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Newton polytopes and symmetric Grothendieck polynomials

Polytopes de Newton et polynômes symétriques de Grothendieck

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ABSTRACT

Symmetric Grothendieck polynomials are inhomogeneous versions of Schur polynomials that arise in combinatorial K-theory. A polynomial has *saturated Newton polytope* (SNP) if every lattice point in the polytope is an exponent vector. We show that the Newton polytopes of these Grothendieck polynomials and their homogeneous components have SNP. Moreover, the Newton polytope of each homogeneous component is a permutahedron. This addresses recent conjectures of C. Monical–N. Tokcan–A. Yong and of A. Fink–K. Mészáros–A. St. Dizier in this special case.

RÉSUMÉ

Les polynômes symétriques de Grothendieck sont des versions inhomogènes des polynômes de Schur qui apparaissent dans la K-théorie combinatoire. Un polynôme a un polytope de Newton saturé (SNP) si chaque point entier dans le polytope est un vecteur d'exposant. Nous montrons que les polytopes de Newton de ces polynômes de Grothendieck et leurs composants homogènes ont un SNP. En outre, le polytope de Newton de chaque composant homogène est un permutoèdre. Cela concerne les récentes conjectures de C. Monical–N. Tokcan–A. Yong et de A. Fink–K. Mészáros–A. St. Dizier dans ce cas spécial.

Let $s_\lambda(x_1,\dots,x_n)$ be the *Schur polynomial*, which is the generating series for semistandard Young tableaux of shape λ with entries in $[n] := \{1,2,\dots,n\}$. By the work of C. Lenart [3, Theorem 2.2], the **symmetric Grothendieck polynomial** is given by

$$G_\lambda(x_1,\dots,x_n) = \sum_\mu a_{\lambda\mu} s_\mu(x_1,\dots,x_n). \tag{1}$$

The sum is over partitions μ (identified with their Young diagrams in English notation) with $\leq n$ rows. The quantity $(-1)^{|\mu|-|\lambda|}a_{\lambda,\mu}$ counts the number of rows and columns strictly increasing skew tableaux of shape μ/λ with entries in $[n]$ such that the entries in row r are weakly less than $r-1$.

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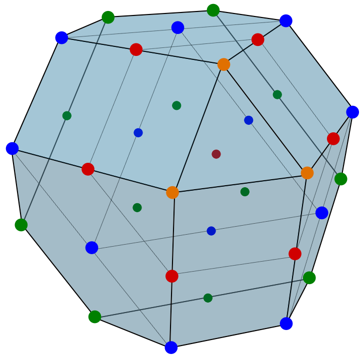

Fig. 1. The Newton polytope of $G_\lambda(x_1, x_2, x_3)$ for $\lambda = (3, 1, 0)$. Each color indicates the degree.

Fig. 1. Le polytope de Newton de $G_\lambda(x_1, x_2, x_3)$ pour $\lambda = (3, 1, 0)$. Chaque couleur indique le degré.

By (1), $G_\lambda(x_1, \dots, x_n)$ is an inhomogeneous deformation of $s_\lambda(x_1, \dots, x_n)$ and is itself symmetric. For example, if $n = 3$ and $\lambda = (3, 1, 0)$,

$$G_\lambda(x_1, x_2, x_3) = s_{(3,1)} - (2s_{(3,1,1)} + s_{(3,2,0)}) + 2s_{(3,2,1)} - s_{(3,2,2)}.$$

These polynomials appear in the study of the K-theoretic Schubert calculus; we refer the reader to [3,1] and the references therein for additional discussion.

More generally, A. Lascoux–M.-P. Schützenberger [2] recursively defined (possibly nonsymmetric) Grothendieck polynomials associated with any permutation $w \in \mathfrak{S}_n$. We mention that A. Buch [1] discovered the *set-valued tableaux* formula for $G_\lambda(x_1, \dots, x_n)$; this formula is often taken as a definition in the literature. (Recently, C. Monical [6] found a bijection between the aforementioned rules of C. Lenart and of A. Buch.)

The **Newton polytope** of a polynomial $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha \in \mathbb{C}[x_1, \dots, x_n]$ is the convex hull of its exponent vectors, i.e. $\mathsf{Newton}(f) = \mathsf{conv}(\{\alpha : c_\alpha \neq 0\}) \subseteq \mathbb{R}^n$. In [7], f is said to have **saturated Newton polytope** (SNP) if $c_\alpha \neq 0$ whenever $\alpha \in \mathsf{Newton}(f)$. A study of SNP and algebraic combinatorics was given in [7].

If $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0)$, the **permutahedron** $\mathcal{P}_\lambda \subseteq \mathbb{R}^n$ is the convex hull of the $\mathfrak{S}_n$-orbit of λ. This theorem extends the old fact that $\mathsf{Newton}(s_\lambda(x_1, \dots, x_n)) = \mathcal{P}_\lambda$:

Theorem 0.1. *$G_\lambda(x_1, \dots, x_n)$ has SNP. In addition, each homogeneous component has SNP with Newton polytope (see Fig. 1) being a permutahedron (as specified below in (3)).*

The first assertion addresses [7, Conjecture 5.5] for the case when the permutation w is **Grassmannian** at position n; that is $w(i) < w(i + 1)$ unless $i = n$. The second assertion responds, in this case, to a conjecture of A. Fink–K. Mészáros–A. St. Dizier [5, Conjecture 5.1]. In [5], these conjectures were proved for the case when $w = 1w'$, where w' is a **dominant permutation**, i.e., w' is 132-avoiding.

Proof of the Theorem: Let $\mu^{(0)} := \lambda$. For $1 \leq k \leq n$ define $\mu^{(k)}$ to be $\mu^{(k-1)}$ with a box added in the northmost row r such that $\mu^{(k-1)}_r - \mu^{(0)}_r < r - 1$ and the addition of the box gives a Young diagram. Stop when no such r exists or $k = n$. Suppose we obtain N such partitions.

Recall that the **dominance order** on partitions of a fixed size is defined by $\theta \leq_D \delta$ if $\sum_{j=1}^t \theta_j \leq \sum_{j=1}^t \delta_j$ for $t \geq 1$.

Claim A 1. *$\mu^{(k)}$ is the $\leq_D$-maximum among all shapes μ of size $|\lambda| + k$ such that $a_{\lambda,\mu} \neq 0$.*

Proof of Claim A: The skew shape $\mu^{(k)}/\lambda$ consists of the k boxes added to λ. We can inductively define a skew tableau T_k of this shape by adding the minimum possible label to T_{k-1} (in the box $\mu^{(k)}/\mu^{(k-1)}$) that maintains row and column strictness. It is straightforward that this tableau exists and witnesses $a_{\lambda,\mu^{(k)}} \neq 0$.

Let μ be a shape such that $a_{\lambda,\mu} \neq 0$. Then $\mu_i \leq \lambda_i + (i - 1)$ for all i. Suppose that $\mu \not\leq_D \mu^{(k)}$ and let $r (> 1)$ be the first row such that $\mu_1 + \cdots + \mu_r > \mu^{(k)}_1 + \cdots + \mu^{(k)}_r$. Then

$$\mu_1 + \cdots + \mu_{r-1} \leq \mu^{(k)}_1 + \cdots + \mu^{(k)}_{r-1} \text{ and } \mu_r > \mu^{(k)}_r.$$

This contradicts the construction of $\mu^{(k)}$ because by $\mu^{(k)}_r - \mu^{(0)}_r < \mu_r - \mu^{(0)}_r \leq r - 1$ $\mu^{(k)}$ must have another box in row r. $\square$

R. Rado's theorem [8] states that for two partitions θ, δ of the same size,

$$\mathcal{P}_\theta \subseteq \mathcal{P}_\delta \iff \theta \leq_D \delta. \tag{2}$$

The Theorem's second assertion is immediate from (2) and Claim A. In fact, if $G_\lambda[k]$ denotes the degree k homogeneous component of $G_\lambda(x_1,\dots,x_n)$, then

$$\mathrm{Newton}(G_\lambda[k]) = \mathcal{P}_{\mu^{(k)}}. \tag{3}$$

Let $v^{(k)} \in \mathcal{P}_{\mu^{(k)}}$. Thus $v^{(k)}$ is a nonnegative vector (being a convex combination of nonnegative vectors). Rado's theorem implies that $v^{(k)}$ is *majorized* by $\mu^{(k)}$. That is, the rearrangement $(v^{(k)})^{\downarrow}$ of the components of $v^{(k)}$ into decreasing order satisfies

$$(v^{(k)})^{\downarrow} \leq_D \mu^{(k)}. \tag{4}$$

Suppose

$$v = \sum_{k=0}^{N} c_k v^{(k)} \quad \text{where} \quad \sum_{k=0}^{N} c_k = 1 \text{ and } c_k \geq 0$$

is a convex combination of the vectors $v^{(k)}$.

Claim B 1. *v is majorized by $\overline{\mu} := \sum_{k=0}^{N} c_k \mu^{(k)}$.*

Proof of Claim B: Let $v^\star := \sum_{k=0}^{N} c_k (v^{(k)})^{\downarrow}$. By (4), for any $t \geq 1$ we have $c_k \sum_{j=1}^{t} (v^{(k)})_j^{\downarrow} \leq c_k \sum_{j=1}^{t} \mu_j^{(k)}$. By summing both sides over all k and interchanging the order of summation, we conclude $v^\star$ is majorized by $\overline{\mu}$. It is a standard property of majorization that $a+b$ is majorized by $a^{\downarrow} + b^{\downarrow}$ [4, Proposition A.1.b]. Thus v is majorized by $v^\star$. Now use that this majorization (being a preorder) is transitive. □

Claim C 1. *Suppose $|\overline{\mu}| - |\mu^{(0)}| = K$, then $\overline{\mu}$ is majorized by $\mu^{(K)}$.*

Proof of Claim C: Let $r_k :=$ row on which k-th box gets added to $\mu^{(k)}$ (so $\mu^{(k)} = \mu^{(0)} + e_{r_1} + \cdots + e_{r_k}$, where e_i is a standard basis vector).

Lemma 0.2. *For any (row) r*

$$\overline{\mu}_1 + \cdots + \overline{\mu}_r = \mu_1^{(0)} + \cdots + \mu_r^{(0)} + c_1 + 2c_2 + \cdots + \ell c_\ell + \cdots + \ell c_N,$$

where ℓ is the largest i such that $r_i = r$.

Proof: Suppose we added boxes $a, a+1, \dots, a+b$ to row r of $\mu^{(0)}$ in order to obtain $\mu^{(N)}$.

We write

$$\overline{\mu}_r = \underbrace{c_0\mu_r^{(0)} + \cdots + c_{a-1}\mu_r^{(a-1)}}_{①} + \underbrace{c_a\mu_r^{(a)} + \cdots + c_{a+b}\mu_r^{(a+b)}}_{②} + \underbrace{c_{a+b+1}\mu_r^{(a+b+1)} + \cdots + c_N\mu_r^{(N)}}_{③}.$$

Next, $\mu_r^{(0)} = \cdots = \mu_r^{(a-1)}$, so

$$① = (c_0 + \cdots + c_{a-1})\mu_r^{(0)}.$$

Since $\mu_r^{(a+i)} = \mu_r^{(0)} + (i+1)$ for $i = 0, \dots, b$, then

$$② = (c_a + \cdots + c_{a+b})\mu_r^{(0)} + (c_a + 2c_{a+1} + \cdots + (b+1)c_{a+b}).$$

Finally, $\mu_r^{(a+b)} = \mu_r^{(a+b+1)} = \cdots = \mu_r^{(n)}$, so

$$③ = (c_{a+b+1} + \cdots + c_N)\mu_r^{(0)} + (c_{a+b+1} + \cdots + c_N)(b+1).$$

Therefore, we conclude that $\overline{\mu}_r = \mu_r^{(0)} + c_a + 2c_{a+1} + \cdots + (b+1)c_{a+b} + \cdots + (b+1)c_N$. The lemma then follows by a simple induction. □

The following is immediate from the definitions.

Lemma 0.3. *Let $b_r = \mu_r^{(N)} - \mu_r^{(0)}$, i.e. b_r is the number of extra boxes $\mu^{(N)}$ has in row r. For any $r < r_k$*

$$\mu_1^{(k)} + \cdots + \mu_r^{(k)} = \mu_1^{(0)} + \cdots + \mu_r^{(0)} + (b_1 + \cdots + b_r).$$

Let ℓ be the largest i such that $r_i = r$. We consider two cases.

Case 1 ($r < r_K$): Observe that

$$c_1 + 2c_2 + \cdots + \ell c_\ell + \cdots + \ell c_N = (c_1 + \cdots + c_N) + (c_2 + \cdots + c_N) + \cdots + (c_\ell + \cdots + c_N) \le \ell.$$

Since $b_1 + \cdots + b_r$ equals the number of boxes placed from rows 1 through r and the ℓ-th box is the last box placed in row r, then $\ell = b_1 + \cdots + b_r$. Combining this equality with the inequality just derived, we see

$$c_1 + 2c_2 + \cdots + \ell c_\ell + \cdots + \ell c_N \le b_1 + \cdots + b_r. \tag{5}$$

By (5) together with Lemmas 0.2 and 0.3,

$$\overline{\mu}_1 + \cdots + \overline{\mu}_r \le \mu_1^{(K)} + \cdots + \mu_r^{(K)}.$$

Case 2 ($r \ge r_K$): Here, we notice that

$$\mu_1^{(K)} + \cdots + \mu_r^{(K)} = \mu_1^{(0)} + \cdots + \mu_r^{(0)} + K. \tag{6}$$

Observe that

$$c_1 + 2c_2 + \cdots + \ell c_\ell + \cdots + \ell c_N \le c_1 + 2c_2 + \cdots + Nc_N = K, \tag{7}$$

where the equality follows from Lemma 0.2. Apply Lemma 0.2 to the left-hand side of (7) and use (6) to replace K on the right-hand side, to conclude that $\overline{\mu}_1 + \cdots + \overline{\mu}_r \le \mu_1^{(K)} + \cdots + \mu_r^{(K)}$. Hence, in either case, $\overline{\mu} \le_D \mu^{(K)}$, as required. □

Let $w_1, \ldots, w_M$ be any exponent vectors of $G_\lambda(x_1, \ldots, x_n)$. SNPness means that if $w \in \mathrm{conv}\{w_1, \ldots, w_M\}$ is a lattice point, then $[x^w]G_\lambda(x_1, \ldots, x_n)) \neq 0$. Suppose that $|w| = K$. Without loss of generality, $M = N$, and there is a unique vector w_k with $|w_k| = k$. Then by Claim B, w is majorized by $\overline{\mu}$. Claim C says $\overline{\mu}$ is majorized by $\mu^{(K)}$ and hence $w^{\downarrow} \le_D \mu^{(K)}$. By (2) we conclude $w \in \mathcal{P}_{\mu^{(K)}}$, which by (3) completes the proof of the Theorem. Indeed, we have shown that $\mathrm{Newton}(G_\lambda(x_1, \ldots, x_n)) = \bigcup_{k=0}^{N} \mathcal{P}_{\mu^{(k)}}$. □

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