free Lipschitz vector fields.

R É S U M É

Dans cette note, nous prouvons le théorème du flot tubulaire pour les champs vectoriels Lipschitz à divergence nulle.

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1. Introduction and basic definitions

1.1. Introduction

Given a regular orbit of a C^r flow ($r \geq 1$), it is always possible, using a change of coordinates, to straighten out all orbits in a certain neighborhood of the orbit. This is a very simple, yet important result called *the flowbox theorem*, and its proof uses basically the inverse function theorem (see, e.g., [16, pp. 40]). This theorem describes completely the local behavior of the orbits in a neighborhood of a regular orbit and shows that, locally, first integrals always exist. However, since the change of coordinates is given implicitly, there is no guarantee that it preserves certain geometric invariants of the flow like, for example, the conservation of a volume form or of a symplectic form. We may wonder why there is the need of preservation of some invariants? Actually, when working with perturbations of flows/vector fields, it is nice to have good coordinates to perform perturbations explicitly; furthermore, once we perturb maintaining the invariant (volume form, symplectic form), we would like to 'return' to the initial scenario and so we are keenly interested that these change of coordinates keep the geometric invariant unchanged, otherwise they are completely useless. With respect to the Hamiltonian vector field context, the proof of the flowbox theorem goes back to classic textbooks by Abraham and Marsden [1] and also by Robinson [17], with some revisited approaches by the author and Dias [6], and more recently by Cabral [9]. Considering the preservation

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of the volume form, the flowbox theorem proof was firstly given by the author in [4] (see also the multidimensional case in [5]), and afterwards different approaches were given by Barbarosie [3] and by Castro and Oliveira [10].

Nevertheless, when we work with vector fields, whether they are divergence-free, or Hamiltonian or even without any invariant restriction at all, in order to have the Picard–Lindelöf uniqueness of integrability into a flow, we impose only Lipschitz continuity. So it is natural to ask if previous mentioned results also work in the broader regularity class of Lipschitz vector fields. Boldt and Calcaterra [8] gave a satisfactory answer regarding Lipschitz vector fields. Since this work applies only to general (i.e. not divergence-free) vector fields, it was not clear that the change of coordinates would preserve volume when applied to the special case of divergence-free vector fields. In the present paper, we present a proof of the result described in the title. We expect that this basic tool can be useful to complete the theory of continuous flows in the volume-preserving case, as it is presented in the recent work [7].

As it is usual in these type of results, the regularity of the change of coordinates obtained is the same as the one of the vector field. So we only expect to obtain a lipeomorphism (a bijective Lipschitz map with Lipschitz inverse) for the change of coordinates. Indeed, despite the fact that Boldt and Calcaterra's lipeomorphim does not keep invariant the volume necessarily, in [8, Example 5] (see Example 1), an example is presented of a vector field, which curiously is divergence-free, and such that no change of coordinates (volume-preserving or not) shall be differentiable.

1.2. Basic definitions

Let M be a connected, closed and C^∞ Riemannian manifold of dimension $n \geq 2$. Since along this paper we deal with divergence-free vector fields, we assume that M is also a volume-manifold with a volume form $\mathscr{V}\colon TM^n \to \mathbb{R}$ where TM stands for the tangent bundle. Furthermore, we equip M with an atlas $\mathcal{A} = \{(\varphi_i, U_i)_i\}$ of M (cf. [15]), such that $(\varphi_i)_*\mathscr{V} = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \ldots \wedge \mathrm{d}x_n$, where x_i are the canonical coordinates in the Euclidean space, $\varphi_i\colon U_i \to \mathbb{R}^n$ a local C^∞ diffeomorphism and U_i an open subset of M. The fact that M is compact guarantees that $\mathcal{A}$ can be taken finite, say $\mathcal{A} = \{(\varphi_i, U_i)\}_{i=1}^k$. We call *Lebesgue measure* the measure associated with $\mathscr{V}$ and denote it by ν. More precisely, we let

$$\nu(\mathcal{B}) = \nu_{\mathscr{V}}(\mathcal{B}) := \int_{\varphi(\mathcal{B})} \mathscr{V}_{\varphi^{-1}(x)}(D\varphi_1^{-1} \cdot e_1, \ldots, D\varphi_n^{-1} \cdot e_n)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

for some Borelian $\mathcal{B} \subset M$ where $\{e_1, \ldots, e_n\}$ is the canonical base of $\mathbb{R}^n$. Let $d(\cdot,\cdot)$ stands for the metric associated with the Riemannian structure.

We say that a function $F\colon \mathbb{R}^n \to \mathbb{R}$ is *Lipschitz* (or Lipschitz continuous) if there exists $L > 0$ such that $\|F(x) - F(y)\| \leq L\|x - y\|$ for all $x, y \in \mathbb{R}^n$. A C^r vector field X $(r \geq 0)$ is a C^r map $X\colon M \to TM$ so that $X(x) \in T_xM$. Let X be written in the coordinates associated with $\mathcal{A}$ such that $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$. If, for every $i = 1, \ldots, n$, each function X_i is Lipschitz continuous, then X is said to be a *Lipschitz vector field*. The integral family of curves, $X^t\colon M \to M$, associated with X satisfies $X^{t+s}(x) = X^t(X^s(x))$ and $X^0(x) = x$ for all $t, s \in \mathbb{R}$ and $x \in M$ and is called the *flow* associated with X. In [13, Theorem 3.41 & Lemma 3.42], it is proved that Lipschitz vector fields integrate Lipschitz flows. Rademacher's theorem ([12, Theorem 3.1.6]) yields that Lipschitz functions admit derivatives for ν-a.e. (almost every) point. The divergence of a vector field, $\nabla \cdot X\colon M \to \mathbb{R}$, where $\nabla := \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$, is a well-defined function on a ν-full measure subset of M if we assume X to be a Lipschitz vector field. We say that a Lipschitz vector field X is *divergence-free* if $\nabla \cdot X = 0$ for ν-a.e. $x \in M$. We denote this set by $\mathfrak{X}_\nu^{0,1}(M)$. We endow $\mathfrak{X}_\nu^{0,1}(M)$ with the norm $\|\cdot\|_{0,1}$ defined by $\|X\|_{0,1} := \max\left\{\sup_{p\in M}\|X(p)\|, \sup_{p,q\in M, p\neq q} \frac{\|X(p)-X(q)\|}{d(p,q)}\right\}$. When a vector field X is of class C^r $(r \geq 1)$, we say that X is *divergence-free* if $\nabla \cdot X = 0$ for all $x \in M$.

2. The Abel–Jacobi–Liouville formula for $\mathfrak{X}_\nu^{0,1}(M)$

As we already said, Lipschitz vector fields are uniquely integrable and, for each time t, the map X^t is Lipschitz continuous, thus DX_x^t exists for ν-a.e. $x \in M$. In fact, X^t is a lipeomorphism with respect to the space variable. We say that a Lipschitz flow $X^t\colon M \to M$ is volume-preserving if, for any Borelian $\mathcal{B} \subseteq M$ and any $t \in \mathbb{R}$, we have $\nu(X^t(\mathcal{B})) = \nu(\mathcal{B})$. From the Change of Variables Theorem, this definition is equivalent to the one that assures that for any $\tau \in \mathbb{R}$ and for ν-a.e. point $x \in M$, we have $\det(DX_x^\tau) = 1$.

The relation between the volume-preserving property of the flow and the divergence-freeness of the vector field is embodied in Proposition 1. This is a kind of Abel–Jacobi–Liouville's formula, but for the Lipschitz class. For the C^r class $(r \geq 2)$, the proof of this formula is easy and the proof for C^1 vector fields usually follows from a C^1-approximation of C^2 vector fields and a limit argument (see, e.g., [14, Theorem 3.2]). Unfortunately, we can not use this argument because vector fields in $\mathfrak{X}_\nu^{0,1}(M)$ are not $\|\cdot\|_{0,1}$-approximable by vector fields in $X \in \mathfrak{X}_\nu^1(M)$, as we can see in the following simple example.

Example 1. Take $X(x, y) = (X_1(x, y), X_2(x, y)) = (1 + |y|, 0)$ in $\mathfrak{X}_\nu^{0,1}(\mathbb{R}^2)$ and use [15] to transport it to $M = \mathbb{S}^2$, defining a vector field in $\mathfrak{X}_\nu^{0,1}(M)$. Assume, by contradiction, that there exists a C^1 vector field $Y(x, y) = (Y_1(x, y), Y_2(x, y)) \in \mathfrak{X}_\nu^1(M)$ such that $\frac{\partial Y_1}{\partial y}|_{(0,0)}$ exists and $\|X - Y\|_{0,1} < 1$. Let us define, for $y \in (-1, 1)$, $\alpha(y) = Y_1(0, y)$, $\beta(y) = X_1(0, y) = 1 + |y|$ and

$$\Delta_y := \frac{|\alpha(y)-\beta(y)-(\alpha(0)-\beta(0))|}{|y-0|} = \left|\frac{\alpha(y)-\alpha(0)}{|y|} - \left(\frac{\beta(y)-\beta(0)}{|y|}\right)\right| = \left|\frac{\alpha(y)-\alpha(0)}{|y|} - 1\right|.$$

We observe that

$$\lim_{y\to 0^+} \frac{\alpha(y)-\alpha(0)}{|y|} = \alpha'(0) = \frac{\partial Y_1}{\partial y}|_{(0,0)} \quad \text{and} \quad \lim_{y\to 0^-} \frac{\alpha(y)-\alpha(0)}{|y|} = -\alpha'(0) = -\frac{\partial Y_1}{\partial y}|_{(0,0)}.$$

Hence, one of these numbers $\alpha'(0)$ or $-\alpha'(0)$ is ≤ 0, which contradics $\|X-Y\|_{0,1} < 1$ above.

To prove Proposition 1, we use the next result:

Theorem 2.1. *(Dacorogna and Moser [11, Theorem 2]) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^∞ boundary. Let $g \in C^\infty(\overline{\Omega}, \mathbb{R})$ be such that $\int_\Omega g = 0$. Then, there exists a vector field $V \in C^\infty(\overline{\Omega}, \mathbb{R}^n)$ and with the same regularity at the boundary such that $\nabla \cdot V(x) = g(x)$ for $x \in \Omega$ and $V(x) = 0$ on the boundary $\partial\Omega$.*

Proposition 1. *If $X \in \mathfrak{X}^{0,1}_\nu(M)$ and $\tau \in \mathbb{R}$, then, for any Borelian $\mathcal{B}$, we have $\nu(\mathcal{B}) = \nu(X^\tau(\mathcal{B}))$.*

Proof. The proof resemble [2, Theorem 2.2]. Use [15] and cover M with volume-preserving charts $\{\varphi_i : U_i \to \mathbb{R}^n\}_{i=1}^k$.
Local argument: Fix $\varphi_1 : U_1 \to \mathbb{R}^n$. Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ map compactly supported in $\overline{B(0,1)}$ and such that $\int_{\mathbb{R}^n} \eta = 1$. We apply the convolution operator of the scaled Friedrich's mollifier $\eta_\epsilon(x) := \epsilon^{-n}\eta\left(\frac{x}{\epsilon}\right)$ on a small ball of radius $\epsilon > 0$ to each component of a Lipschitz vector field X in the local chart φ_1 in U_1. Thus, $X_i * \eta_\epsilon(x) = \int_{\mathbb{R}^n} X_i(y)\eta_\epsilon(x-y)\,d\nu(y)$ for each $i = 1, ..., n$ and so the vector field $X^1_\epsilon = X * \eta_\epsilon = (X_1 * \eta_\epsilon, ..., X_n * \eta_\epsilon)$ on U_1 is of class C^∞. We have also that if X_i has Lipschitz constant L, then $X_i * \eta_\epsilon$ has also Lipschitz constant L. Observe that:

$$\begin{aligned}
\nabla \cdot X^1_\epsilon(x) &= \nabla \cdot (X_1 * \eta_\epsilon, ..., X_n * \eta_\epsilon) = \frac{\partial}{\partial x_1}(X_1 * \eta_\epsilon) + ... + \frac{\partial}{\partial x_n}(X_n * \eta_\epsilon) \\
&= \frac{\partial}{\partial x_1}\left(\int_{\mathbb{R}^n} X_1(y)\eta_\epsilon(x-y)\,d\nu(y)\right) + ... + \frac{\partial}{\partial x_n}\left(\int_{\mathbb{R}^n} X_n(y)\eta_\epsilon(x-y)\,d\nu(y)\right) \\
&= \int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_1}X_1(y)\right)\eta_\epsilon(x-y)\,d\nu(y) + ... + \int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_n}X_n(y)\right)\eta_\epsilon(x-y)\,d\nu(y) \\
&= \int_{\mathbb{R}^n}\left(\frac{\partial}{\partial x_1}X_1(y) + ... + \frac{\partial}{\partial x_n}X_n(y)\right)\eta_\epsilon(x-y)\,d\nu(y) = \int_{\mathbb{R}^n}(\nabla \cdot X(y))\,\eta_\epsilon(x-y)\,d\nu(y) = 0.
\end{aligned}$$

Since $\lim_{\epsilon\to 0} \eta_\epsilon(x) = \delta(x)$, where δ is the Dirac delta function, and $X_i * \delta = X_i$ we have that X^1_ϵ converges locally to X in the C^0-topology.
Global argument: Take a partition of the unit $\{\xi_i\}_{i=1}^k$ subordinated to $\{U_i\}_{i=1}^k$ and supported in V_i for each $i = 1, ..., k$. Consider an open set $W_i \subset V_i \subset U_i$ such that $\varphi_i(W_i) = B(0, \frac{1}{3})$, $\varphi_i(V_i) = B(0, \frac{2}{3})$, $\xi_i|_{W_i} = 1$ and $\Omega := M \setminus int(\cup_{i=1}^k W_i)$ is a manifold with C^∞ boundary. By the local argument, we consider, for $i = 1, ..., k$, local defined vector fields X^i_ϵ arbitrarily C^0-close to X. Let us define a (not necessarily divergence-free) C^∞ vector field in the whole M by

$$Y(x) := \sum_{i=1}^k \xi_i(x) X^i_\epsilon(x),$$

and the C^∞ map $g(x) = \nabla \cdot Y(x)$. We claim that $\int_\Omega g(x)\,d\nu(x) = 0$. Indeed, using the Divergence Theorem twice,

$$\begin{aligned}
\int_\Omega g(x)\,d\nu(x) &= \int_\Omega \nabla \cdot Y(x)\,d\nu(x) = \int_{\partial\Omega} Y \cdot \vec{n}(x)\,dS_{n-1}(x) = \sum_{i=1}^k \int_{\partial W_i} \xi_i(x) X^i_\epsilon \cdot \vec{n}(x)\,dS_{n-1}(x) \\
&= \sum_{i=1}^k \int_{\partial W_i} X^i_\epsilon \cdot \vec{n}(x)\,dS_{n-1}(x) = \sum_{i=1}^k \int_{W_i} \nabla \cdot X^i_\epsilon(x)\,d\nu(x) = 0.
\end{aligned}$$

We are in the conditions of applying Theorem 2.1. So there exists a C^∞ vector field $V : \Omega \to T_\Omega M$ such that $\nabla \cdot V(x) = g(x)$ for $x \in \Omega$, $V(x) = 0$ on $\partial\Omega$ and C^∞ at $\partial\Omega$. Notice that:

$$
\begin{aligned}
g(x) &= \nabla \cdot Y(x) = \nabla \cdot \sum_{i=1}^{k} \xi_i(x) X_\epsilon^i(x) = \sum_{i=1}^{k} \Big(\nabla \xi_i(x) \cdot X_\epsilon^i(x) + \xi_i(x) \nabla \cdot X_\epsilon^i(x) \Big) = \sum_{i=1}^{k} \nabla \xi_i(x) \cdot X_\epsilon^i(x) \\
&= \sum_{i=1}^{k-1} \nabla \xi_i(x) \cdot X_\epsilon^i(x) + \nabla \xi_k(x) \cdot X_\epsilon^k(x) = \sum_{i=1}^{k-1} \nabla \xi_i(x) \cdot X_\epsilon^i(x) + \left(\nabla (1 - \sum_{i=1}^{k-1} \xi_i(x)) \right) \cdot X_\epsilon^k(x) \\
&= \sum_{i=1}^{k-1} \nabla \xi_i(x) \cdot (X_\epsilon^i(x) - X_\epsilon^k(x)).
\end{aligned}
$$

Therefore, if we take $X_\epsilon^i(x)$ $(i = 1, \dots, k)$ sufficiently C^0-close to X, then we obtain g arbitrarily C^0-close to zero. We also have that V is C^0-close to zero. It is also easy to see that Y is C^0-close to X.

Finally, we define $Z := Y - V$. Clearly, $\nabla \cdot Z = \nabla \cdot Y - \nabla \cdot V = \nabla \cdot Y - g = 0$ and $Z \in \mathfrak{X}_\nu^\infty(M)$, since V is C^∞ at the boundary of Ω. Since Y is C^0-arbitrarily close to X and V is C^0-arbitrarily close to the zero vector field we obtain that Z is C^0-arbitrarily close to X.

Now, we pick any small cube $\mathcal{C}$ and we claim that $\nu(\mathcal{C}) = \nu(X^\tau(\mathcal{C}))$. So, consider a sequence $\{Z_n\} \subset \mathfrak{X}_\nu^\infty(M)$ such that $Z_n \to X$ in the C^0 sense. Clearly $\nu(\mathcal{C}) = \nu(Z_n^\tau(\mathcal{C}))$ for all n. Since X^τ is Lipschitz we have $\nu(X^\tau(\partial\mathcal{C})) = 0$ guaranteeing no raise of volume on the boundary and so $\nu(\mathcal{C}) = \nu(X^\tau(\mathcal{C}))$. □

3. Proof of the main result

Once again we appeal to another very useful result by Dacorogna and Moser to obtain our main theorem, i.e. a conservative local change of coordinates that trivializes the action of the flow.

Theorem 3.1. *(Dacorogna and Moser [11, Theorem 1]) Let $\Omega = B(x, r)$ and $f, g \in C^{0,1}(\overline{\Omega})$ two positive functions. Then, there exists a diffeomorphism[1] φ with $\varphi, \varphi^{-1} \in C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^n)$, where $\alpha < 1$, satisfying*

$$g(\varphi(x)) \det D\varphi_x = \lambda f(x), \tag{1}$$

for all $x \in \Omega$ where $\lambda = \int g / \int f$. We also have $\varphi = Id$ at $\partial\Omega$.

We say that two vector fields $X_1 \colon U_1 \to TU_1$ and $X_2 \colon U_2 \to TU_2$ are *locally topologically conjugate near* $p_1 \in U_1$ and $p_2 \in U_2$ if there exist two open neighborhoods $O_i \ni p_i$ $(i = 1, 2)$ and a homeomorphism $\phi \colon O_1 \to O_2$ with $\phi(p_1) = p_2$ such that for any $x \in O_1$ and a small interval I containing 0 the integral curve $\sigma_x \colon I \to O_1$ defined by $\sigma_x(0) = x$ and $\frac{d}{dt}\sigma_x(t) = X_1(\sigma_x(t))$ for all $t \in I$ (i.e. defined by $X_1^t(x)$ for $t \in I$) is a solution associated with X_1 if and only if the integral curve $\phi \circ \sigma_x \colon I \to O_2$ is a solution associated with X_2.

Theorem 1. *(Flowbox theorem for Lipschitz divergence-free vector fields)*
Let be given $X \in \mathfrak{X}_\nu^{0,1}(M)$, a non-singular point $p_1 \in M$ and the trivial vector field $T(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) = (1, 0, \dots, 0)$ on canonical coordinates $(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$ of $\mathbb{R}^n$.

(i) *Then, X and T are locally topologically conjugate near p_1 and $p_2 = \hat{0}$. The homeomorphism ϕ that gives the conjugacy is a lipeomorphism.*
(ii) *Then, X and $T_c = cT$ are locally topologically (volume-preserving) conjugate near p_1 and $p_2 = \hat{0}$ for some $c = c(X, p_1) > 0$. The homeomorphism Φ, which gives the conjugacy, is a volume-preserving lipeomorphism.*

Proof. The item (i) is precisely [8, Theorem 4] where $\phi \colon O_1 \ni p_1 \to O_2 \ni \hat{0}$. Assume, using Moser coordinates [15], that X evolves in $\mathbb{R}^n$ with coordinates $(x_1, \dots, x_n)$, $p_1 = 0$ and $X(p_1) \in \{(x, 0, \dots, 0) \in \mathbb{R}^n \colon x \in \mathbb{R}\}$. Like in [8] we let $\Pi = X(p_1)^\perp$. Take $r > 0$ sufficiently small such that $\Omega := B(0, r) \subset \Pi$ and $\Omega \subset O_1$. Using the same notation as in [8] for each $x \in O_1$, there exists a unique $t_x \in (-T, T)$ such that $\sigma_x(-t_x)$ is in a very small $(n-1)$-dimensional ball centered in 0 inside Π. The lipeomorphism ϕ is defined by $\phi(x) = \sigma_x(-t_x) + t_x(1, 0, \dots, 0)$ and so, we have $\phi(\Omega) = \hat{\Omega}$ where $\hat{\Omega} := (1, 0, \dots, 0)^\perp$ in coordinates $(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$. We have $\phi^{-1}(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) = X^{x_1}(0, \hat{x}_2, \dots, \hat{x}_n)$.

We define the C^∞ function

$$
\begin{array}{rccc}
f \colon & \hat{\Omega} & \longrightarrow & \mathbb{R} \\
& (\hat{x}_2, \dots, \hat{x}_n) & \longmapsto & 1
\end{array}
$$

and the Lipschitz continuous function

[1] The optimal gain in smoothness from $f, g \in C^{0,1}$ to $\varphi \in C^{1,1}$ (typical of Dacorogna and Moser theorem) is not assured in the Lipschitz case. Nevertheless, $C^{1,\alpha}$ regularity is sufficient for our purposes.

$$\begin{array}{rccc} g: & \Omega & \longrightarrow & \mathbb{R} \\ & (x_2,\dots,x_n) & \longmapsto & X_1(0,x_2,\dots,x_n), \end{array}$$

where $X_1(x_1,x_2,\dots,x_n)$ is the projection into the first coordinate of $X(x_1,x_2,\dots,x_n)$. Since the functions f and g have the regularity required in Theorem 3.1, we apply this theorem to $\hat{\Omega}=B(0,r)\subseteq\mathbb{R}^{n-1}$ so there exists a $C^{1,\alpha}$ diffeomorphism ($\alpha<1$)

$$\begin{array}{rccc} \varphi: & \hat{\Omega} & \longrightarrow & \varphi(\hat{\Omega})=\Omega\subseteq\mathbb{R}^{n-1} \\ & (\hat{x}_2,\dots,\hat{x}_n) & \longmapsto & (\varphi_1(\hat{x}_2,\dots,\hat{x}_n),\dots,\varphi_{n-1}(\hat{x}_2,\dots,\hat{x}_n)) \end{array}$$

satisfying the partial differential equation

$$g(\varphi(\hat{x}_2,\dots,\hat{x}_n))\det D\varphi_{(\hat{x}_2,\dots,\hat{x}_n)}=\lambda, \tag{2}$$

for all $(\hat{x}_2,\dots,\hat{x}_n)\in\hat{\Omega}$ where $\lambda=\int g/\int 1$, and $\varphi|_{\partial\hat{\Omega}}$ is the identity. Now, we define the change of coordinates by:

$$\begin{array}{rccc} \Psi: & O_2\subset\mathbb{R}\times\hat{\Omega} & \longrightarrow & O_1 \\ & \hat{x}=(\hat{x}_1,\hat{x}_2,\dots,\hat{x}_n) & \longmapsto & X^{\lambda^{-1}\hat{x}_1}((0,\varphi(\hat{x}_2,\dots,\hat{x}_n)) \end{array}$$

where $X^{\lambda^{-1}\overline{x}_1}((0,\varphi(\hat{x}_2,\dots,\hat{x}_n))=(X_1^{\lambda^{-1}\hat{x}_1}(0,\varphi(\hat{x}_2,\dots,\hat{x}_n)),\dots,X_n^{\lambda^{-1}\hat{x}_1}(0,\varphi(\hat{x}_2,\dots,\hat{x}_n)))$. Observe that O_1 can diminish due to the parameter λ (we keep the notation O_1). Since Ψ is the composition of a $C^{1,\alpha}$ diffeomorphism and a lipeomorphism, we have that Ψ is of class $C^{0,1}$. Thus we can apply Rademacher's theorem and take derivatives for ν-a.e. We begin by claiming that:

$$\det D\Psi_{(0,\hat{x}_2,\dots,\hat{x}_n)}=1 \text{ for } \nu\text{-a.e. } (0,\hat{x}_2,\dots,\hat{x}_n)\in O_2. \tag{3}$$

Note that, taking $x_1=0$ for $j=2,\dots,n$ and $i=2,\dots,n$, we have the partial derivative at $(0,\hat{x}_2,\dots,\hat{x}_n)$ given by:

$$\frac{\partial}{\partial\hat{x}_i}X_j^{\lambda^{-1}\hat{x}_1}(0,\varphi(\hat{x}_2,\dots,\hat{x}_n))=\frac{\partial\varphi_j}{\partial\hat{x}_i}(\hat{x}_2,\dots,\hat{x}_n), \tag{4}$$

and for $j=1$ and $i=2,\dots,n$ we have the partial derivative at $(0,\hat{x}_2,\dots,\hat{x}_n)$ given by:

$$\frac{\partial}{\partial\hat{x}_i}X_1^{\lambda^{-1}\hat{x}_1}(0,\varphi(\hat{x}_2,\dots,\hat{x}_n))=0. \tag{5}$$

Let us compute the derivatives when $t=\hat{x}_1=0$. Taking into account that the first column is the time-derivative of a flow i.e. the vector field, and also (4) and (5), we obtain,

$$D\Psi_{(0,\hat{x}_2,\dots,\hat{x}_n)}=\begin{pmatrix} \lambda^{-1}X_1(X^0((0,\varphi(\hat{x}_2,\dots,\hat{x}_n))) & 0 & \dots & 0 \\ \lambda^{-1}X_2(X^0((0,\varphi(\hat{x}_2,\dots,\hat{x}_n))) & \frac{\partial\varphi_1}{\partial\hat{x}_2}|_{(\hat{x}_2,\dots,\hat{x}_n)} & \dots & \frac{\partial\varphi_1}{\partial\hat{x}_n}|_{(\hat{x}_2,\dots,\hat{x}_n)} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda^{-1}X_n(X^0((0,\varphi(\hat{x}_2,\dots,\hat{x}_n))) & \frac{\partial\varphi_{n-1}}{\partial\hat{x}_2}|_{(\hat{x}_2,\dots,\hat{x}_n)} & \dots & \frac{\partial\varphi_{n-1}}{\partial\hat{x}_n}|_{(\hat{x}_2,\dots,\hat{x}_n)} \end{pmatrix}. \tag{6}$$

Using (2) and Laplace's expansion along the first line, we conclude that,

$$\det(D\Psi_{(0,\hat{x}_2,\dots,\hat{x}_n)})=\lambda^{-1}X_1((0,\varphi(\hat{x}_2,\dots,\hat{x}_n)))\det D\varphi_{(\hat{x}_2,\dots,\hat{x}_n)}=g(\varphi(\hat{x}_2,\dots,\hat{x}_n))\lambda^{-1}\det D\varphi_{(\hat{x}_2,\dots,\hat{x}_n)}=1,$$

therefore (3) is proved. Now we will check that $\det D\Psi_{(\hat{x}_1^0,\hat{x}_2^0,\dots,\hat{x}_n^0)}=1$ for ν-a.e. $(\hat{x}_1^0,\hat{x}_2^0,\dots,\hat{x}_n^0)\in O_2$. Notice that

$$\Psi(\hat{x}_1,\hat{x}_2,\dots,\hat{x}_n)=X^{\lambda^{-1}\hat{x}_1^0}[X^{\lambda^{-1}(\hat{x}_1-\hat{x}_1^0)}((0,\varphi(\hat{x}_2,\dots,\hat{x}_n)))]=X^{\lambda^{-1}\hat{x}_1^0}[\Psi(\hat{x}_1-\hat{x}_1^0,\hat{x}_2,\dots,\hat{x}_n)],$$

so, for ν-a.e. point, we have (modulo an 'identification' on the fiber bundle) that

$$D\Psi_{(\hat{x}_1,\hat{x}_2,\dots,\hat{x}_n)}=DX^{\lambda^{-1}\hat{x}_1^0}_{\Psi(\hat{x}_1-\hat{x}_1^0,\hat{x}_2,\dots,\hat{x}_n)}D\Psi_{(\hat{x}_1-\hat{x}_1^0,\hat{x}_2,\dots,\hat{x}_n)}. \tag{7}$$

Evaluating $D\Psi_{(\hat{x}_1,\hat{x}_2,\dots,\hat{x}_n)}$ at $\hat{x}_1=\hat{x}_1^0$ we get:

$$D\Psi_{(\hat{x}_1^0,\hat{x}_2,\dots,\hat{x}_n)}=DX^{\lambda^{-1}\hat{x}_1^0}_{\Psi(0,\hat{x}_2,\dots,\hat{x}_n)}D\Psi_{(0,\hat{x}_2,\dots,\hat{x}_n)}. \tag{8}$$

Since X^t is volume-preserving, using (3) and Proposition 1, we conclude that $\det D\Psi_{(\hat{x}_1^0,\hat{x}_2^0,\dots,\hat{x}_n^0)}=1$ for ν-a.e. Finally, take $\Phi=\Psi^{-1}$. $\square$

Remark 3.1. *Consider the constant vector field* $T_c := (\lambda, 0, \ldots, 0)$ *(say* $c = \lambda$*). Let* $(x_1, x_2, \ldots, x_n) = \Psi(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$*. Recalling that* $DX_x^t \cdot X(x) = X(X^t(x))$*, for* ν*-a.e.* $(x_1, x_2, \ldots, x_n)$*, we obtain:*

$$\begin{aligned}\Psi_* T_c(x_1, x_2, \ldots, x_n) &= D\Psi_{(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)} T_c \,\Psi^{-1}(x_1, x_2, \ldots, x_n) = D\Psi_{(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)} T_c(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) \\ &\overset{(8)}{=} DX^{\lambda^{-1}\hat{x}_1}_{\Psi(0, \hat{x}_2, \ldots, \hat{x}_n)} D\Psi_{(0, \hat{x}_2, \ldots, \hat{x}_n)}(\lambda, 0, \ldots, 0) \overset{(6)}{=} DX^{\lambda^{-1}\hat{x}_1}_{\Psi(0, \hat{x}_2, \ldots, \hat{x}_n)} X(\Psi(0, \hat{x}_2, \ldots, \hat{x}_n)) \\ &= X(X^{\lambda^{-1}\hat{x}_1}(\Psi(0, \hat{x}_2, \ldots, \hat{x}_n))) = X(X^{\lambda^{-1}\hat{x}_1}((0, \varphi(\hat{x}_2, \ldots, \hat{x}_n))) = X(\Psi(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)).\end{aligned}$$

Taking $\Phi = \Psi^{-1}$*, we obtain* $T_c = \Phi_* X$*, where the pull-back is defined for a* ν*-a.e. point.*

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