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Partial differential equations/Theory of signals

Rigidity of optimal bases for signal spaces

Rigidité des bases optimales pour les espaces de signaux

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A R T I C L E   I N F O

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A B S T R A C T

We discuss optimal L^2-approximations of functions controlled in the H^1-norm. We prove that the basis of eigenfunctions of the Laplace operator with Dirichlet boundary condition is the only orthonormal basis (b_i) of L^2 that provides an optimal approximation in the sense of

$$\left\| f - \sum_{i=1}^{n} (f, b_i) b_i \right\|_{L^2}^2 \le \frac{\|\nabla f\|_{L^2}^2}{\lambda_{n+1}} \quad \forall f \in H_0^1(\Omega), \quad \forall n \ge 1.$$

This solves an open problem raised by Y. Aflalo, H. Brezis, A. Bruckstein, R. Kimmel, and N. Sochen (Best bases for signal spaces, C. R. Acad. Sci. Paris, Ser. I 354 (12) (2016) 1155–1167).

R É S U M É

On s'intéresse à l'approximation optimale pour la norme L^2 de fonctions contrôlées en norme H^1. On prouve que la base des fonctions propres du laplacien avec condition de Dirichlet au bord est l'unique base orthonormale (b_i) de L^2 qui réalise une approximation optimale au sens de

$$\left\| f - \sum_{i=1}^{n} (f, b_i) b_i \right\|_{L^2}^2 \le \frac{\|\nabla f\|_{L^2}^2}{\lambda_{n+1}} \quad \forall f \in H_0^1(\Omega), \quad \forall n \ge 1.$$

Ceci résout un problème ouvert posé par Y. Aflalo, H. Brezis, A. Bruckstein, R. Kimmel et N. Sochen (Best bases for signal spaces, C. R. Acad. Sci. Paris, Ser. I 354 (12) (2016) 1155–1167).

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1. Introduction and main result

This note is a follow-up of the papers by Y. Aflalo, H. Brezis and R. Kimmel [2] and Y. Aflalo, H. Brezis, A. Bruckstein, R. Kimmel and N. Sochen [1].

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. Let $e = (e_i)$ be an orthonormal basis of $L^2(\Omega)$ consisting of the eigenfunctions of the Laplace operator with Dirichlet boundary condition:

$$\begin{cases} -\Delta e_i = \lambda_i e_i & \text{in } \Omega, \\ e_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

where $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ is the ordered sequence of eigenvalues repeated according to their multiplicity.

We first recall a very standard result:

Theorem 1.1. *We have, for all $n \ge 1$,*

$$\left\| f - \sum_{i=1}^{n} (f, e_i) e_i \right\|_{L^2}^2 \le \frac{\| \nabla f \|_{L^2}^2}{\lambda_{n+1}} \qquad \forall f \in H_0^1(\Omega). \tag{2}$$

Here and throughout the rest of this paper $(\cdot, \cdot)$ denotes the scalar product in $L^2(\Omega)$.

Indeed, we may write

$$\left\| f - \sum_{i=1}^{n} (f, e_i) e_i \right\|_{L^2}^2 = \left\| \sum_{i=n+1}^{+\infty} (f, e_i) e_i \right\|_{L^2}^2 = \sum_{i=n+1}^{+\infty} (f, e_i)^2. \tag{3}$$

On the other hand,

$$\|\nabla f\|_{L^2}^2 = \sum_{i=1}^{+\infty} \lambda_i (f, e_i)^2 \ge \sum_{i=n+1}^{+\infty} \lambda_i (f, e_i)^2 \ge \lambda_{n+1} \sum_{i=n+1}^{+\infty} (f, e_i)^2. \tag{4}$$

Combining (3) and (4) yields (2). □

The authors of [2] and [1] have investigated the "optimality" in various directions of the basis (e_i), with respect to inequality (2). Here is one of their results restated in a slightly more general form.

Theorem 1.2 *(Theorem 3.1 in [2]). There is no integer $n \ge 1$, no constant $0 \le \alpha < 1$ and no sequence $(\psi_i)_{1 \le i \le n}$ in $L^2(\Omega)$ such that*

$$\left\| f - \sum_{i=1}^{n} (f, \psi_i) \psi_i \right\|_{L^2}^2 \le \frac{\alpha}{\lambda_{n+1}} \|\nabla f\|_{L^2}^2 \qquad \forall f \in H_0^1(\Omega). \tag{5}$$

The proof in [2] relies on the Fischer–Courant max–min principle (see Remark 3.3 below). For the convenience of the reader, we present a very elementary proof based on a simple and efficient device originally due to H. Poincaré [5, pp. 249–250] (and later rediscovered by many people, e.g., H. Weyl [7, p. 445] and R. Courant [3, pp. 17–18]; see also H. Weinberger [6, p. 56] and P. Lax [4, p. 319]).

Suppose not, and set

$$f = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n + c_{n+1} e_{n+1} \tag{6}$$

where $c = (c_1, c_2, \cdots, c_n, c_{n+1}) \in \mathbb{R}^{n+1}$. The under-determined linear system

$$(f, \psi_i) = 0, \qquad \forall i = 1, \cdots, n \tag{7}$$

of n equations with $n+1$ unknowns admits a non-trivial solution. Inserting f into (5) yields

$$\lambda_{n+1} \sum_{i=1}^{n+1} c_i^2 \le \alpha \sum_{i=1}^{n+1} \lambda_i c_i^2 \le \alpha \lambda_{n+1} \sum_{i=1}^{n+1} c_i^2. \tag{8}$$

Therefore $\sum_{i=1}^{n+1} c_i^2 = 0$ and thus $c = 0$. A contradiction. This proves Theorem 1.2. □

The authors of [1] were thus led to investigate the question of whether inequality (2) holds *only* for the orthonormal bases consisting of eigenfunctions corresponding to ordered eigenvalues. They established that a "discrete", i.e., finite-dimensional, version does hold; see [1, Theorem 2.1] and Remark 3.2 below. But their proof of "uniqueness" could not be adapted to the infinite-dimensional case (because it relied on a "descending" induction). It was raised there as an open problem (see [1, p. 1166]). Our next result solves this problem.

Theorem 1.3. *Let (b_i) be an orthonormal basis of $L^2(\Omega)$ such that, for all $n \geq 1$,*

$$\left\| f - \sum_{i=1}^{n} (f, b_i) b_i \right\|_{L^2}^2 \leq \frac{\|\nabla f\|_{L^2}^2}{\lambda_{n+1}} \quad \forall f \in H_0^1(\Omega). \tag{9}$$

Then, (b_i) consists of an orthonormal basis of eigenfunctions of $-\Delta$ with corresponding eigenvalues (λ_i).

2. Proof of Theorem 1.3

A basic ingredient of the argument is the following lemma:

Lemma 2.1. *Assume that (9) holds for all $n \geq 1$ and all $f \in H_0^1(\Omega)$, and that*

$$\lambda_i < \lambda_{i+1} \tag{10}$$

for some $i \geq 1$. Then

$$(b_j, e_k) = 0, \qquad \forall j, k \text{ such that } 1 \leq j \leq i < k. \tag{11}$$

Proof. Fix $k > i$. Let l be the largest integer $l \leq k-1$ such that

$$\lambda_l < \lambda_{l+1}. \tag{12}$$

Clearly

$$i \leq l \tag{13}$$

and

$$\lambda_{l+1} = \lambda_{l+2} = \cdots = \lambda_k. \tag{14}$$

Applying (9) for $n = l$, we get

$$\left\| f - \sum_{i=1}^{l} (f, b_i) b_i \right\|_{L^2}^2 \leq \frac{\|\nabla f\|_{L^2}^2}{\lambda_{l+1}} \quad \forall f \in H_0^1(\Omega). \tag{15}$$

We use again Poincaré's "magic trick". Take f of the form

$$f = c_1 e_1 + \cdots + c_l e_l + c e_k \tag{16}$$

such that

$$(f, b_j) = 0 \qquad \forall j = 1, \cdots, l. \tag{17}$$

This is a system of l linear equations with $l+1$ unknowns, so that there are nontrivial solutions. We may as well assume that

$$c_1^2 + \cdots + c_l^2 + c^2 = 1. \tag{18}$$

By (15) and (14), we have

$$\lambda_{l+1} \leq \lambda_1 c_1^2 + \cdots + \lambda_l c_l^2 + \lambda_k c^2 = \lambda_1 c_1^2 + \cdots + \lambda_l c_l^2 + \lambda_{l+1} c^2. \tag{19}$$

From (18) we get

$$\lambda_{l+1}(c_1^2 + \cdots + c_l^2) \leq \lambda_1 c_1^2 + \cdots + \lambda_l c_l^2. \tag{20}$$

Thus

$$(\lambda_{l+1} - \lambda_1) c_1^2 + \cdots + (\lambda_{l+1} - \lambda_l) c_l^2 \leq 0. \tag{21}$$

By (12) the coefficients $\lambda_{l+1} - \lambda_i$ are positive for every $i = 1, \cdots, l$. Therefore

$$c_1 = \cdots = c_l = 0. \tag{22}$$

Hence $c = \pm 1$ so that $f = \pm e_k$ and by (17)

$$(b_j, e_k) = 0 \qquad \forall j = 1, \cdots, l. \tag{23}$$

The conclusion follows from (23) and (13). $\square$

Before we present the proof in the general case, for the convenience of the reader, we start with the case of simple eigenvalues. Since $\lambda_1 < \lambda_2$ then, by the lemma,

$$(b_1, e_k) = 0 \qquad \forall k \geq 2. \tag{24}$$

Thus $b_1 = \pm e_1$. Next we apply the lemma with $\lambda_2 < \lambda_3$. We have that

$$(b_2, e_k) = 0 \qquad \forall k \geq 3. \tag{25}$$

Also, we have that

$$(b_2, e_1) = \pm(b_2, b_1) = 0. \tag{26}$$

Therefore $b_2 = \pm e_2$. Similarly, we have that $b_i = \pm e_i$ for $i \geq 3$.

We now turn to the general case:

Proof of Theorem 1.3. As above we have $b_1 = \pm e_1$. Consider the first index $i \geq 2$ such that $\lambda_i < \lambda_{i+1}$. Call it i_1. From the lemma we have that

$$(b_j, e_k) = 0 \qquad \forall j, k \text{ such that } 1 \leq j \leq i_1 < k. \tag{27}$$

Therefore $b_2, \cdots, b_{i_1} \in \operatorname{span}(e_2, \cdots, e_{i_1})$. Hence, each b_j with $2 \leq j \leq i_1$ is an eigenfunction of $-\Delta$ with corresponding eigenvalue $\lambda = \lambda_2 = \cdots = \lambda_{i_1}$. Therefore, due to dimensions, $b_2, \cdots, b_{i_1}$ is an orthonormal basis of

$$\operatorname{span}(b_2, \cdots, b_{i_1}) = \operatorname{span}(e_2, \cdots, e_{i_1}) = \ker(-\Delta - \lambda_{i_1} I); \tag{28}$$

in particular each

$$e_k \in \operatorname{span}(b_1, \cdots, b_{i_1}) \qquad k = 1, \cdots, i_1. \tag{29}$$

Consider the next block

$$\lambda = \lambda_{i_1+1} = \cdots = \lambda_{i_2} < \lambda_{i_2+1}. \tag{30}$$

From the lemma we have that

$$(b_j, e_k) = 0 \qquad \forall j, k \text{ such that } 1 \leq j \leq i_2 < k. \tag{31}$$

We also know that for $j \geq i_1 + 1$,

$$(b_j, e_k) = 0 \qquad k = 1, \cdots, i_1 \tag{32}$$

because of (29). Combining (31) and (32) yields

$$(b_j)_{i_1+1 \leq j \leq i_2} \in \operatorname{span}(e_j)_{i_1+1 \leq j \leq i_2}. \tag{33}$$

As above, we conclude, using (30), that $b_{i_1+1}, \cdots, b_{i_2}$ is an orthonormal basis of

$$\operatorname{span}(b_j)_{i_1+1 \leq j \leq i_2} = \operatorname{span}(e_j)_{i_1+1 \leq j \leq i_2} = \ker(-\Delta - \lambda_{i_2} I). \tag{34}$$

Similarly for the next blocks. $\square$

3. Final remarks

Remark 3.1. We call the attention of the reader to the fact that the functions b_i are only assumed to be in $L^2(\Omega)$ and we deduce from Theorem 1.3 that (surprisingly) they belong to $H_0^1(\Omega) \cap \mathcal{C}^\infty(\Omega)$.

Remark 3.2. Theorem 1.3 holds in a more general setting. Let V and H be Hilbert spaces such that $V \subset H$ with compact and dense inclusion ($\dim H \leq +\infty$). Let $a : V \times V \to \mathbb{R}$ be a continuous bilinear symmetric form for which there exist constants $C, \alpha > 0$ such that, for all $v \in V$,

$$a(v, v) \geq 0,$$
$$a(v, v) + C|v|_H^2 \geq \alpha \|v\|_V^2.$$

Let $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ be the sequence of eigenvalues associated with the orthonormal (in H) eigenfunctions $e_1, e_2, \cdots \in V$, i.e.,

$$a(e_i, v) = \lambda_i (e_i, v) \qquad \forall v \in V,$$

where $(\cdot, \cdot)$ denotes the scalar product in H. We point out that, in this general setting, it may happen that $\lambda_1 = 0$ (e.g., $-\Delta$ with Neumann boundary conditions); and λ_1 may have multiplicity > 1. Recall that, for every $n \geq 1$ and $f \in V$:

$$\lambda_{n+1}\left|f-\sum_{i=1}^{n}(e_i,f)e_i\right|_H^2 \le a(f,f). \tag{35}$$

Let (b_i) be an orthonormal basis of H such that for all $n \ge 1$ and $f \in V$

$$\lambda_{n+1}\left|f-\sum_{i=1}^{n}(b_i,f)b_i\right|_H^2 \le a(f,f). \tag{36}$$

Then, (b_i) consists of an orthonormal basis of eigenfunctions of a with corresponding eigenvalues (λ_i). The proof is identical to the one above.

When $\dim H < +\infty$ and $V = H$, this result is originally due to [1]. The proof of rigidity was quite different and could not be adapted to the infinite-dimensional case. It was raised there as an open problem.

Remark 3.3. Recall that the usual Fischer–Courant max–min principle asserts that for every $n \ge 1$, we have

$$\lambda_{n+1} = \max_{\substack{M\subset L^2(\Omega)\\ M \text{ linear space}\\ \dim M = n}} \ \min_{\substack{0\neq f\in H_0^1(\Omega)\\ f\in M^\perp}} \frac{\|\nabla f\|_{L^2}^2}{\|f\|_{L^2}^2}, \tag{37}$$

(see, e.g., [4] or [6]). Our technique sheds some light about the structure of the maximizers in (37). Let (b_i) be an orthonormal sequence in $L^2(\Omega)$ such that, for every $n \ge 1$,

$$\lambda_{n+1} = \min_{\substack{0\neq f\in H_0^1(\Omega)\\ f\in M_n^\perp}} \frac{\|\nabla f\|_{L^2}^2}{\|f\|_{L^2}^2} \qquad \text{where } M_n = \operatorname{span}(b_1, b_2, \cdots, b_n). \tag{38}$$

Then, each b_i is an eigenfunction associated with λ_i. This is an easy consequence of the proof of Theorem 1.3.

Remark 3.4 *(rigidity of the tail).* Assume that (9) holds only for $n = k, k+1, \cdots$. Let the eigenvalues be simple. Applying the same reasoning as in our proof gives

$$\operatorname{span}(b_1, \cdots, b_n) = \operatorname{span}(e_1, \cdots, e_n) \qquad n = k, k+1, \cdots \tag{39}$$

The same argument as before yields $b_i = \pm e_i$ for $i = k+1, k+2, \cdots$. Concerning the b_i's for $i \le k$, we only know that $b_1, \cdots, b_k \in \operatorname{span}(e_1, \cdots, e_k)$ and therefore they are smooth. A similar result holds if the eigenvalues are not simple.

Remark 3.5. We now turn to the reverse situation, i.e., we assume that (9) holds only for $1 \le n \le k$. In this case (9) yields very little information on the b_i's. Consider for example the case $n = k = 1$. In other words, assume that $b = b_1 \in L^2(\Omega)$ is such that $\|b\|_{L^2} = 1$ and

$$\| f - (f,b)b\|_{L^2}^2 \le \frac{1}{\lambda_2}\|\nabla f\|_{L^2}^2 \qquad \forall f \in H_0^1(\Omega). \tag{40}$$

Of course, (40) holds with $b = e_1$. From Lemma 2.1, we know that (40) implies that

$$(e_2, b) = 0. \tag{41}$$

Clearly, (41) is not sufficient. Indeed, take $b = e_3$. Then, (41) holds but (40) fails for $f = e_1$. We do not have a simple characterization of the functions b satisfying (40). But we can construct a large family of functions b (which need not be smooth) such that (40) holds. Assume that $0 < \lambda_1 \le \lambda_2 < \lambda_3$. Let $\chi \in L^2(\Omega)$ be any function such that

$$(e_1, \chi) = 0, \tag{42}$$

$$(e_2, \chi) = 0, \tag{43}$$

$$\|\chi\|_{L^2}^2 = 1. \tag{44}$$

Set

$$b = \alpha e_1 + \varepsilon\chi \qquad \alpha^2 + \varepsilon^2 = 1, \text{ with } 0 < \varepsilon < 1. \tag{45}$$

Claim: there exists $\varepsilon_0 > 0$, depending on $(\lambda_i)_{1\le i\le 3}$, such that for every $0 < \varepsilon < \varepsilon_0$ (40) holds. We have, for $f \in H_0^1(\Omega)$, and with $c_i = (f, e_i)$,

$$\frac{1}{\lambda_2}\|\nabla f\|_{L^2}^2 - \|f - (f,b)b\|_{L^2}^2 = \frac{1}{\lambda_2}\|\nabla f\|_{L^2}^2 - \left(\|f\|_{L^2}^2 - (f,b)^2\right) \tag{46}$$

$$= \sum_{i=1}^{+\infty}\frac{\lambda_i}{\lambda_2}c_i^2 - \sum_{i=1}^{+\infty}c_i^2 + (f,b)^2. \tag{47}$$

On the other hand

$$(f,b)^2 = (\alpha(f,e_1) + \varepsilon(f,\chi))^2 \tag{48}$$

$$= \alpha^2 c_1^2 + 2\alpha\varepsilon(f,e_1)(f,\chi) + \varepsilon^2(f,\chi)^2 \tag{49}$$

$$= \alpha^2 c_1^2 + 2\alpha\varepsilon(f - c_2e_2, e_1)(f - c_2e_2, \chi) + \varepsilon^2(f,\chi)^2 \tag{50}$$

$$\ge \alpha^2 c_1^2 - 2\varepsilon\|f - c_2e_2\|_{L^2}^2 \tag{51}$$

$$= \alpha^2 c_1^2 - 2\varepsilon\sum_{i\ne 2}c_i^2. \tag{52}$$

Going back to (47), using (45) and choosing $\varepsilon < \varepsilon_0$ small enough, yields

$$\frac{1}{\lambda_2}\|\nabla f\|_{L^2}^2 - \|f - (f,b)b\|_{L^2}^2$$

$$\ge \left(\frac{\lambda_1}{\lambda_2} - 2\varepsilon - \varepsilon^2\right)c_1^2 + \sum_{i=3}^{+\infty}\left(\frac{\lambda_i}{\lambda_2} - 1 - 2\varepsilon\right)c_i^2 \tag{53}$$

$$\ge 0. \tag{54}$$

Remark 3.6. In the general setting of Remark 3.2, it may happen that $0 = \lambda_1 < \lambda_2$. Suppose now that $b \in H$ is such that $\|b\|_H = 1$ and

$$\|\, f - (f,b)b\|_H^2 \le \frac{1}{\lambda_2}a(f,f) \qquad \forall f \in V. \tag{55}$$

Claim: we have $b = \pm e_1$. Indeed, let $f = e_1$ in (55) we have that

$$\|\, e_1 - (e_1,b)b\|_H^2 \le \frac{\lambda_1}{\lambda_2} = 0. \tag{56}$$

Therefore $b = \pm e_1$.

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