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Partial differential equations

Observation estimate for kinetic transport equations by diffusion approximation

Inégalité d'observation pour des équations cinétiques linéaires par l'approximation de diffusion

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A R T I C L E I N F O

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A B S T R A C T

We study the unique continuation property for the neutron transport equation and for a simplified model of the Fokker–Planck equation in a bounded domain with absorbing boundary condition. An observation estimate is derived. It depends on the smallness of the mean free path and the frequency of the velocity average of the initial data. The proof relies on the well-known diffusion approximation under convenience scaling and on the basic properties of this diffusion. Eventually, we propose a direct proof for the observation at one time of parabolic equations. It is based on the analysis of the heat kernel.

R É S U M É

L'objet de cet article est l'observation (et aussi la continuation unique) pour des solutions d'équations cinétiques linéaires avec, comme opérateur de collision, soit un modèle simplifié de l'équation de la neutronique, soit un opérateur de Fokker–Planck linéarisé. À l'aide de l'approximation de la diffusion, une inégalité d'observation en un temps donné est obtenue. Elle dépend du libre parcours moyen (ou de l'opacité du milieu) et de la fréquence de la moyenne de la donnée initiale. En plus de l'approximation de la diffusion, on utilise l'inégalité d'observation en temps fixé pour la diffusion. Pour cette dernière, on propose une nouvelle démonstration directe avec des estimations à poids utilisant la paramétrix à l'ordre zéro du noyau de la chaleur.

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1. Introduction

This article is devoted to the question of unique continuation for linear kinetic transport equation with a scattering operator in the diffusive limit. Let Ω be a bounded open subset of $\mathbb{R}^d$, $d>1$, with boundary $\partial\Omega$ of class C^2. Consider in $\{(x,v)\in\Omega\times\mathbb{S}^{d-1}\}\times\mathbb{R}_t^+$ the transport equation in the v direction with a scattering operator S and the absorbing boundary condition

$$\begin{cases}\partial_t f+\dfrac{1}{\epsilon}v\cdot\nabla f+\dfrac{a}{\epsilon^2}S(f)=0 & \text{in } \Omega\times\mathbb{S}^{d-1}\times(0,+\infty)\ , \\ f=0 & \text{on } \left(\partial\Omega\times\mathbb{S}^{d-1}\right)_-\times(0,+\infty)\ , \\ f(\cdot,\cdot,0)=f_0\in L^2(\Omega\times\mathbb{S}^{d-1})\ , & \end{cases} \tag{1.1}$$

where $\epsilon\in(0,1]$ is a small parameter and $a\in L^\infty(\Omega)$ is a scattering opacity satisfying $0<c_{\min}\leq a(x)\leq c_{\max}<\infty$. Here, $\nabla=\nabla_x$ and $\left(\partial\Omega\times\mathbb{S}^{d-1}\right)_-=\left\{(x,v)\in\partial\Omega\times\mathbb{S}^{d-1};\, v\cdot\vec{n}_x<0\right\}$ where $\vec{n}_x$ is the unit outward normal field at $x\in\partial\Omega$.

Two standard examples of scattering operators $S:f\mapsto Sf$ are given below:

- the neutron scattering operator,

$$Sf=f-\langle f\rangle \text{ where } \langle f\rangle(x,t)=\frac{1}{\left|\mathbb{S}^{d-1}\right|}\int_{\mathbb{S}^{d-1}} f(x,v,t)\,\mathrm{d}v;$$

- the Fokker–Planck scattering operator,

$$Sf=-\frac{1}{d-1}\Delta_{\mathbb{S}^{d-1}}f, \text{ where } \Delta_{\mathbb{S}^{d-1}} \text{ is the Laplace–Beltrami operator on } \mathbb{S}^{d-1}.$$

Recall that such operators have the properties of self-adjointness and $Sv=v$, which imply that $\langle v\cdot\nabla Sf\rangle=\langle v\cdot\nabla f\rangle$.

Let ω be a nonempty open subset of Ω. Suppose that we observe the solution f at time $T>0$ and on ω, i.e. $f(x,v,T)_{|(x,v)\in\omega\times\mathbb{S}^{d-1}}$ is known. A classical inverse problem consists in recovering at least one solution, and in particular its initial data, which fits the observation on $\omega\times\mathbb{S}^{d-1}\times\{T\}$. Our problem of unique continuation is: with how many initial data will the corresponding solution achieve the given observation $f(x,v,T)_{|(x,v)\in\omega\times\mathbb{S}^{d-1}}$? Here ϵ is a small parameter and it is natural to focus on the limit solution. This is the diffusion approximation saying that the solution f converges to a solution to a parabolic equation when ϵ tends to 0 (see [3,8,16,2,5,6,4]). In this framework, two remarks can be made:

- for our scattering operator, there holds

$$\|f-\langle f\rangle\|_{L^2(\Omega\times\mathbb{S}^{d-1}\times\mathbb{R}_t^+)}\leq\epsilon\frac{1}{\sqrt{2c_{\min}}}\|f_0\|_{L^2(\Omega\times\mathbb{S}^{d-1})}.$$

For the operator of neutron transport, one uses a standard energy method by multiplying both sides of the first line of (1.1) by f and integrating over $\Omega\times\mathbb{S}^{d-1}\times(0,T)$. For the Fokker–Planck scattering operator, one combines the standard energy method as above and Poincaré inequality

$$\|f-\langle f\rangle\|_{L^2(\Omega\times\mathbb{S}^{d-1}\times\mathbb{R}_t^+)}\leq\frac{1}{\sqrt{d-1}}\left\|\nabla_{\mathbb{S}^{d-1}}f\right\|_{L^2(\Omega\times\mathbb{S}^{d-1}\times\mathbb{R}_t^+)}\leq\epsilon\frac{1}{\sqrt{2c_{\min}}}\|f_0\|_{L^2(\Omega\times\mathbb{S}^{d-1})}.$$

- In the sense of distributions in Ω, for any $t\geq 0$, the average of f solves the following parabolic equation

$$\partial_t\langle f\rangle-\frac{1}{d}\nabla\cdot\left(\frac{1}{a}\nabla\langle f\rangle\right)=\nabla\cdot\left(\frac{1}{a}\langle(v\otimes v)\nabla(f-\langle f\rangle)\rangle\right)+\epsilon\nabla\cdot\left(\frac{1}{a}\langle v\partial_t f\rangle\right). \tag{1.2}$$

Indeed, multiplying by $\dfrac{\epsilon}{a}v$ the equation $\partial_t f+\dfrac{1}{\epsilon}v\cdot\nabla f+\dfrac{a}{\epsilon^2}Sf=0$ and taking the average over $\mathbb{S}^{d-1}$, using $\partial_t\langle f\rangle+\dfrac{1}{\epsilon}\langle v\cdot\nabla f\rangle=0$, $\langle v\cdot\nabla Sf\rangle=\langle v\cdot\nabla f\rangle$ and $\langle v\,(v\cdot\nabla\langle f\rangle)\rangle=\frac{1}{d}\nabla\langle f\rangle$, one obtains, for any $t\geq 0$ and any $\varphi\in C_0^\infty(\Omega)$,

$$\int_\Omega\partial_t\langle f\rangle\varphi\,\mathrm{d}x+\frac{1}{d}\int_\Omega\frac{1}{a}\nabla\langle f\rangle\cdot\nabla\varphi\,\mathrm{d}x+\int_\Omega\frac{1}{a}\langle v\,(v\cdot\nabla(f-\langle f\rangle)+\epsilon\,\partial_t f)\rangle\cdot\nabla\varphi\,\mathrm{d}x=0.$$

Moreover, we prove that the boundary condition on $\langle f\rangle$ is small in some adequate norm with respect to ϵ. In the sequel, any estimate will be explicit with respect to ϵ.

Backward uniqueness for parabolic equations has a long history (see [9,25]). Lions and Malgrange [21] used the method of Carleman estimates. Later, Bardos and Tartar [7] gave some improvements by using the log convexity method of Agmon and Nirenberg. More recently, motivated by control theory and inverse problems (see [15,22]), Carleman estimates became an important tool to achieve an observability inequality (see [13,12,11,18–20]). In [24], the desired observability inequality is deduced from the observation estimate at one point in time that is obtained by studying the frequency function in the spirit of the log convexity method. In particular, one can quantify the following unique continuation property (see [10,23]): If $u(x,t) = e^{t\Delta} u_0(x)$ with $u_0 \in L^2(\Omega)$ and $u(\cdot,T) = 0$ on ω, then $u_0 \equiv 0$.

Our main result below involves the regularity of the nonzero initial data f_0 measured in term of two quantities. Let $p > 2$,

$$\mathbb{M}_p := \frac{\|f_0\|_{L^{2p}(\Omega\times\mathbb{S}^{d-1})}}{\|\langle f_0\rangle\|_{L^2(\Omega)}} \text{ and } \mathbb{F} := \frac{\|\langle f_0\rangle\|^2_{L^2(\Omega)}}{\|\langle f_0\rangle\|^2_{H^{-1}(\Omega)}} .$$

Observe in particular that $\mathbb{F}$ is the most natural evaluation of the frequency of the velocity average of the initial data.

Theorem 1.1. *Suppose that $a \in C^2(\overline{\Omega})$ and $f_0 \in L^{2p}(\Omega\times\mathbb{S}^{d-1})$ with $\mathbb{M}_p + \mathbb{F} < +\infty$ for some $p > 2$. Then the unique solution f to (1.1) satisfies, for any $T > 0$,*

$$\left(1 - \epsilon^{\frac{1}{2p}}(1 + T^{\frac{p-1}{2p}} C_p)\mathbb{M}_p e^{\sigma(f_0,T)}\right) \|\langle f_0\rangle\|_{L^2(\Omega)} \le e^{\sigma(f_0,T)} \|\langle f\rangle(\cdot,T)\|_{L^2(\omega)}$$

with $C_p = \left(\frac{p-1}{p-2}\right)^{\frac{p-1}{2p}}$ and $\sigma(f_0,T) = c\left(1 + \frac{1}{T} + T\mathbb{F}\right)$, where c only depends on (Ω,ω,d,a).

By a direct application of our main result, we have the following corollary.

Corollary 1.2. *Let $a \in C^2(\overline{\Omega})$ and $f_0 \in L^{2p}(\Omega\times\mathbb{S}^{d-1})$ with $\mathbb{M}_p + \mathbb{F} < +\infty$ for some $p > 2$. Suppose that $f_0 \ge 0$. Then there is $\epsilon_0 \in (0,1)$ depending on $(\mathbb{M}_p, \mathbb{F}, \Omega, \omega, d, p, T, a)$ such that if $f(\cdot,\cdot,T) = 0$ on $\omega \times \mathbb{S}^{d-1}$ for some $\epsilon \le \epsilon_0$, then $f_0 \equiv 0$.*

This paper is organized as follows: the proof of the main result is given in the next section. It requires two important results: an approximation diffusion convergence of the average of f; an observation estimate at one point in time for the diffusion equation with homogeneous Dirichlet boundary condition. In Section 3, we prove the approximation theorem stated in Section 2. In Section 4, a direct proof of the observation inequality at one point in time for parabolic equations is proposed. Finally, in an appendix, we prove a backward estimate for the diffusion equation and a trace estimate for the kinetic transport equation.

2. Proof of main Theorem 1.1

The main task in the proof of Theorem 1.1 consists of the two following propositions. Below we denote by $u \in C([0,T], L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega))$ any solution to the diffusion equation

$$\partial_t u - \frac{1}{d}\nabla\cdot\left(\frac{1}{a}\nabla u\right) = 0 \tag{2.1}$$

with $a \in C^2(\overline{\Omega})$ and $0 < c_{\min} \le a(x) \le c_{\max} < \infty$.

The first proposition is a quantitative unique continuation of the diffusion equation.

Proposition 2.1. *There are $C > 0$ and $\mu \in (0,1)$ such that any solution $u \in C([0,T], L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega))$ of (2.1) satisfies*

$$\int_\Omega |u(x,T)|^2\, dx \le \left(Ce^{\frac{C}{T}} \int_\omega |u(x,T)|^2\, dx\right)^{1-\mu} \left(\int_\Omega |u(x,0)|^2\, dx\right)^{\mu} .$$

Here C and μ only depend on (a,Ω,ω,d).

As an immediate application, combining with the following backward estimate for diffusion equation (see Appendix),

$$\|u(\cdot,0)\|_{L^2(\Omega)} \le \frac{c}{\sqrt{T}} e^{cT\frac{\|u(\cdot,0)\|^2_{L^2(\Omega)}}{\|u(\cdot,0)\|^2_{H^{-1}(\Omega)}}} \|u(\cdot,T)\|_{L^2(\Omega)}, \tag{2.2}$$

we have the following corollary.

Corollary 2.2. *For any nonzero $u \in C\left([0,T], L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right)$ solution to (2.1), one has*

$$\|u(\cdot,0)\|_{L^2(\Omega)} \leq e^{C\left(1+\frac{1}{T}+T\frac{\|u(\cdot,0)\|^2_{L^2(\Omega)}}{\|u(\cdot,0)\|^2_{H^{-1}(\Omega)}}\right)} \|u(\cdot,T)\|_{L^2(\omega)}$$

where C only depends on (a,Ω,ω,d).

The second proposition deals with the diffusion approximation for the linear kinetic transport equation.

Proposition 2.3. *Assume $f_0 \in L^{2p}(\Omega \times \mathbb{S}^{d-1})$ for some $p > 2$ and consider $u \in C\left(0,T; H_0^1(\Omega)\right)$ solution to (2.1) with initial data $u(\cdot,0) = \langle f_0 \rangle$, then for any $T > 0$ and any $\chi \in C_0^\infty(\omega)$, the solution f to (1.1) satisfies*

$$\left\|\chi\left(\langle f\rangle_{|t=T} - u(\cdot,T)\right)\right\|_{H^{-1}(\Omega)} \leq \epsilon^{\frac{1}{2p}}\left(1+T^{\frac{p-1}{2p}}C_p\right)C\|f_0\|_{L^{2p}(\Omega\times\mathbb{S}^{d-1})}$$

where $C_p = \left(\frac{p-1}{p-2}\right)^{\frac{p-1}{2p}}$ and $C > 0$ only depends on (Ω,d,a,χ).

The proof of Proposition 2.1 and Proposition 2.3 is given in sections 4 and 3, respectively. Now we start the proof of Theorem 1.1.

Let $\chi \in C_0^\infty(\omega)$. On the one hand, since $u(\cdot,0) = \langle f_0\rangle$, we have, by Corollary 2.2,

$$\|\langle f_0\rangle\|_{L^2(\Omega)} \leq e^{C\left(1+\frac{1}{T}+T\frac{\|\langle f_0\rangle\|^2_{L^2(\Omega)}}{\|\langle f_0\rangle\|^2_{H^{-1}(\Omega)}}\right)} \|\chi u(\cdot,T)\|_{L^2(\Omega)} .$$

On the other hand, by regularizing effect, we see that

$$\|\chi u(\cdot,T)\|_{L^2(\Omega)} \leq \|\chi u(\cdot,T)\|^{1/2}_{H^{-1}(\Omega)} \|\chi u(\cdot,T)\|^{1/2}_{H_0^1(\Omega)} \leq C\|\chi u(\cdot,T)\|^{1/2}_{H^{-1}(\Omega)}\left(1+\frac{1}{T^{1/4}}\right)\|\langle f_0\rangle\|^{1/2}_{L^2(\Omega)} .$$

Therefore, the two above facts yield

$$\begin{aligned}\|\langle f_0\rangle\|_{L^2(\Omega)} &\leq e^{C\left(1+\frac{1}{T}+T\frac{\|\langle f_0\rangle\|^2_{L^2(\Omega)}}{\|\langle f_0\rangle\|^2_{H^{-1}(\Omega)}}\right)}\left(\left\|\chi\left(u(\cdot,T)-\langle f\rangle_{|t=T}\right)\right\|_{H^{-1}(\Omega)} + \left\|\chi\langle f\rangle_{|t=T}\right\|_{H^{-1}(\Omega)}\right)\\ &\leq e^{C\left(1+\frac{1}{T}+T\frac{\|\langle f_0\rangle\|^2_{L^2(\Omega)}}{\|\langle f_0\rangle\|^2_{H^{-1}(\Omega)}}\right)}\left(\epsilon^{\frac{1}{2p}}\left(1+T^{\frac{p-1}{2p}}C_p\right)\|f_0\|_{L^{2p}(\Omega\times\mathbb{S}^{d-1})} + \left\|\chi\langle f\rangle_{|t=T}\right\|_{H^{-1}(\Omega)}\right)\end{aligned}$$

where in the last line we used Proposition 2.3. This completes the proof.

3. Estimates for the diffusion approximation

Below we give precise error estimates for the diffusion approximation.

Theorem 3.1. *Let $a \in C^1\left(\overline{\Omega}\right)$ such that $0 < c_{\min} \leq a(x) \leq c_{\max} < \infty$. Assume $f_0 \in L^{2p}(\Omega\times\mathbb{S}^{d-1})$ for some $p > 2$ and consider $u \in C\left(0,T; H_0^1(\Omega)\right)$ solution to (2.1) with initial data $u(\cdot,0) = \langle f_0\rangle$, then for any $T > 0$, the solution f to (1.1) satisfies*

$$\left\|\langle f\rangle_{|t=T} - u(\cdot,T)\right\|_{H^{-1}(\Omega)} + \|\langle f\rangle - u\|_{L^2(\Omega\times(0,T))} \leq \epsilon^{\frac{1}{2p}}\left(1+T^{\frac{p-1}{2p}}C_p\right)C\|f_0\|_{L^{2p}(\Omega\times\mathbb{S}^{d-1})}$$

where $C_p = \left(\frac{p-1}{p-2}\right)^{\frac{p-1}{2p}}$ and $C > 0$ only depends on $(\Omega, d, c_{\min}, c_{\max}, \|\nabla a\|_\infty)$.

In the literature, there are at least two ways to get diffusion approximation estimates:

- use a Hilbert expansion: the solution f to the transport problem can be formally written as $f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \ldots$, and we substitute this expansion into the governing equations in order to prove the existence of $f_0, f_1, f_2, \ldots$. Next we set $F = f - (f_0 + \epsilon f_1)$ and check that it solves a transport problem for which the energy method can be used. This way requires well-prepared initial data, which is $f_0 = \langle f_0\rangle$ to avoid initial layers;

– use the moment method: the zeroth and first moments of f are respectively $\langle f\rangle$ and $\langle vf\rangle$. First, we check that $f-\langle f\rangle$ is small in some adequate norm with respect to ϵ. Next, by computing the zeroth and first moments of the equation solved by f (as it was done in the introduction), we derive that $\langle f\rangle$ solves a parabolic problem for which the energy method can be used. This way and a new ϵ uniform estimate on the trace (see Proposition 3.2 below) give Theorem 3.1. Notice that since only the average of f is involved, the proof requires no analysis of the initial layer near $t=0$.

Proposition 3.2. *If $f_0\in L^{2p}(\Omega\times\mathbb{S}^{d-1})$ for some $p>2$, then the solution f to (1.1) satisfies*

$$\|f\|_{L^2(\partial\Omega\times\mathbb{S}^{d-1}\times(0,T))}\leq CT^{\frac{p-1}{2p}}\epsilon^{\frac{1}{2p}}C_p\|f_0\|_{L^{2p}(\Omega\times\mathbb{S}^{d-1})}$$

where $C_p=\left(\frac{p-1}{p-2}\right)^{\frac{p-1}{2p}}$ and $C>0$ only depends on (Ω,d).

Proposition 3.2 is proved in the Appendix. The proof of Theorem 3.1 starts as follows. Let $w_\epsilon=\langle f\rangle-u$ where u solves

$$\begin{cases}\partial_t u-\frac{1}{d}\nabla\cdot\left(\frac{1}{a}\nabla u\right)=0 & \text{in } \Omega\times(0,+\infty)\,,\\ u=0 & \text{on } \partial\Omega\times(0,+\infty)\,,\\ u(\cdot,0)=\langle f_0\rangle\in L^2(\Omega)\,. & \end{cases}$$

By (1.2) and a density argument, w_ϵ solves, for any $t\geq 0$ and any $\varphi\in H_0^1(\Omega)$,

$$\int_\Omega \partial_t w_\epsilon\varphi\,\mathrm{d}x+\frac{1}{d}\int_\Omega\nabla w_\epsilon\cdot\frac{1}{a}\nabla\varphi\,\mathrm{d}x=-\int_\Omega\langle v\,(v\cdot\nabla(f-\langle f\rangle))\rangle\cdot\frac{1}{a}\nabla\varphi\,\mathrm{d}x-\epsilon\int_\Omega\langle v\partial_t f\rangle\cdot\frac{1}{a}\nabla\varphi\,\mathrm{d}x \tag{3.1}$$

with boundary condition $w_\epsilon=\langle f\rangle$ on $\partial\Omega\times\mathbb{R}_t^+$ and initial data $w_\epsilon(\cdot,0)=0$. We choose

$$\varphi=\left(-\frac{1}{d}\nabla\cdot\left(\frac{1}{a}\nabla\right)\right)^{-1}w_\epsilon\,.$$

By integrations by parts, the identity (3.1) becomes:

$$\begin{aligned}\frac{1}{2d}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{1}{a}|\nabla\varphi|^2\,\mathrm{d}x+\left\|\frac{1}{d}\nabla\cdot\left(\frac{1}{a}\nabla\right)\varphi\right\|^2_{L^2(\Omega)}&=-\int_{\partial\Omega}\langle f\rangle\frac{1}{a}\partial_n\varphi\,\mathrm{d}x\\ &-\int_\Omega\langle v\,(v\cdot\nabla(f-\langle f\rangle))\rangle\cdot\frac{1}{a}\nabla\varphi\,\mathrm{d}x-\epsilon\int_\Omega\langle v\partial_t f\rangle\cdot\frac{1}{a}\nabla\varphi\,\mathrm{d}x\,.\end{aligned} \tag{3.2}$$

First, the contribution of the boundary data is estimated: one has, by a classical trace theorem

$$-\int_{\partial\Omega}\langle f\rangle\frac{1}{a}\partial_n\varphi\,\mathrm{d}x\leq C_1\|f\|_{L^2(\partial\Omega\times\mathbb{S}^{d-1})}\left\|\nabla\cdot\left(\frac{1}{a}\nabla\right)\varphi\right\|_{L^2(\Omega)}$$

where the positive constant C_1 depends on $\|\nabla a\|_\infty$.

Secondly, the contribution of the term

$$\int_\Omega\langle v\,(v\cdot\nabla(f-\langle f\rangle))\rangle\cdot\frac{1}{a}\nabla\varphi\,\mathrm{d}x$$

is estimated: by integration by parts and using $\nabla\varphi=\partial_n\varphi\vec{n}_x$ on $\partial\Omega$, one has

$$\begin{aligned}\int_\Omega\langle v\,(v\cdot\nabla(f-\langle f\rangle))\rangle\cdot\frac{1}{a}\nabla\varphi\,\mathrm{d}x=&-\frac{1}{|\mathbb{S}^{d-1}|}\int_{\Omega\times\mathbb{S}^{d-1}}(f-\langle f\rangle)v\cdot\nabla\left(v\cdot\frac{1}{a}\nabla\varphi\right)\mathrm{d}x\,\mathrm{d}v\\ &+\frac{1}{|\mathbb{S}^{d-1}|}\int_{\partial\Omega\times\mathbb{S}^{d-1}}(v\cdot\vec{n}_x)^2(f-\langle f\rangle)\frac{1}{a}\partial_n\varphi\,\mathrm{d}x\,\mathrm{d}v\end{aligned}$$

which implies

$$\int_\Omega \langle v\,(v\cdot\nabla(f-\langle f\rangle))\rangle\cdot\frac{1}{a}\nabla\varphi\,\mathrm{d}x \le C_1\,\|f-\langle f\rangle\|_{L^2(\Omega\times\mathbb{S}^{d-1})}\left\|\nabla\cdot\left(\frac{1}{a}\nabla\right)\varphi\right\|_{L^2(\Omega)} + C_1\,\|f\|_{L^2(\partial\Omega\times\mathbb{S}^{d-1})}\left\|\nabla\cdot\left(\frac{1}{a}\nabla\right)\varphi\right\|_{L^2(\Omega)}$$

with some constant $C_1>0$ depending on $\|\nabla a\|_\infty$.

Thirdly, the contribution of the term $\epsilon\int_\Omega\langle v\partial_t f\rangle\cdot\frac{1}{a}\nabla\varphi\,\mathrm{d}x$ is estimated: from the identities

$$\begin{aligned}
&\epsilon\int_\Omega\langle v\partial_t f\rangle\cdot\frac{1}{a}\nabla\varphi\,\mathrm{d}x\\
&=\frac{1}{|\mathbb{S}^{d-1}|}\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\times\mathbb{S}^{d-1}} f v\cdot\frac{1}{a}\nabla\varphi\,\mathrm{d}x\,\mathrm{d}v-\frac{1}{|\mathbb{S}^{d-1}|}\int_{\Omega\times\mathbb{S}^{d-1}} f v\cdot\frac{1}{a}\nabla(\epsilon\,\partial_t\varphi)\,\mathrm{d}x\,\mathrm{d}v\\
&=\frac{1}{|\mathbb{S}^{d-1}|}\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\times\mathbb{S}^{d-1}} f v\cdot\frac{1}{a}\nabla\varphi\,\mathrm{d}x\,\mathrm{d}v-\frac{1}{|\mathbb{S}^{d-1}|}\int_{\Omega\times\mathbb{S}^{d-1}} (f-\langle f\rangle)\, v\cdot\frac{1}{a}\nabla(\epsilon\,\partial_t\varphi)\,\mathrm{d}x\,\mathrm{d}v
\end{aligned}$$

and

$$\begin{aligned}
\epsilon\,\partial_t\,\varphi&=\left(-\frac{1}{d}\nabla\cdot\left(\frac{1}{a}\nabla\right)\right)^{-1}(\epsilon\,\partial_t\,w_\epsilon)=\left(-\frac{1}{d}\nabla\cdot\left(\frac{1}{a}\nabla\right)\right)^{-1}(-\langle v\cdot\nabla f\rangle-\epsilon\,\partial_t\,u)\\
&=\left(-\frac{1}{d}\nabla\cdot\left(\frac{1}{a}\nabla\right)\right)^{-1}\langle -v\cdot\nabla(f-\langle f\rangle)\rangle+\epsilon\,u\,,
\end{aligned}$$

we see that

$$\begin{aligned}
\epsilon\int_\Omega\langle v\partial_t f\rangle\cdot\frac{1}{a}\nabla\varphi\,\mathrm{d}x&=\frac{1}{|\mathbb{S}^{d-1}|}\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\times\mathbb{S}^{d-1}} f v\cdot\frac{1}{a}\nabla\varphi\,\mathrm{d}x\,\mathrm{d}v\\
&\quad+\frac{1}{|\mathbb{S}^{d-1}|}\int_{\Omega\times\mathbb{S}^{d-1}}(f-\langle f\rangle)\,v\cdot\frac{1}{a}\nabla\left(\left(-\frac{1}{d}\nabla\cdot\left(\frac{1}{a}\nabla\right)\right)^{-1}\langle v\cdot\nabla(f-\langle f\rangle)\rangle\right)\mathrm{d}x\,\mathrm{d}v\\
&\quad-\epsilon\frac{1}{|\mathbb{S}^{d-1}|}\int_{\Omega\times\mathbb{S}^{d-1}}(f-\langle f\rangle)\,v\cdot\frac{1}{a}\nabla u\,\mathrm{d}x\,\mathrm{d}v\\
&\le\frac{1}{|\mathbb{S}^{d-1}|}\epsilon\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\times\mathbb{S}^{d-1}} f v\cdot\frac{1}{a}\nabla\varphi\,\mathrm{d}x\,\mathrm{d}v+C\,\|f-\langle f\rangle\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}+\epsilon^2C\,\|\nabla u\|^2_{L^2(\Omega)}\,.
\end{aligned}$$

Combining the three above contributions with (3.2), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{1}{a}|\nabla\varphi|^2\,\mathrm{d}x+\left\|\nabla\cdot\left(\frac{1}{a}\nabla\right)\varphi\right\|^2_{L^2(\Omega)}\le\epsilon\,C\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\times\mathbb{S}^{d-1}} f v\cdot\frac{1}{a}\nabla\varphi\,\mathrm{d}x\,\mathrm{d}v+\epsilon^2C\,\|\nabla u\|^2_{L^2(\Omega)}+C\left(\|f\|^2_{L^2(\partial\Omega\times\mathbb{S}^{d-1})}+\|f-\langle f\rangle\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}\right).$$

Integrating the above over $(0,T)$, we observe with $\varphi=\left(-\frac{1}{d}\nabla\cdot\left(\frac{1}{a}\nabla\right)\right)^{-1}w_\epsilon$ and $w_\epsilon=\langle f\rangle-u$ that

$$\begin{aligned}
&\|w_\epsilon(\cdot,T)\|^2_{H^{-1}(\Omega)}+\|w_\epsilon\|^2_{L^2(\Omega\times(0,T))}\\
&\le\epsilon C\left(\|f_{|t=T}\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}+\|u_{|t=T}\|^2_{L^2(\Omega)}+\|f_0\|^2_{L^2(\Omega\times\mathbb{S}^{d-1})}+\|u_0\|^2_{L^2(\Omega)}\right)\\
&\quad+\epsilon^2C\,\|\nabla u\|^2_{L^2(\Omega\times(0,T))}+C\left(\|f\|^2_{L^2(\partial\Omega\times\mathbb{S}^{d-1}\times(0,T))}+\|f-\langle f\rangle\|^2_{L^2(\Omega\times\mathbb{S}^{d-1}\times(0,T))}\right).
\end{aligned}$$

Next, we use the trace estimate in Proposition 3.2,

$$\int_{\Omega\times\mathbb{S}^{d-1}}|f(x,v,T)|^2\,\mathrm{d}x\,\mathrm{d}v+\frac{2c_{\min}}{\epsilon^2}\int_0^T\int_{\Omega\times\mathbb{S}^{d-1}}|f-\langle f\rangle|^2\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t\le\int_{\Omega\times\mathbb{S}^{d-1}}|f_0|^2\,\mathrm{d}x\,\mathrm{d}v$$

and

$$\int_\Omega |u(x,T)|^2 \, dx + \frac{2}{dc_{max}} \int_0^T \int_\Omega |\nabla u|^2 \, dx \, dt \leq \int_\Omega |\langle f_0 \rangle|^2 \, dx$$

to get that

$$\|(\langle f \rangle - u)(\cdot, T)\|_{H^{-1}(\Omega)} + \|\langle f \rangle - u\|_{L^2(\Omega \times (0,T))} \leq \sqrt{\epsilon} C \|f_0\|_{L^2(\Omega \times \mathbb{S}^{d-1})} + \epsilon^{\frac{1}{2p}} T^{\frac{p-1}{2p}} C_p C \|f_0\|_{L^{2p}(\Omega \times \mathbb{S}^{d-1})} .$$

This completes the proof.

4. Observation estimates for the diffusion equation

In this section, we establish an observation estimate at one point in time for parabolic equations with space–time coefficients (see Theorem 4.1 below). Clearly, Proposition 2.1 is a direct application of Theorem 4.1 when the coefficients are time-independent. Such an estimate is an interpolation inequality. Hölder-type inequalities of such form already appear in [17] for elliptic operators by Carleman inequalities. It applies to the observability for the heat equation in a manifold and to the estimate of Lebeau–Robbiano on sums of eigenfunctions. On the other hand, for parabolic operators, Escauriaza, Fernandez and Vessella proved such an interpolation estimate far from the boundary by some adequate Carleman estimates [10]. Here, our approach is completely new and uses properties of the heat kernel with a parametrix of order 0. With a diffusion operator, it is natural to make appear the geodesic distance. Since we are interested in the unique continuation at a time $T > 0$ for parabolic equations, we will see that time-dependent coefficients and lower-order terms do not affect the simplicity of the proof.

Theorem 4.1. *Let Ω be a bounded open set in $\mathbb{R}^n$, $n \geq 1$, either convex or C^2 and connected. Let ω be a nonempty open subset of Ω, and $T > 0$. Let A be a $n \times n$ symmetric positive-definite matrix with $C^2\left(\overline{\Omega} \times [0,T]\right)$ coefficients. Let $b = (b_0, b_1) \in (L^\infty(\Omega \times (0,T)))^{n+1}$. There are $C > 0$ and $\mu \in (0,1)$ such that any solution to*

$$\begin{cases} \partial_t u - \nabla \cdot (A \nabla u) + b_1 \cdot \nabla u + b_0 u = 0 & \text{in } \Omega \times (0,T) , \\ u = 0 & \text{on } \partial\Omega \times (0,T) , \\ u(\cdot, 0) \in L^2(\Omega) , \end{cases}$$

satisfies

$$\int_\Omega |u(x,T)|^2 \, dx \leq \left(C e^{\frac{C}{T}} \int_\omega |u(x,T)|^2 \, dx \right)^{1-\mu} \left(\left(1 + e^{C_b T}\right) \int_\Omega |u(x,0)|^2 \, dx \right)^{\mu} .$$

Moreover, C and μ only depend on $(A, b, \Omega, \omega, n)$. Here $C_b = C\left(\|b_1\|^2_{L^\infty} + \|b_0\|_{L^\infty}\right)$.

The proof of Theorem 4.1 uses a covering argument and a propagation of interpolation inequalities along a chain of balls (also called propagation of smallness): first we extend $A(\cdot, T)$ to a C^2 function on $\mathbb{R}^n$ denoted A_T. Next, for each $x_0 \in \mathbb{R}^n$, there are a neighborhood of x_0 and a function $x \mapsto \boldsymbol{d}(x, x_0)$ on which the following four properties hold:

1. $\frac{1}{C} |x - x_0| \leq \boldsymbol{d}(x, x_0) \leq C |x - x_0|$ for some $C \geq 1$ depending on (x_0, A_T) ;
2. $x \mapsto \boldsymbol{d}^2(x, x_0)$ is C^2 ;
3. $A_T(x) \nabla \boldsymbol{d}(x, x_0) \cdot \nabla \boldsymbol{d}(x, x_0) = 1$;
4. $\frac{1}{2} A_T(x) \nabla^2 \boldsymbol{d}^2(x, x_0) = I_n + O(\boldsymbol{d}(x, x_0))$.

Here ∇^2 denotes the Hessian matrix and $\boldsymbol{d}(x, x_0)$ is the geodesic distance connecting x to x_0. The proof of the above properties for $\boldsymbol{d}(x, x_0)$ is a consequence of Gauss's lemma for C^2 metrics (see [14, page 7]).

Now we are able to define $B_R = \{x; \boldsymbol{d}(x, x_0) < R\}$ the ball of center x_0 and radius R. We will choose $x_0 \in \Omega$ such that one of the two following assumptions hold: (i) $\overline{B_r} \subset \Omega$ for any $r \in (0, R]$; (ii) $B_r \cap \partial\Omega \neq \emptyset$ for any $r \in [R_0, R]$ where $R_0 > 0$ and $A(\cdot, t) \nabla \boldsymbol{d}^2 \cdot \nu \geq 0$ on $\partial\Omega \cap B_R$ for any $t \in [T - \tau, T]$ where $R > 0$ and $\tau \in (0,1)$ are sufficiently small. Here ν is the unit outward normal vector to $\partial\Omega \cap B_R$.

The case (i) deals with the propagation in the interior domain by a chain of balls strictly included in Ω. We can choose R sufficiently small for B_R to be a strictly convex set. The analysis near the boundary $\partial\Omega$ requires the assumptions of (ii). To deal with (ii), we recall the two following facts (see [26, page 532]): $\forall y \in \partial\Omega$, $\exists x_0 \in \Omega$, $\exists R > 0$, $y \in B_R$ and $(x - x_0) \cdot \nu(x) > 0$ for any $x \in \partial\Omega \cap B_R$; For any matrix $A(x_0, T)$ with the above x_0, there is a change of coordinates such that the new solution, still denoted by u, solves a parabolic equation in $\Omega \cap B_R$ with coefficients, still denoted by (A, b), such that $A(x_0, T)(x - x_0) \cdot \nu > 0$ on $\partial\Omega \cap B_R$. Now, it implies the desired assumption:

$$A(\cdot, t)\nabla \boldsymbol{d}^2 \cdot \nu = (A(\cdot, t) - A(x_0, T))\nabla \boldsymbol{d}^2 \cdot \nu + A(x_0, T)\left(\nabla \boldsymbol{d}^2 - (x - x_0)\right) \cdot \nu$$
$$+ A(x_0, T)(x - x_0) \cdot \nu \geq 0 \text{ on } \partial\Omega \cap B_R \text{ for any } t \in [T - \tau, T] .$$

However when $\Omega \subset \mathbb{R}^n$ is a convex domain or a star-shaped domain with respect to $x_0 \in \Omega$, we only need to propagate the estimate in the interior domain.

If, further, $A = I_n$, then $\boldsymbol{d}(x, x_0) = |x - x_0|$ and it is well defined for any $x \in \Omega$. From [24] such observation at one point in time implies the observability for the heat equation. From [1] such observation at one time is equivalent to the estimate of Lebeau–Robbiano type on the sums of eigenfunctions. Eventually a careful evaluation of the constants gives the following estimates.

Theorem 4.2. *Suppose that $\Omega \subset \mathbb{R}^n$ is a convex domain or a star-shaped domain with respect to $x_0 \in \Omega$ such that $\{x; |x - x_0| < r\} \Subset \Omega$ for some $r \in (0, 1)$. Then for any $u_0 \in L^2(\Omega)$, $T > 0$, $(a_i)_{i \geq 1} \in \mathbb{R}$, $\lambda > 0$, $\varepsilon \in (0, 1)$, one has*

$$\left\| e^{T\Delta} u_0 \right\|_{L^2(\Omega)} \leq \frac{1}{r^n} \frac{1}{r^{\varepsilon(n-2)}} e^{\frac{C}{T} \frac{1}{r^{6\varepsilon}}} \int_0^T \left\| e^{t\Delta} u_0 \right\|_{L^2(|x - x_0| < r)} \mathrm{d}t$$

and

$$\sum_{\lambda_i \leq \lambda} |a_i|^2 \leq \frac{1}{r^{2n(1+\varepsilon)}} e^{C \frac{1}{r^{2\varepsilon}} \sqrt{\lambda}} \int_{|x - x_0| < r} \left| \sum_{\lambda_i \leq \lambda} a_i e_i(x) \right|^2 \mathrm{d}x$$

where $C > 0$ is a constant only depending on $\left(\varepsilon, n, \max\left\{|x - x_0|; x \in \overline{\Omega}\right\}\right)$. Here (λ_i, e_i) denotes the eigenbasis of the Laplace operator with Dirichlet boundary condition.

In the next subsection, we state some preliminary lemmas and corollaries. In subsection 4.2, we prove Theorem 4.1. Subsection 4.3 is devoted to the proof of Theorem 4.2. In the three last subsections, we prove the lemmas.

4.1. Preliminary results

In this subsection, we present some lemmas and corollaries that will be used for the proof of Theorem 4.1. The strategy consists in using a logarithmic convexity method (see Lemma 4.3 below) with some weight function (see Corollary 4.6 below) inspired by the heat kernel. In order to check a kind of logarithmic convexity for a suitable functional, some boundary terms require to be dropped or to have the good sign. This is possible under a type of local star-shaped assumption (see comments after Theorem 4.1). Such localization process makes appear the functions F_1, F_2 in Lemma 4.3 and will be treated thanks to the technical Lemma 4.7 below.

The following lemma allows us to solve the differential inequalities and makes appear the Hölder type of inequality in Theorem 4.1.

Lemma 4.3. *Let $\hbar > 0$, $T > 0$ and $F_1, F_2 \in C([0, T])$. Consider two positive functions $y, N \in C^1([0, T])$ such that*

$$\begin{cases} \left|\frac{1}{2} y'(t) + N(t) y(t)\right| \leq \left(\frac{1}{2} N(t) + \frac{C_0}{T - t + \hbar} + C_1\right) y(t) + F_1(t) y(t) \\ N'(t) \leq \left(\frac{1 + C_0}{T - t + \hbar} + C_1\right) N(t) + F_2(t) \end{cases}$$

where $C_0, C_1 \geq 0$. Then for any $0 \leq t_1 < t_2 < t_3 \leq T$, one has

$$y(t_2)^{1+M} \leq y(t_3) y(t_1)^M e^D \left(\frac{T - t_1 + \hbar}{T - t_3 + \hbar}\right)^{3C_0(1+M)}$$

with

$$M = 3 \frac{\displaystyle\int_{t_2}^{t_3} \frac{e^{tC_1}}{(T - t + \hbar)^{1+C_0}} \mathrm{d}t}{\displaystyle\int_{t_1}^{t_2} \frac{e^{tC_1}}{(T - t + \hbar)^{1+C_0}} \mathrm{d}t} \quad \text{and} \quad D = 3(1 + M)\left[(t_3 - t_1)\left(C_1 + \int_{t_1}^{t_3} |F_2| \mathrm{d}t\right) + \int_{t_1}^{t_3} |F_1| \mathrm{d}t\right].$$

The proof of Lemma 4.3 will be given at subsection 4.4.

Corollary 4.4. *Under the assumptions of Lemma 4.3, for any $\hbar > 0$ and $\ell > 1$ such that $\ell\hbar < \min(1/2, T/4)$, one has*

$$y(T-\ell\hbar)^{1+M_\ell} \le y(T)\, y(T-2\ell\hbar)^{M_\ell}\, e^{D_\ell}\,(2\ell+1)^{3C_0(1+M_\ell)}$$

where $D_\ell = 3(1+M_\ell)\left(C_1 + \int_{t_1}^{t_3} (|F_1| + |F_2|)\,dt\right)$, $M_\ell \le 3e^{C_1}\frac{(\ell+1)^{C_0}}{1-\left(\frac{2}{3}\right)^{C_0}}$ *if* $C_0 > 0$ *and* $M_\ell \le 3e^{C_1}\frac{\ln(\ell+1)}{\ln(3/2)}$ *if* $C_0 = 0$.

Proof. Apply Lemma 4.3 with $t_3 = T$, $t_2 = T-\ell\hbar$, $t_1 = T-2\ell\hbar$, with $\ell\hbar <$min$(1/2, T/4)$. Here when $C_0 > 0$

$$M_\ell = 3\frac{\int_{T-\ell\hbar}^{T} \frac{e^{tC_1}}{(T-t+\hbar)^{1+C_0}}dt}{\int_{T-2\ell\hbar}^{T-\ell\hbar} \frac{e^{tC_1}}{(T-t+\hbar)^{1+C_0}}dt} \le 3\,e^{2\ell\hbar C_1}\frac{(\ell+1)^{C_0}-1}{1-\left(\frac{\ell+1}{2\ell+1}\right)^{C_0}} \le 3\,e^{C_1}\frac{(\ell+1)^{C_0}}{1-\left(\frac{2}{3}\right)^{C_0}} \text{ for } \ell > 1\,.$$

And when $C_0 = 0$

$$M_\ell = 3\frac{\int_{T-\ell\hbar}^{T} \frac{e^{tC_1}}{(T-t+\hbar)}dt}{\int_{T-2\ell\hbar}^{T-\ell\hbar} \frac{e^{tC_1}}{(T-t+\hbar)}dt} \le 3\,e^{2\ell\hbar C_1}\frac{\ln(\ell+1)}{\ln\left(\frac{2\ell+1}{\ell+1}\right)} \le 3\,e^{C_1}\frac{\ln(\ell+1)}{\ln(3/2)} \text{ for } \ell > 1\,.$$

The following lemma establishes the differential inequalities associated with the parabolic equations in any open set $\vartheta \subset \mathbb{R}^n$. □

Lemma 4.5. *For any $\Phi \in C^2\left(\overline{\Omega}\times[0,T]\right)$, $z \in H^1\left(0,T;H_0^1(\vartheta)\right)$, one has*

$$\frac{1}{2}\frac{d}{dt}\int_\vartheta |z|^2 e^\Phi dx + \int_\vartheta A\nabla z\cdot\nabla z e^\Phi dx$$
$$= \frac{1}{2}\int_\vartheta |z|^2(\partial_t\Phi + \nabla\cdot(A\nabla\Phi) + A\nabla\Phi\cdot\nabla\Phi)\,e^\Phi dx + \int_\vartheta z(\partial_t z - \nabla\cdot(A\nabla z))\,e^\Phi dx$$

and, for some C only depending on $\|(A,\partial_x A,\partial_t A)\|_{L^\infty}$,

$$\frac{d}{dt}\frac{\int_\vartheta A\nabla z\cdot\nabla z e^\Phi dx}{\int_\vartheta |z|^2 e^\Phi dx} \le \frac{-2\int_\vartheta A\nabla^2\Phi A\nabla z\cdot\nabla z e^\Phi dx}{\int_\vartheta |z|^2 e^\Phi dx} + \frac{\int_{\partial\vartheta}(A\nabla z\cdot\nabla z)(A\nabla\Phi\cdot\nu)\,e^\Phi dx}{\int_\vartheta |z|^2 e^\Phi dx}$$
$$+ \frac{\int_\vartheta |\partial_t z - \nabla\cdot(A\nabla z)|^2 e^\Phi dx}{\int_\vartheta |z|^2 e^\Phi dx} + C\frac{\int_\vartheta (1+|\nabla\Phi|)\,|\nabla z|^2 e^\Phi dx}{\int_\vartheta |z|^2 e^\Phi dx}$$
$$+ \frac{\int_\vartheta A\nabla z\cdot\nabla z\,(\partial_t\Phi + \nabla\cdot(A\nabla\Phi) + A\nabla\Phi\cdot\nabla\Phi)\,e^\Phi dx}{\int_\vartheta |z|^2 e^\Phi dx}$$

$$- \frac{\int_\vartheta A\nabla z \cdot \nabla z e^\Phi \,dx}{\int_\vartheta |z|^2 e^\Phi \,dx} \times \frac{\int_\vartheta |z|^2 \left(\partial_t \Phi + \nabla\cdot(A\nabla\Phi) + A\nabla\Phi\cdot\nabla\Phi\right) e^\Phi \,dx}{\int_\vartheta |z|^2 e^\Phi \,dx} .$$

Lemma 4.5 is proved in subsection 4.5.

Corollary 4.6. *Let $R > 0$ be sufficiently small and $z \in H^1\left(0, T; H_0^1(\Omega\cap B_R)\right)$ with $B_R = \{x; \boldsymbol{d}(x, x_0) < R\}$. Introduce for $t \in (0, T]$, $\mathcal{P}z = \partial_t z - \nabla\cdot(A\nabla z)$,*

$$G_\hbar(x,t) = \frac{1}{(T-t+\hbar)^{n/2}} e^{-\frac{\boldsymbol{d}^2(x,x_0)}{4(T-t+\hbar)}} \quad \forall x \in B_R ,$$

and

$$N_\hbar(t) = \frac{\int_{\Omega\cap B_R} A(x,t)\nabla z(x,t)\cdot\nabla z(x,t) G_\hbar(x,t)\,dx}{\int_{\Omega\cap B_R} |z(x,t)|^2 G_\hbar(x,t)\,dx}$$

whenever $\int_{\Omega\cap B_R} |z(x,t)|^2 \,dx \neq 0$. Then, the following two properties hold: for some $c \geq 0$

i)

$$\left| \frac{1}{2}\frac{d}{dt} \int_{\Omega\cap B_R} |z(x,t)|^2 G_\hbar(x,t)\,dx + N_\hbar(t) \int_{\Omega\cap B_R} |z(x,t)|^2 G_\hbar(x,t)\,dx \right|$$
$$\leq \left(\frac{cR}{T-t+\hbar} + c\right) \int_{\Omega\cap B_R} |z(x,t)|^2 G_\hbar(x,t)\,dx + \int_{\Omega\cap B_R} |z(x,t)\mathcal{P}z(x,t)| G_\hbar(x,t)\,dx.$$

ii) When $A\nabla\boldsymbol{d}^2\cdot\nu \geq 0$ on $\partial\Omega\cap B_R$,

$$\frac{d}{dt}N_\hbar(t) \leq \left(\frac{1+cR}{T-t+\hbar} + c\right) N_\hbar(t) + \frac{\int_{\Omega\cap B_R} |\mathcal{P}z(x,t)|^2 G_\hbar(x,t)\,dx}{\int_{\Omega\cap B_R} |z(x,t)|^2 G_\hbar(x,t)\,dx} .$$

Proof. Apply Lemma 4.5 with $\vartheta = \Omega\cap B_R$ and

$$\Phi(x,t) = -\frac{\boldsymbol{d}^2(x,x_0)}{4(T-t+\hbar)} - \frac{n}{2}\ln(T-t+\hbar) .$$

Recall that $\boldsymbol{d}(x, x_0)$ is the geodesic distance connecting x to x_0 associated with $A_T(x) = A(x, T)$. Also, R is sufficiently small for B_R to be a strictly convex set. Under the assumption $A\nabla\boldsymbol{d}^2\cdot\nu \geq 0$ on $\partial\Omega\cap B_R$, the boundary term in Lemma 4.5 is non-positive. It remains to bound $\partial_t\Phi + \nabla\cdot(A\nabla\Phi) + A\nabla\Phi\cdot\nabla\Phi$, $-2A\nabla^2\Phi A\nabla z\cdot\nabla z$ and $|\nabla\Phi|$. First, we deal with $|\nabla\Phi|$ by noticing that, for some $C_{A_T} > 0$ depending on the ellipticity constant of A, it holds

$$C_{A_T}|\nabla\Phi|^2 \leq A_T\nabla\Phi\cdot\nabla\Phi = \frac{\boldsymbol{d}^2(x,x_0)}{4(T-t+\hbar)^2} .$$

Next a straightforward computation gives: $\partial_t\Phi(x,t) = -\frac{\boldsymbol{d}^2(x,x_0)}{4(T-t+\hbar)^2} + \frac{n}{2(T-t+\hbar)}$,

$$\nabla\cdot(A_T\nabla\Phi) = -\frac{n}{2(T-t+\hbar)} + \frac{O(\boldsymbol{d}(x,x_0))}{T-t+\hbar}$$

and

$$\partial_t \Phi + \nabla \cdot (A_T \nabla \Phi) + A_T \nabla \Phi \cdot \nabla \Phi = \frac{O(\boldsymbol{d}(x, x_0))}{T - t + \hbar} .$$

This implies

$$\begin{aligned} \partial_t \Phi + \nabla \cdot (A \nabla \Phi) + A \nabla \Phi \cdot \nabla \Phi &= \frac{O(\boldsymbol{d}(x, x_0))}{T - t + \hbar} + \nabla \cdot \left((A - A_T)(A_T)^{-1} A_T \nabla \Phi \right) + (A - A_T) \nabla \Phi \cdot \nabla \Phi \\ &= \frac{O(\boldsymbol{d}(x, x_0))}{T - t + \hbar} + O(1) , \end{aligned}$$

where in the last equality we used $\|A(\cdot, t) - A_T\| \leq \|\partial_t A\| (T - t + \hbar)$. Finally, we have

$$\begin{aligned} -2A\nabla^2 \Phi A \nabla z \cdot \nabla z &= \frac{1}{2(T - t + \hbar)} \left(A_T \nabla^2 \boldsymbol{d}^2 A \nabla z \cdot \nabla z + (A - A_T) \nabla^2 \boldsymbol{d}^2 A \nabla z \cdot \nabla z \right) \\ &= A \nabla z \cdot \nabla z \left(\frac{1 + O(\boldsymbol{d}(x, x_0))}{T - t + \hbar} + O(1) \right) . \end{aligned}$$

Clearly, $G_\hbar(x, t) = \mathrm{e}^{\Phi(x,t)}$. This completes the proof of Corollary 4.6. □

The following lemma will be used to deal with the delocalized terms, under the notations and assumptions of Theorem 4.1. Its proof will be given at subsection 4.6.

Lemma 4.7. *Let $\rho \in (0, R)$ and $0 < \varepsilon < \rho/2$. There are constants $c_1 > 1$ and $c_2, c_3 > 0$ only depending on $(\rho, \varepsilon, A, b)$ such that for any $T - \theta \leq t \leq T$, one has*

$$\frac{\displaystyle\int_0^T \int_\Omega |u(x, s)|^2 \, \mathrm{d}x \, \mathrm{d}s + \int_\Omega |u(x, 0)|^2 \, \mathrm{d}x}{\displaystyle\int_{\Omega \cap B_\rho} |u(x, t)|^2 \, \mathrm{d}x} \leq c_1 \, \mathrm{e}^{\frac{c_1}{\theta}}$$

where

$$\frac{1}{\theta} = c_2 \ln \left(\mathrm{e}^{c_3 \left(1 + \frac{1}{T}\right)} \frac{\displaystyle\int_0^T \int_\Omega |u(x, s)|^2 \, \mathrm{d}x \, \mathrm{d}s + \int_\Omega |u(x, 0)|^2 \, \mathrm{d}x}{\displaystyle\int_{\Omega \cap B_{\rho - 2\varepsilon}} |u(x, T)|^2 \, \mathrm{d}x} \right) ,$$

with $0 < \theta \leq \min(1, T/2)$.

The interested reader may wish here to compare this lemma with [10, Lemma 5].

4.2. *Proof of Theorem 4.1*

We divide the proof of Theorem 4.1 into nine steps.

Step 1: recall that from $x \mapsto A(x, T)$, we have defined the geodesic distance $\boldsymbol{d}(x, x_0)$ and the ball $B_R = \left\{ x \in \mathbb{R}^n ; \boldsymbol{d}(x, x_0) < R \right\}$. Also, $R > 0$ is sufficiently small for B_R to be a strictly convex set, and $x_0 \in \Omega$ is chosen such that one of the two following assumptions holds: (*i*) $\overline{B_r} \subset \Omega$ for any $r \in (0, R]$; (*ii*) $B_r \cap \partial\Omega \neq \emptyset$ for any $r \in [R_0, R]$ where $R_0 > 0$ and $A(\cdot, t) \nabla \boldsymbol{d}^2 \cdot \nu \geq 0$ on $\partial\Omega \cap B_R$ for any $t \in [T - \tau, T]$ where $\tau \in (0, 1)$ is sufficiently small.

Further, let $\hbar > 0$ and

$$G_\hbar(x, t) = \frac{1}{(T - t + \hbar)^{n/2}} \mathrm{e}^{-\frac{\boldsymbol{d}^2(x, x_0)}{4(T - t + \hbar)}} \quad \forall x \in B_R .$$

Step 2: we will introduce the notation $\mathbb{U}$. By energy estimates and regularizing effect, for any $t > 0$, one has

$$\int_\Omega |u(x,t)|^2 \,\mathrm{d}x + \int_\Omega A\nabla u(x,t)\cdot\nabla u(x,t)\,\mathrm{d}x$$
$$\leq C_{A,b}\left(1+\frac{1}{t}\right)\left(\int_0^t\int_\Omega |u(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega |u(x,0)|^2\,\mathrm{d}x\right).$$

Here and from now, $C_{A,b}$ is a positive constant depending only on the ellipticity constant of A and on $\|(b_0,b_1)\|_{L^\infty}$, whose value may change from line to line. From now, denote

$$\mathbb{U} := \int_0^T\int_\Omega |u(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega |u(x,0)|^2\,\mathrm{d}x\,.$$

Step 3: let $\psi \in C_0^\infty(B_R)$ with $\psi = 1$ on $B_{R-\varepsilon}$ for some small positive $\varepsilon < R/4$. Denote $z = \psi u$ and $\mathcal{P}z = \partial_t z - \nabla\cdot(A\nabla z)$. Therefore, a direct computation gives

$$\mathcal{P}z = -b_1\cdot\nabla z - b_0 z + (b_1\cdot\nabla\psi - \nabla\cdot(A\nabla\psi))\,u - 2A\nabla\psi\cdot\nabla u\,.$$

Consequently, the following estimates hold for some $C_A > 0$ depending on the ellipticity constant of A:

$$\begin{cases} |z\mathcal{P}z| \leq \frac{1}{2}|A\nabla z\cdot\nabla z| + \left(C_A\|b_1\|^2_{L^\infty} + \|b_0\|_{L^\infty} + 1\right)|z|^2 + |\mathcal{Q}(\psi,u)|^2 \\ |\mathcal{P}z|^2 \leq C_A\|b_1\|^2_{L^\infty}|A\nabla z\cdot\nabla z| + 4\|b_0\|^2_{L^\infty}|z|^2 + 2|\mathcal{Q}(\psi,u)|^2\,, \end{cases}$$

where

$$\mathcal{Q}(\psi,u) := (b_1\cdot\nabla\psi - \nabla\cdot(A\nabla\psi))\,u - 2A\nabla\psi\cdot\nabla u.$$

Step 4: from the above estimates, the inequalities in Corollary 4.6 become

$$\left|\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\cap B_R}|z|^2 G_\hbar\,\mathrm{d}x + N_\hbar\int_{\Omega\cap B_R}|z|^2 G_\hbar\,\mathrm{d}x\right| \leq \left(\frac{1}{2}N_\hbar + \frac{C_{A,b}}{T-t+\hbar} + C_{A,b}\right)\int_{\Omega\cap B_R}|z|^2 G_\hbar\,\mathrm{d}x + \int_{\Omega\cap B_R}|\mathcal{Q}(\psi,u)|^2 G_\hbar\,\mathrm{d}x$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}N_\hbar(t) \leq \left(\frac{1+C_0}{T-t+\hbar} + C_{A,b}\right)N_\hbar(t) + 4\|b_0\|^2_{L^\infty} + 2\frac{\int_{\Omega\cap B_R}|\mathcal{Q}(\psi,u)|^2 G_\hbar\,\mathrm{d}x}{\int_{\Omega\cap B_R}|z|^2 G_\hbar\,\mathrm{d}x}\,,$$

where $0 < C_0 < 1$ by taking R sufficiently small.

Step 5: let $\ell > 1$ be such that $\ell\hbar <$min$(\tau/2, T/4)$. By Corollary 4.4 with $y(t) = \int_{\Omega\cap B_R}|z(x,t)|^2 G_\hbar(x,t)\,\mathrm{d}x$, $N(t) = N_\hbar(t)$,

$$F_1(t) = \frac{\int_{\Omega\cap B_R}|\mathcal{Q}(\psi,u)(x,t)|^2 G_\hbar(x,t)\,\mathrm{d}x}{\int_{\Omega\cap B_R}|z(x,t)|^2 G_\hbar(x,t)\,\mathrm{d}x}$$

and $F_2(t) = 4\|b_0\|^2_{L^\infty} + 2F_1(t)$, knowing that $N'(t) \leq \left(\frac{1+C_0}{T-t+\hbar} + C_{A,b}\right)N(t) + F_2(t)$, one can deduce the following interpolation inequality with $M_\ell \leq 3\mathrm{e}^{C_{A,b}}\frac{(\ell+1)^{C_0}}{1-\left(\frac{2}{3}\right)^{C_0}}$ and $0 < C_0 < 1$:

$$y(T-\ell\hbar)^{1+M_\ell} \leq y(T)\,y(T-2\ell\hbar)^{M_\ell}\,(2\ell+1)^{3C_0(1+M_\ell)}\,\mathrm{e}^{D_\ell}$$

that is

$$\left(\int_{\Omega\cap B_R} |z(x,T-\ell\hbar)|^2 \,\mathrm{e}^{\frac{-\boldsymbol{d}^2(x,x_0)}{4(\ell+1)\hbar}}\,\mathrm{d}x\right)^{1+M_\ell}$$

$$\leq (\ell+1)^{n/2}(2\ell+1)^{3C_0(1+M_\ell)} \int_{\Omega\cap B_R} |z(x,T)|^2 \,\mathrm{e}^{\frac{-\boldsymbol{d}^2(x,x_0)}{4\hbar}}\,\mathrm{d}x$$

$$\times \left(\int_{\Omega\cap B_R} |z(x,T-2\ell\hbar)|^2 \,\mathrm{e}^{\frac{-\boldsymbol{d}^2(x,x_0)}{4(2\ell+1)\hbar}}\,\mathrm{d}x\right)^{M_\ell} \mathrm{e}^{3(1+M_\ell)\left(C_{A,b}+\int_{T-2\ell\hbar}^{T}(|F_1|+|F_2|)\,\mathrm{d}t\right)}.$$

Step 6: we will estimate F_1 and F_2. From the definition of F_1,

$$|F_1(t)| \leq \mathrm{e}^{\frac{(R-2\varepsilon)^2}{4(T-t+\hbar)}}\,\mathrm{e}^{-\frac{(R-\varepsilon)^2}{4(T-t+\hbar)}} \frac{\int_{\Omega\cap\{R-\varepsilon\leq \boldsymbol{d}(x,x_0)\}} |\mathcal{Q}(\psi,u)(x,t)|^2\,\mathrm{d}x}{\int_{\Omega\cap B_{R-2\varepsilon}} |u(x,t)|^2\,\mathrm{d}x}.$$

Since $\mathrm{e}^{\frac{(R-2\varepsilon)^2}{4(T-t+\hbar)}}\,\mathrm{e}^{-\frac{(R-\varepsilon)^2}{4(T-t+\hbar)}} = \mathrm{e}^{-\frac{\varepsilon(2R-3\varepsilon)}{4(T-t+\hbar)}} \leq \mathrm{e}^{-\frac{\varepsilon(2R-3\varepsilon)}{12\ell\hbar}}$ for $t\in[T-2\ell\hbar,T]$ with $\ell>1$, one has, by step 2, when $t\in[T-2\ell\hbar,T]$

$$|F_1(t)| \leq \mathrm{e}^{-\frac{\varepsilon(2R-3\varepsilon)}{12\ell\hbar}} \frac{c_4 C_{A,b}\left(1+\frac{1}{t}\right)\mathbb{U}}{\int_{\Omega\cap B_{R-2\varepsilon}} |u(x,t)|^2\,\mathrm{d}x}$$

where $c_4>1$ is a constant only dependent on (A,b,R,ε). By Lemma 4.7 with $\rho = R-2\varepsilon$,

$$\int_{T-2\ell\hbar}^{T} |F_1(t)|\,\mathrm{d}t \leq c_4 \mathrm{e}^{-\frac{\varepsilon(2R-3\varepsilon)}{12\ell\hbar}} C_{A,b}\left(1+\frac{1}{T}\right) c_1 \mathrm{e}^{\frac{c_1}{\theta}} \text{ if } 2\ell\hbar \leq \theta\,.$$

Since $F_2(t) = 4\|b_0\|^2_{L^\infty} + 2F_1(t)$, we conclude that for any $2\ell\hbar \leq \theta\frac{\varepsilon(2R-3\varepsilon)}{6c_1}$

$$\int_{T-2\ell\hbar}^{T} (|F_1(t)|+|F_2(t)|)\,\mathrm{d}t \leq 2c_4\left(1+\frac{1}{T}\right),$$

where $c_4>1$ is a constant only dependent on (A,b,R,ε). Recall that θ and $c_1>1$ are given in Lemma 4.7.

Step 7: Combining the conclusion of step 5 and step 6, we can deduce that there is $c_5 := \frac{\varepsilon(2R-3\varepsilon)}{6c_1} \in (0,1)$ such that for any $2\ell\hbar \leq c_5\theta$

$$\left(\int_{\Omega\cap B_R} |z(x,T-\ell\hbar)|^2 \,\mathrm{e}^{\frac{-\boldsymbol{d}^2(x,x_0)}{4(\ell+1)\hbar}}\,\mathrm{d}x\right)^{1+M_\ell} \leq \mathrm{e}^{C_{A,b}(1+M_\ell)\left(1+\frac{1}{T}\right)}(2\ell+1)^{3C_0(1+M_\ell)+n/2}\int_{\Omega\cap B_R} |z(x,T)|^2 \,\mathrm{e}^{\frac{-\boldsymbol{d}^2(x,x_0)}{4\hbar}}\,\mathrm{d}x\,\mathbb{U}^{M_\ell}$$

which implies

$$\left(\int_{\Omega\cap B_R} |z(x,T-\ell\hbar)|^2\,\mathrm{d}x\right)^{1+M_\ell} \leq \mathrm{e}^{C_{A,b}(1+M_\ell)\left(1+\frac{1}{T}\right)}(2\ell+1)^{3C_0(1+M_\ell)+n/2}\,\mathrm{e}^{\frac{R^2}{4(\ell+1)\hbar}(1+M_\ell)}$$

$$\times \int_{\Omega\cap B_R} |z(x,T)|^2 \,\mathrm{e}^{\frac{-\boldsymbol{d}^2(x,x_0)}{4\hbar}}\,\mathrm{d}x\,\mathbb{U}^{M_\ell}.$$

Step 8: we split $\int_{\Omega\cap B_R} |z(x,T)|^2 \,\mathrm{e}^{\frac{-\boldsymbol{d}^2(x,x_0)}{4\hbar}}\,\mathrm{d}x$ into two parts. For any $0<r<R/2$ such that $B_r \Subset \Omega$,

$$\int_{\Omega\cap B_R} |z(x,T)|^2 e^{\frac{-d^2(x,x_0)}{4\hbar}} dx \leq \int_{B_r} |u(x,T)|^2 dx + e^{\frac{-r^2}{4\hbar}} C_{A,b}\left(1+\frac{1}{T}\right)\mathbb{U}.$$

Consequently, we have

$$\left(\int_{\Omega\cap B_R} |z(x,T-\ell\hbar)|^2 dx\right)^{1+M_\ell} \leq e^{C_{A,b}(1+M_\ell)\left(1+\frac{1}{T}\right)} (2\ell+1)^{3C_0(1+M_\ell)+n/2}\,\mathbb{U}^{M_\ell} \times \left(e^{\frac{R^2}{4(\ell+1)\hbar}(1+M_\ell)} \int_{B_r} |u(x,T)|^2 dx + e^{\frac{R^2}{4(\ell+1)\hbar}(1+M_\ell)} e^{\frac{-r^2}{4\hbar}}\mathbb{U}\right).$$

Now, choose $\ell > 1$ such that $\frac{R^2}{4(\ell+1)}(1+M_\ell) \leq \frac{r^2}{8}$ (knowing that $M_\ell \leq 3e^{C_1}\frac{(\ell+1)^{C_0}}{1-\left(\frac{2}{3}\right)^{C_0}}$ for $\ell > 1$ and $C_0 < 1$). Therefore, there is $K > 1$ such that, for any $T > 0$ and any $\hbar \leq \frac{c_5}{2\ell}\theta$,

$$\left(\int_{\Omega\cap B_{R-\varepsilon}} |u(x,T-\ell\hbar)|^2 dx\right)^{1+K} \leq K e^{\frac{K}{T}}\mathbb{U}^K \left(e^{\frac{r^2}{8\hbar}} \int_{B_r} |u(x,T)|^2 dx + e^{\frac{-r^2}{8\hbar}}\mathbb{U}\right).$$

Step 9: by step 2 and Lemma 4.7 with $\rho = R - 2\varepsilon$, since $\ell\hbar \leq \theta$,

$$\int_\Omega |u(x,T)|^2 dx \leq C_{A,b}\left(1+\frac{1}{T}\right)\mathbb{U} \leq C_{A,b}\left(1+\frac{1}{T}\right) c_1 e^{\frac{c_1}{\theta}} \int_{\Omega\cap B_{R-\varepsilon}} |u(x,T-\ell\hbar)|^2 dx.$$

As a consequence, with the conclusion of step 8, for any $\hbar \leq \frac{c_5}{2\ell}\theta$, one obtains:

$$\left(\int_\Omega |u(x,T)|^2 dx\right)^{1+K} \leq e^{\frac{(1+K)c_1}{\theta}} K e^{\frac{K}{T}}\mathbb{U}^K \left(e^{\frac{r^2}{8\hbar}} \int_{B_r} |u(x,T)|^2 dx + e^{\frac{-r^2}{8\hbar}}\mathbb{U}\right).$$

On the other hand, for any $\hbar \in \left(\frac{c_5}{2\ell}\theta, \frac{1}{\ell}\min(\tau/2, T/4)\right)$, one has $1 \leq e^{\frac{-r^2}{8\hbar}} e^{\frac{r^2\ell}{4c_5\theta}}$. And for any $\hbar \geq \frac{1}{\ell}\min(\tau/2, T/4)$, there holds $1 \leq e^{\frac{-r^2}{8\hbar}}\left(e^{\frac{r^2\ell}{4\tau}} + e^{\frac{r^2\ell}{4T}}\right)$. Finally, there is $K > 1$ such that for any $T > 0$ and any $\hbar > 0$,

$$\left(\int_\Omega |u(x,T)|^2 dx\right)^{1+K} \leq e^{\frac{K}{\theta}} K e^{\frac{K}{T}}\mathbb{U}^K \left(e^{\frac{r^2}{8\hbar}} \int_{B_r} |u(x,T)|^2 dx + e^{\frac{-r^2}{8\hbar}}\mathbb{U}\right).$$

Next, choose $\hbar > 0$ such that $e^{\frac{r^2}{8\hbar}} := 2e^{\frac{K}{\theta}} K e^{\frac{K}{T}} \left(\frac{\mathbb{U}}{\int_\Omega |u(x,T)|^2 dx}\right)^{1+K}$ that is

$$e^{\frac{K}{\theta}} K e^{\frac{K}{T}} e^{\frac{-r^2}{8\hbar}} \mathbb{U}^{1+K} = \frac{1}{2}\left(\int_\Omega |u(x,T)|^2 dx\right)^{1+K}$$

so that

$$\int_\Omega |u(x,T)|^2 dx \leq 2K e^{\frac{K}{T}} e^{\frac{K}{\theta}} \left(e^{\frac{K}{T}} \int_{B_r} |u(x,T)|^2 dx\right)^{\frac{1}{2+2K}} \mathbb{U}^{\frac{1+2K}{2+2K}}.$$

Recall that, by Lemma 4.7 with $\rho = R - 2\varepsilon$,

$$e^{\frac{K}{\theta}}=\left(e^{c_3\left(1+\frac{1}{T}\right)}\frac{\mathbb{U}}{\int_{\Omega\cap B_{R-4\varepsilon}}|u(x,T)|^2\,dx}\right)^{Kc_2}.$$

Further, by a standard energy method and Poincaré inequality, it holds

$$\mathbb{U}\le\left(1+C_Ae^{C_bT}\right)\int_\Omega|u(x,0)|^2\,dx$$

where $C_b=C_A\left(\|b_1\|^2_{L^\infty}+\|b_0\|_{L^\infty}\right)$ and $C_A>0$ depends on the ellipticity constant of A. Finally, we obtain

$$\int_{\Omega\cap B_{R-4\varepsilon}}|u(x,T)|^2\,dx\le Ke^{\frac{K}{T}}\left(\left(1+e^{C_bT}\right)\int_\Omega|u(x,0)|^2\,dx\right)^{\frac{K}{1+K}}\left(\int_{B_r}|u(x,T)|^2\,dx\right)^{\frac{1}{1+K}}$$

for some positive constant K only depending on (A,b,ε,R,r,n). By an adequate covering of Ω by balls $B_{R-4\varepsilon}$, where x_0 and R are chosen such that $A\nabla\boldsymbol{d}^2\cdot\nu\ge 0$ on $\partial\Omega\cap B_R$ and by a propagation of smallness based on the previous estimate, we get the desired observation inequality at one point in time for parabolic equations.

4.3. Proof of Theorem 4.2

From [24] and from [1], Theorem 4.2 can be deduced with the following claim: let $\Omega\subset\mathbb{R}^n$ be a convex domain or a star-shaped domain with respect to $x_0\in\Omega$ such that $B_r:=\{x;\,|x-x_0|<r\}\Subset\Omega$, $r<R:=\max_{x\in\overline{\Omega}}|x-x_0|$. Then it holds

$$\int_\Omega|u(x,T)|^2\,dx\le\left(2^{n+2}\ell^ne^{\frac{r^2\ell}{4T}}\int_{B_r}|u(x,T)|^2\,dx\right)^{1-\mu}\left(\int_\Omega|u(x,0)|^2\,dx\right)^{\mu},$$

with

$$1-\mu=\frac{1}{2\left(1+\frac{r^2\ell}{R^2}\right)}\text{ and }\frac{r^2\ell}{R^2}=\ell^\varepsilon\frac{2^{2+\varepsilon}}{\varepsilon\ln(3/2)}\ \forall\varepsilon\in(0,1)\ .$$

The proof of the claim follows the same strategy than the one of Theorem 4.1. Let $\hbar>0$ and $x_0\in\Omega$. Here, $t\in(0,T]$,

$$G_\hbar(x,t)=\frac{1}{(T-t+\hbar)^{n/2}}e^{-\frac{|x-x_0|^2}{4(T-t+\hbar)}}\text{ and }N_\hbar(t)=\frac{\int_\Omega|\nabla u(x,t)|^2G_\hbar(x,t)\,dx}{\int_\Omega|u(x,t)|^2G_\hbar(x,t)\,dx},$$

the differential inequalities are:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|u(x,t)|^2G_\hbar(x,t)\,dx+N_\hbar(t)\int_\Omega|u(x,t)|^2G_\hbar(x,t)\,dx=0\,;$$

When Ω is convex or star-shaped w.r.t. x_0,

$$\frac{d}{dt}N_\hbar(t)\le\frac{1}{T-t+\hbar}N_\hbar(t)\ .$$

By solving such differential inequalities as in the proof of Corollary 4.4, we have: for any $0<t_1<t_2<t_3\le T$,

$$\left(\int_\Omega|u(x,t_2)|^2G_\hbar(x,t_2)\,dx\right)^{1+M}\le\int_\Omega|u(x,t_3)|^2G_\hbar(x,t_3)\,dx\left(\int_\Omega|u(x,t_1)|^2G_\hbar(x,t_1)\,dx\right)^{M}$$

where

$$M = \frac{-\ln(T - t_3 + \hbar) + \ln(T - t_2 + \hbar)}{-\ln(T - t_2 + \hbar) + \ln(T - t_1 + \hbar)} .$$

Choose $t_3 = T$, $t_2 = T - \ell\hbar$, $t_1 = T - 2\ell\hbar$ with $0 < 2\ell\hbar < T$ and $\ell > 1$, and denote

$$M_\ell = \frac{\ln(\ell+1)}{\ln\left(\frac{2\ell+1}{\ell+1}\right)} ,$$

then

$$\left(\int_\Omega |u(x, T-\ell\hbar)|^2 \, \mathrm{e}^{-\frac{|x-x_0|^2}{4(\ell+1)\hbar}} \, \mathrm{d}x\right)^{1+M_\ell} \le (\ell+1)^{n/2} \int_\Omega |u(x,T)|^2 \, \mathrm{e}^{-\frac{|x-x_0|^2}{4\hbar}} \, \mathrm{d}x \left(\int_\Omega |u(x,0)|^2 \, \mathrm{d}x\right)^{M_\ell}$$

which implies

$$\left(\int_\Omega |u(x,T)|^2 \, \mathrm{d}x\right)^{1+M_\ell} \le (\ell+1)^{n/2} \, \mathrm{e}^{\frac{R^2(1+M_\ell)}{4(\ell+1)\hbar}} \int_\Omega |u(x,T)|^2 \, \mathrm{e}^{-\frac{|x-x_0|^2}{4\hbar}} \, \mathrm{d}x \left(\int_\Omega |u(x,0)|^2 \, \mathrm{d}x\right)^{M_\ell} .$$

Next, we split $\int_\Omega |u(x,T)|^2 \, \mathrm{e}^{-\frac{|x-x_0|^2}{4\hbar}} \, \mathrm{d}x$ into two parts:

$$\int_\Omega |u(x,T)|^2 \, \mathrm{e}^{-\frac{|x-x_0|^2}{4\hbar}} \, \mathrm{d}x \le \int_{B_r} |u(x,T)|^2 \, \mathrm{d}x + \mathrm{e}^{-\frac{r^2}{4\hbar}} \int_\Omega |u(x,0)|^2 \, \mathrm{d}x .$$

Therefore,

$$\left(\int_\Omega |u(x,T)|^2 \, \mathrm{d}x\right)^{1+M_\ell} \le (\ell+1)^{n/2} \left(\int_\Omega |u(x,0)|^2 \, \mathrm{d}x\right)^{M_\ell} \times \left(\mathrm{e}^{\frac{R^2(1+M_\ell)}{4(\ell+1)\hbar}} \int_{B_r} |u(x,T)|^2 \, \mathrm{d}x + \mathrm{e}^{\frac{R^2(1+M_\ell)}{4(\ell+1)\hbar}} \, \mathrm{e}^{-\frac{r^2}{4\hbar}} \int_\Omega |u(x,0)|^2 \, \mathrm{d}x\right) .$$

But for $\ell > 1$, $\frac{(1+M_\ell)}{4(\ell+1)} \le \frac{1}{2(\ell+1)} \frac{\ln(\ell+1)}{\ln\left(\frac{2\ell+1}{\ell+1}\right)} \le \frac{1}{2\ln(3/2)} \frac{\ln(2\ell)}{\ell} \le \frac{2^\varepsilon}{2\varepsilon\ln(3/2)} \frac{1}{\ell^{1-\varepsilon}}$ $\forall\varepsilon \in (0,1)$. Our choice of ℓ:

$$\ell := \left(\frac{R^2}{r^2}\right)^{1/(1-\varepsilon)} \left(\frac{2^{2+\varepsilon}}{\varepsilon\ln(3/2)}\right)^{1/(1-\varepsilon)} \quad \forall\varepsilon \in (0,1)$$

gives

$$\frac{R^2(1+M_\ell)}{4(\ell+1)\hbar} \le \frac{r^2}{8\hbar} \text{ and } M_\ell \le \frac{r^2\ell}{R^2} .$$

Further, it implies that for any $2\ell\hbar < T$,

$$\left(\int_\Omega |u(x,T)|^2 \, \mathrm{d}x\right)^{1+\frac{r^2\ell}{R^2}} \le (\ell+1)^{n/2} \left(\int_\Omega |u(x,0)|^2 \, \mathrm{d}x\right)^{\frac{r^2\ell}{R^2}} \left(\mathrm{e}^{\frac{r^2}{8\hbar}} \int_{B_r} |u(x,T)|^2 \, \mathrm{d}x + \mathrm{e}^{-\frac{r^2}{8\hbar}} \int_\Omega |u(x,0)|^2 \, \mathrm{d}x\right) .$$

On the other hand, for any $2\ell\hbar \ge T$, $1 \le \mathrm{e}^{\frac{r^2\ell}{4T}} \mathrm{e}^{-\frac{r^2}{8\hbar}}$. Therefore, for any $\hbar > 0$

$$\left(\int_\Omega |u(x,T)|^2 \, \mathrm{d}x\right)^{1+\frac{r^2\ell}{R^2}} \le (\ell+1)^{n/2} \left(\int_\Omega |u(x,0)|^2 \, \mathrm{d}x\right)^{\frac{r^2\ell}{R^2}} \left(\mathrm{e}^{\frac{r^2}{8\hbar}} \int_{B_r} |u(x,T)|^2 \, \mathrm{d}x + \mathrm{e}^{\frac{r^2\ell}{4T}} \, \mathrm{e}^{-\frac{r^2}{8\hbar}} \int_\Omega |u(x,0)|^2 \, \mathrm{d}x\right) .$$

Finally, we choose $\hbar > 0$ such that

$$\mathrm{e}^{\frac{r^2}{8\hbar}} := 2\left(\ell+1\right)^{n/2}\mathrm{e}^{\frac{r^2\ell}{4T}}\left(\frac{\int_{\Omega}\left|u\left(x,0\right)\right|^2\,\mathrm{d}x}{\int_{\Omega}\left|u\left(x,T\right)\right|^2\,\mathrm{d}x}\right)^{1+\frac{r^2\ell}{R^2}}$$

so that

$$\int_{\Omega}\left|u\left(x,T\right)\right|^2\,\mathrm{d}x \le \left(4\left(\ell+1\right)^n\mathrm{e}^{\frac{r^2\ell}{4T}}\int_{B_r}\left|u\left(x,T\right)\right|^2\,\mathrm{d}x\right)^{\frac{1}{2\left(1+\frac{r^2\ell}{R^2}\right)}}\left(\int_{\Omega}\left|u\left(x,0\right)\right|^2\,\mathrm{d}x\right)^{\frac{1+2\frac{r^2\ell}{R^2}}{2\left(1+\frac{r^2\ell}{R^2}\right)}} .$$

This completes the proof.

4.4. *Proof of Lemma 4.3*

We shall distinguish two cases: $t\in[t_1,t_2]$; $t\in[t_2,t_3]$. For $t_1\le t\le t_2$, we integrate $\left((T-t+\hbar)^{1+C_0}\mathrm{e}^{-tC_1}N\left(t\right)\right)'\le (T-t+\hbar)^{1+C_0}\mathrm{e}^{-tC_1}F_2\left(t\right)$ over (t,t_2) to get

$$\left(\frac{T-t_2+\hbar}{T-t+\hbar}\right)^{1+C_0}\mathrm{e}^{-C_1(t_2-t)}N\left(t_2\right)-\int_{t_1}^{t_2}\left|F_2\left(s\right)\right|\mathrm{d}s\le N\left(t\right) .$$

Then we solve $y'+2\alpha\left(t\right)y\le 0$ with

$$\alpha\left(t\right)=\frac{1}{2}\left(\frac{T-t_2+\hbar}{T-t+\hbar}\right)^{1+C_0}\mathrm{e}^{-C_1(t_2-t)}N\left(t_2\right)-\frac{C_0}{T-t+\hbar}-C_1-\int_{t_1}^{t_2}\left|F_2\right|\mathrm{d}s-\left|F_1\left(t\right)\right|$$

and integrate it over (t_1,t_2) to obtain

$$\mathrm{e}^{N\left(t_2\right)\int_{t_1}^{t_2}\left(\frac{T-t_2+\hbar}{T-t+\hbar}\right)^{1+C_0}\mathrm{e}^{-C_1(t_2-t)}\mathrm{d}t}\le\frac{y\left(t_1\right)}{y\left(t_2\right)}\left(\frac{T-t_1+\hbar}{T-t_2+\hbar}\right)^{2C_0}\mathrm{e}^{2(t_2-t_1)\left(C_1+\int_{t_1}^{t_2}\left|F_2\right|\mathrm{d}s\right)+2\int_{t_1}^{t_2}\left|F_1\right|\mathrm{d}s} .$$

For $t_2\le t\le t_3$, we integrate $\left((T-t+\hbar)^{1+C_0}\mathrm{e}^{-tC_1}N\left(t\right)\right)'\le (T-t+\hbar)^{1+C_0}\mathrm{e}^{-tC_1}F_2\left(t\right)$ over (t_2,t) to get

$$N\left(t\right)\le\mathrm{e}^{C_1(t-t_2)}\left(\frac{T-t_2+\hbar}{T-t+\hbar}\right)^{1+C_0}\left(N\left(t_2\right)+\int_{t_2}^{t_3}\left|F_2\left(s\right)\right|\mathrm{d}s\right) .$$

Then we solve $0\le y'+3\alpha\left(t\right)y$ with

$$\alpha\left(t\right)=\left(\frac{T-t_2+\hbar}{T-t+\hbar}\right)^{1+C_0}\mathrm{e}^{C_1(t-t_2)}\left(N\left(t_2\right)+\int_{t_2}^{t_3}\left|F_2\right|\mathrm{d}s\right)+\frac{C_0}{T-t+\hbar}+C_1+\left|F_1\left(t\right)\right|$$

and integrate it over (t_2,t_3) to obtain

$$y\left(t_2\right)\le\mathrm{e}^{3\left(N\left(t_2\right)+\int_{t_2}^{t_3}\left|F_2\right|\mathrm{d}s\right)\int_{t_2}^{t_3}\left(\frac{T-t_2+\hbar}{T-t+\hbar}\right)^{1+C_0}\mathrm{e}^{C_1(t-t_2)}\mathrm{d}t}$$
$$\times\, y\left(t_3\right)\left(\frac{T-t_2+\hbar}{T-t_3+\hbar}\right)^{3C_0}\mathrm{e}^{3(t_3-t_2)C_1+3\int_{t_2}^{t_3}\left|F_1\right|\mathrm{d}s} .$$

Finally, combining the case $t_1\le t\le t_2$ and the case $t_2\le t\le t_3$, we have

$$y(t_2) \le y(t_3)\left(\frac{y(t_1)}{y(t_2)}\right)^M \left(\frac{T-t_2+\hbar}{T-t_3+\hbar}\right)^{3C_0} \left(\frac{T-t_1+\hbar}{T-t_2+\hbar}\right)^{2MC_0} e^{(2M(t_2-t_1)+3(t_3-t_2))C_1}$$
$$\times e^{2M\int_{t_1}^{t_2}|F_1|\,ds+3\int_{t_2}^{t_3}|F_1|\,ds}\, e^{2M(t_2-t_1)\int_{t_1}^{t_2}|F_2|\,ds}\, e^{M(t_2-t_1)\int_{t_2}^{t_3}|F_2|\,ds}$$

with

$$M = 3\frac{\int_{t_2}^{t_3}\frac{e^{tC_1}}{(T-t+\hbar)^{1+C_0}}dt}{\int_{t_1}^{t_2}\frac{e^{tC_1}}{(T-t+\hbar)^{1+C_0}}dt},$$

which implies the desired estimate.

4.5. Proof of Lemma 4.5

The aim of this section is to prove the differential inequalities for the parabolic equations stated in Lemma 4.5. For any $z \in H^1\left(0,T;H_0^1(\vartheta)\right)$, a weak solution to $\partial_t z - \nabla\cdot(A\nabla z) = g$ with $g \in L^2(\Omega\times(0,T))$, we apply the following formula

$$\int_\vartheta \partial_t z\varphi\,dx + \int_\vartheta A\nabla z\cdot\nabla\varphi\,dx = \int_{\partial\vartheta} A\nabla z\cdot\nu\varphi\,dx + \int_\vartheta g\varphi\,dx \tag{4.5.1}$$

with different functions φ: $\varphi = ze^\Phi$, $\varphi = \partial_t ze^\Phi$ and $\varphi = A\nabla z\cdot\nabla\Phi e^\Phi$. Here ν is the unit outward normal vector to $\partial\vartheta$ and $\Phi = \Phi(x,t)$ is a sufficiently smooth function which will be chosen later. When $\varphi = ze^\Phi$, (4.5.1) becomes

$$\int_\vartheta A\nabla z\cdot\nabla ze^\Phi\,dx = -\int_\vartheta\left(\partial_t z + A\nabla z\cdot\nabla\Phi - \frac{1}{2}g\right)ze^\Phi\,dx + \frac{1}{2}\int_\vartheta gze^\Phi\,dx \tag{4.5.2}$$

and we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\vartheta |z|^2 e^\Phi\,dx &= \int_\vartheta z\partial_t ze^\Phi\,dx + \frac{1}{2}\int_\vartheta |z|^2\,\partial_t\Phi e^\Phi\,dx\\
&= -\int_\vartheta A\nabla z\cdot\nabla ze^\Phi\,dx - \int_\vartheta A\nabla z\cdot\nabla\Phi ze^\Phi\,dx + \int_\vartheta gze^\Phi\,dx + \frac{1}{2}\int_\vartheta |z|^2\,\partial_t\Phi e^\Phi\,dx\\
&= -\int_\vartheta A\nabla z\cdot\nabla ze^\Phi\,dx - \frac{1}{2}\int_\vartheta A\nabla\left(z^2\right)\cdot\nabla\Phi e^\Phi\,dx + \int_\vartheta gze^\Phi\,dx + \frac{1}{2}\int_\vartheta |z|^2\,\partial_t\Phi e^\Phi\,dx\\
&= -\int_\vartheta A\nabla z\cdot\nabla ze^\Phi\,dx + \frac{1}{2}\int_\vartheta |z|^2\left(\partial_t\Phi + \nabla\cdot(A\nabla\Phi) + A\nabla\Phi\cdot\nabla\Phi\right)e^\Phi\,dx + \int_\vartheta gze^\Phi\,dx\,.
\end{aligned} \tag{4.5.3}$$

When $\varphi = \partial_t ze^\Phi$, (4.5.1) becomes

$$\int_\vartheta |\partial_t z|^2 e^\Phi\,dx + \int_\vartheta A\nabla z\cdot\partial_t\nabla ze^\Phi\,dx + \int_\vartheta A\nabla z\cdot\nabla\Phi\partial_t ze^\Phi\,dx = \int_\vartheta g\partial_t ze^\Phi\,dx,$$

which implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\vartheta A\nabla z\cdot\nabla ze^\Phi\,dx - \frac{1}{2}\int_\vartheta \partial_t A\nabla z\cdot\nabla ze^\Phi\,dx\\
&= \int_\vartheta A\nabla z\cdot\partial_t\nabla ze^\Phi\,dx + \frac{1}{2}\int_\vartheta A\nabla z\cdot\nabla z\partial_t\Phi e^\Phi\,dx\\
&= -\int_\vartheta |\partial_t z|^2 e^\Phi\,dx - \int_\vartheta A\nabla z\cdot\nabla\Phi\partial_t ze^\Phi\,dx + \int_\vartheta g\partial_t ze^\Phi\,dx + \frac{1}{2}\int_\vartheta A\nabla z\cdot\nabla z\partial_t\Phi e^\Phi\,dx
\end{aligned}$$

$$= -\int_{\vartheta} \left|\partial_t z + A\nabla z\cdot\nabla\Phi - \frac{1}{2}g\right|^2 e^{\Phi}\,dx + \frac{1}{2}\int_{\vartheta} A\nabla z\cdot\nabla z\partial_t\Phi e^{\Phi}\,dx$$
$$+\int_{\vartheta} (\partial_t z - g)\, A\nabla z\cdot\nabla\Phi e^{\Phi}\,dx + \int_{\vartheta} |A\nabla z\cdot\nabla\Phi|^2 e^{\Phi}\,dx + \int_{\vartheta}\left|\frac{1}{2}g\right|^2 e^{\Phi}\,dx\,.$$

In the last equality, we made appear $\left|\partial_t z + A\nabla z\cdot\nabla\Phi - \frac{1}{2}g\right|$.

Next, we compute $\int_{\vartheta} (\partial_t z - g)\, A\nabla z\cdot\nabla\Phi e^{\Phi}\,dx$ by taking $\varphi = A\nabla z\cdot\nabla\Phi e^{\Phi}$ in (4.5.1): one has, with standard summation notations and $A = (A_{ij})_{1\leq i,j\leq n}$,

$$\int_{\vartheta} (\partial_t z - g)\, A\nabla z\cdot\nabla\Phi e^{\Phi}\,dx + \int_{\vartheta} |A\nabla z\cdot\nabla\Phi|^2 e^{\Phi}\,dx$$
$$= -\int_{\vartheta} A\nabla z\cdot\nabla\left(A\nabla z\cdot\nabla\Phi e^{\Phi}\right)dx + \int_{\partial\vartheta} (A\nabla z\cdot\nu)\,(A\nabla z\cdot\nabla\Phi)\,e^{\Phi}\,dx + \int_{\vartheta} |A\nabla z\cdot\nabla\Phi|^2 e^{\Phi}\,dx$$
$$= -\int_{\vartheta} A_{ij}\partial_{x_j}z\partial_{x_i}A_{k\ell}\partial_{x_\ell}z\partial_{x_k}\Phi e^{\Phi}\,dx - \int_{\vartheta} A\nabla^2\Phi A\nabla z\cdot\nabla z e^{\Phi}\,dx$$
$$-\int_{\vartheta} A\nabla^2 z A\nabla z\cdot\nabla\Phi e^{\Phi}\,dx + \int_{\partial\vartheta} (A\nabla z\cdot\nu)\,(A\nabla z\cdot\nabla\Phi)\,e^{\Phi}\,dx\,.$$

But by one integration by parts

$$-\int_{\vartheta} A\nabla^2 z A\nabla z\cdot\nabla\Phi e^{\Phi}\,dx$$
$$= -\frac{1}{2}\int_{\partial\vartheta} (A\nabla z\cdot\nabla z)\,(A\nabla\Phi\cdot\nu)\,e^{\Phi}\,dx + \frac{1}{2}\int_{\vartheta} \partial_{x_\ell}A_{ij}\partial_{x_j}zA_{k\ell}\partial_{x_i}z\partial_{x_k}\Phi e^{\Phi}\,dx$$
$$+\frac{1}{2}\int_{\vartheta} (A\nabla z\cdot\nabla z)\,\nabla\cdot(A\nabla\Phi)\,e^{\Phi}\,dx + \frac{1}{2}\int_{\vartheta} (A\nabla z\cdot\nabla z)\,(A\nabla\Phi\cdot\nabla\Phi)\,e^{\Phi}\,dx\,.$$

The homogeneous Dirichlet boundary condition on z implies $\nabla z = \nu\partial_\nu z$ on $\partial\vartheta$. Therefore, one deduces

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\vartheta} A\nabla z\cdot\nabla z e^{\Phi}\,dx &= \frac{1}{2}\int_{\vartheta}\partial_t A\nabla z\cdot\nabla z e^{\Phi}\,dx - \int_{\vartheta}\left|\partial_t z + A\nabla z\cdot\nabla\Phi - \frac{1}{2}g\right|^2 e^{\Phi}\,dx \\ &\quad -\int_{\vartheta} A\nabla^2\Phi A\nabla z\cdot\nabla z e^{\Phi}\,dx \\ &\quad +\frac{1}{2}\int_{\vartheta}(A\nabla z\cdot\nabla z)\,(\partial_t\Phi + \nabla\cdot(A\nabla\Phi) + A\nabla\Phi\cdot\nabla\Phi)\,e^{\Phi}\,dx \\ &\quad -\int_{\vartheta} A_{ij}\partial_{x_j}z\partial_{x_i}A_{k\ell}\partial_{x_\ell}z\partial_{x_k}\Phi e^{\Phi}\,dx + \frac{1}{2}\int_{\vartheta}\partial_{x_\ell}A_{ij}\partial_{x_j}zA_{k\ell}\partial_{x_i}z\partial_{x_k}\Phi e^{\Phi}\,dx \\ &\quad +\frac{1}{2}\int_{\partial\vartheta}(A\nabla z\cdot\nabla z)\,(A\nabla\Phi\cdot\nu)\,e^{\Phi}\,dx + \int_{\vartheta}\left|\frac{1}{2}g\right|^2 e^{\Phi}\,dx\,.\end{aligned} \tag{4.5.4}$$

Now, we are able to compute $\frac{d}{dt}\frac{\int_{\vartheta} A\nabla z\cdot\nabla z e^{\Phi}\,dx}{\int_{\vartheta}|z|^2 e^{\Phi}\,dx}$: one has, by (4.5.3) and (4.5.4),

$$\left(\int_\vartheta |z|^2 \mathrm{e}^\Phi \,\mathrm{d}x\right)^2 \frac{\mathrm{d}}{\mathrm{d}t} \frac{\displaystyle\int_\vartheta A\nabla z\cdot\nabla z \mathrm{e}^\Phi \,\mathrm{d}x}{\displaystyle\int_\vartheta |z|^2 \mathrm{e}^\Phi \,\mathrm{d}x}$$

$$\begin{aligned}
&= -2\int_\vartheta A\nabla^2\Phi A\nabla z\cdot\nabla z \mathrm{e}^\Phi \,\mathrm{d}x \int_\vartheta |z|^2 \mathrm{e}^\Phi \,\mathrm{d}x + \int_{\partial\vartheta} (A\nabla z\cdot\nabla z)\,(A\nabla\Phi\cdot\nu)\,\mathrm{e}^\Phi \,\mathrm{d}x \int_\vartheta |z|^2 \mathrm{e}^\Phi \,\mathrm{d}x \\
&\quad - 2\int_\vartheta \left|\partial_t z + A\nabla z\cdot\nabla\Phi - \frac{1}{2}g\right|^2 \mathrm{e}^\Phi \,\mathrm{d}x \int_\vartheta |z|^2 \mathrm{e}^\Phi \,\mathrm{d}x \\
&\quad + 2\int_\vartheta A\nabla z\cdot\nabla z \mathrm{e}^\Phi \,\mathrm{d}x \left(\int_\vartheta A\nabla z\cdot\nabla z \mathrm{e}^\Phi \,\mathrm{d}x - \int_\vartheta g z \mathrm{e}^\Phi \,\mathrm{d}x\right) \\
&\quad + \int_\vartheta \partial_t A\nabla z\cdot\nabla z \mathrm{e}^\Phi \,\mathrm{d}x \int_\vartheta |z|^2 \mathrm{e}^\Phi \,\mathrm{d}x \\
&\quad + 2\left(-\int_\vartheta A_{ij}\partial_{x_j} z \partial_{x_i} A_{k\ell}\partial_{x_\ell} z \partial_{x_k}\Phi \mathrm{e}^\Phi \,\mathrm{d}x + \frac{1}{2}\int_\vartheta \partial_{x_\ell} A_{ij}\partial_{x_j} z A_{k\ell}\partial_{x_i} z \partial_{x_k}\Phi \mathrm{e}^\Phi \,\mathrm{d}x\right) \int_\vartheta |z|^2 \mathrm{e}^\Phi \,\mathrm{d}x \\
&\quad + \int_\vartheta (A\nabla z\cdot\nabla z)\,(\partial_t\Phi + \nabla\cdot(A\nabla\Phi) + A\nabla\Phi\cdot\nabla\Phi)\,\mathrm{e}^\Phi \,\mathrm{d}x \int_\vartheta |z|^2 \mathrm{e}^\Phi \,\mathrm{d}x \\
&\quad - \int_\vartheta A\nabla z\cdot\nabla z \mathrm{e}^\Phi \,\mathrm{d}x \left(\int_\vartheta |z|^2\,(\partial_t\Phi + \nabla\cdot(A\nabla\Phi) + A\nabla\Phi\cdot\nabla\Phi)\,\mathrm{e}^\Phi \,\mathrm{d}x\right) \\
&\quad + 2\int_\vartheta \left|\frac{1}{2}g\right|^2 \mathrm{e}^\Phi \,\mathrm{d}x \int_\vartheta |z|^2 \mathrm{e}^\Phi \,\mathrm{d}x\,.
\end{aligned}$$

Notice that by the Cauchy–Schwarz inequality and (4.5.2), the contribution of the fourth and fifth terms of the above becomes

$$\begin{aligned}
&-\int_\vartheta \left|\partial_t z + A\nabla z\cdot\nabla\Phi - \frac{1}{2}g\right|^2 \mathrm{e}^\Phi \,\mathrm{d}x \int_\vartheta |z|^2 \mathrm{e}^\Phi \,\mathrm{d}x \\
&\quad + \int_\vartheta A\nabla z\cdot\nabla z \mathrm{e}^\Phi \,\mathrm{d}x \left(\int_\vartheta A\nabla z\cdot\nabla z \mathrm{e}^\Phi \,\mathrm{d}x - \int_\vartheta g z \mathrm{e}^\Phi \,\mathrm{d}x\right) \\
&= -\int_\vartheta \left|\partial_t z + A\nabla z\cdot\nabla\Phi - \frac{1}{2}g\right|^2 \mathrm{e}^\Phi \,\mathrm{d}x \int_\vartheta |z|^2 \mathrm{e}^\Phi \,\mathrm{d}x \\
&\quad + \left(-\int_\vartheta \left(\partial_t z + A\nabla z\cdot\nabla\Phi - \frac{1}{2}g\right) z \mathrm{e}^\Phi \,\mathrm{d}x + \frac{1}{2}\int_\vartheta g z \mathrm{e}^\Phi \,\mathrm{d}x\right) \\
&\quad \times \left(-\int_\vartheta \left(\partial_t z + A\nabla z\cdot\nabla\Phi - \frac{1}{2}g\right) z \mathrm{e}^\Phi \,\mathrm{d}x - \frac{1}{2}\int_\vartheta g z \mathrm{e}^\Phi \,\mathrm{d}x\right) \\
&= -\int_\vartheta \left|\partial_t z + A\nabla z\cdot\nabla\Phi - \frac{1}{2}g\right|^2 \mathrm{e}^\Phi \,\mathrm{d}x \int_\vartheta |z|^2 \mathrm{e}^\Phi \,\mathrm{d}x \\
&\quad + \left(\int_\vartheta \left(\partial_t z + A\nabla z\cdot\nabla\Phi - \frac{1}{2}g\right) z \mathrm{e}^\Phi \,\mathrm{d}x\right)^2 - \left(\frac{1}{2}\int_\vartheta g z \mathrm{e}^\Phi \,\mathrm{d}x\right)^2 \\
&\leq 0\,.
\end{aligned}$$

Therefore, one concludes that

$$\frac{d}{dt}\frac{\int_\vartheta A\nabla z\cdot\nabla z e^\Phi\,dx}{\int_\vartheta |z|^2 e^\Phi\,dx} \le \frac{-2\int_\vartheta A\nabla^2\Phi A\nabla z\cdot\nabla z e^\Phi\,dx}{\int_\vartheta |z|^2 e^\Phi\,dx} + \frac{\int_{\partial\vartheta}(A\nabla z\cdot\nabla z)(A\nabla\Phi\cdot\nu)\,e^\Phi\,dx}{\int_\vartheta |z|^2 e^\Phi\,dx}$$

$$+\frac{\int_\vartheta |g|^2 e^\Phi\,dx}{\int_\vartheta |z|^2 e^\Phi\,dx}+\frac{\int_\vartheta \partial_t A\nabla z\cdot\nabla z e^\Phi\,dx}{\int_\vartheta |z|^2 e^\Phi\,dx}$$

$$+\frac{-2\int_\vartheta A_{ij}\partial_{x_j}z\partial_{x_i}A_{k\ell}\partial_{x_\ell}z\partial_{x_k}\Phi e^\Phi\,dx+\int_\vartheta \partial_{x_\ell}A_{ij}\partial_{x_j}zA_{k\ell}\partial_{x_i}z\partial_{x_k}\Phi e^\Phi\,dx}{\int_\vartheta |z|^2 e^\Phi\,dx}$$

$$+\frac{\int_\vartheta A\nabla z\cdot\nabla z\,(\partial_t\Phi+\nabla\cdot(A\nabla\Phi)+A\nabla\Phi\cdot\nabla\Phi)\,e^\Phi\,dx}{\int_\vartheta |z|^2 e^\Phi\,dx}$$

$$-\frac{\int_\vartheta A\nabla z\cdot\nabla z e^\Phi\,dx}{\int_\vartheta |z|^2 e^\Phi\,dx}\times\frac{\int_\vartheta |z|^2\,(\partial_t\Phi+\nabla\cdot(A\nabla\Phi)+A\nabla\Phi\cdot\nabla\Phi)\,e^\Phi\,dx}{\int_\vartheta |z|^2 e^\Phi\,dx}.$$

4.6. *Proof of Lemma 4.7*

Let $0<\varepsilon<\rho/2$ and $\phi\in C_0^\infty(B_\rho)$ be such that $0\le\phi\le 1$, $\phi=1$ on $B_{\rho-\varepsilon}=\{x;\boldsymbol{d}(x,x_0)<\rho-\varepsilon\}$. We multiply the equation $\partial_t u-\nabla\cdot(A\nabla u)+b_1\cdot\nabla u+b_0u=0$ by $\phi^2 u e^{-\boldsymbol{d}(x,x_0)^2/h}$, where $h>0$, and we integrate over $\Omega\cap B_\rho$. We get, by one integration by parts,

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega\cap B_\rho}|\phi u|^2 e^{-\boldsymbol{d}(x,x_0)^2/h}\,dx+\int_{\Omega\cap B_\rho}A\nabla u\cdot\nabla\left(\phi^2 u e^{-\boldsymbol{d}(x,x_0)^2/h}\right)dx$$
$$=-\int_{\Omega\cap B_\rho}(b_1\cdot\nabla u+b_0u)\,\phi^2 u\,e^{-\boldsymbol{d}(x,x_0)^2/h}\,dx.$$

But, $A\nabla u\cdot\nabla\left(\phi^2 u\,e^{-\boldsymbol{d}^2/h}\right)=\left[2\phi uA\nabla\phi\cdot\nabla u+\phi^2A\nabla u\cdot\nabla u+\phi^2u\left(-\frac{2\boldsymbol{d}\nabla\boldsymbol{d}}{h}\right)\cdot A\nabla u\right]e^{-\boldsymbol{d}^2/h}$. Therefore, by the Cauchy–Schwarz inequality, it follows that

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega\cap B_\rho}|\phi u|^2 e^{-\boldsymbol{d}(x,x_0)^2/h}\,dx+\int_{\Omega\cap B_\rho}\phi^2A\nabla u\cdot\nabla u e^{-\boldsymbol{d}(x,x_0)^2/h}\,dx$$
$$\le\frac{1}{2}\int_{\Omega\cap B_\rho}\phi^2A\nabla u\cdot\nabla u e^{-\boldsymbol{d}(x,x_0)^2/h}\,dx+4\int_{\Omega\cap B_\rho}A\nabla\phi\cdot\nabla\phi\,|u|^2 e^{-\boldsymbol{d}(x,x_0)^2/h}\,dx$$
$$+4\int_{\Omega\cap B_\rho}\frac{\boldsymbol{d}^2}{h^2}A\nabla\boldsymbol{d}\cdot\nabla\boldsymbol{d}\,|\phi u|^2 e^{-\boldsymbol{d}(x,x_0)^2/h}\,dx+\|b_0\|_{L^\infty}\int_{\Omega\cap B_\rho}|\phi u|^2 e^{-\boldsymbol{d}(x,x_0)^2/h}\,dx$$

$$+\left(\int_{\Omega\cap B_\rho}\phi^2\left|\nabla u\right|^2 \mathrm{e}^{-\boldsymbol{d}(x,x_0)^2/h}\,\mathrm{d}x\right)^{1/2}\left(\left\|b_1\right\|^2_{L^\infty}\int_{\Omega\cap B_\rho}\left|\phi u\right|^2 \mathrm{e}^{-\boldsymbol{d}(x,x_0)^2/h}\,\mathrm{d}x\right)^{1/2}.$$

Thus, with the fact that $A_T(x)\nabla\boldsymbol{d}(x,x_0)\cdot\nabla\boldsymbol{d}(x,x_0)=1$ and the ellipticity of A, one gets, for some constants $C_A, C_{A,b}>0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega\cap B_\rho}\left|\phi u\right|^2 \mathrm{e}^{-\boldsymbol{d}(x,x_0)^2/h}\,\mathrm{d}x-\left(\frac{\rho^2}{h^2}C_A+C_{A,b}\right)\int_{\Omega\cap B_\rho}\left|\phi u\right|^2 \mathrm{e}^{-\boldsymbol{d}(x,x_0)^2/h}\,\mathrm{d}x$$
$$\leq C_A\mathrm{e}^{-\frac{(\rho-\varepsilon)^2}{h}}\int_{\Omega\cap B_\rho}\left|u(x,t)\right|^2\,\mathrm{d}x\,.$$

Then we have

$$\int_{\Omega\cap B_\rho}\left|\phi u(\cdot,T)\right|^2 \mathrm{e}^{-\boldsymbol{d}(x,x_0)^2/h}\,\mathrm{d}x\leq \mathrm{e}^{\left(\frac{\rho^2}{h^2}C_A+C_{A,b}\right)(T-t)}\int_{\Omega\cap B_\rho}\left|\phi u(\cdot,t)\right|^2 \mathrm{e}^{-\boldsymbol{d}(x,x_0)^2/h}\,\mathrm{d}x$$
$$+C_A\mathrm{e}^{\left(\frac{\rho^2}{h^2}C_A+C_{A,b}\right)(T-t)}\mathrm{e}^{-\frac{(\rho-\varepsilon)^2}{h}}\int_t^T\int_{\Omega\cap B_\rho}\left|u\right|^2\,\mathrm{d}x\,\mathrm{d}s$$

which gives

$$\int_{\Omega\cap B_{\rho-2\varepsilon}}\left|u(x,T)\right|^2\,\mathrm{d}x\leq \mathrm{e}^{\left(\frac{\rho^2}{h^2}C_A+C_{A,b}\right)(T-t)}\mathrm{e}^{\frac{(\rho-2\varepsilon)^2}{h}}\int_{\Omega\cap B_\rho}\left|u(x,t)\right|^2\,\mathrm{d}x$$
$$+C_A\mathrm{e}^{\left(\frac{\rho^2}{h^2}C_A+C_{A,b}\right)(T-t)}\mathrm{e}^{-\frac{(\rho-\varepsilon)^2}{h}}\mathrm{e}^{\frac{(\rho-2\varepsilon)^2}{h}}\int_t^T\int_{\Omega\cap B_\rho}\left|u\right|^2\,\mathrm{d}x\,\mathrm{d}s\,.$$

Let $T/2<T-\delta h\leq t\leq T$; it yields

$$\int_{\Omega\cap B_{\rho-2\varepsilon}}\left|u(x,T)\right|^2\,\mathrm{d}x\leq \mathrm{e}^{C_{A,b}\delta h}\mathrm{e}^{\frac{\rho^2}{h}\delta C_A}\mathrm{e}^{\frac{(\rho-2\varepsilon)^2}{h}}\int_{\Omega\cap B_\rho}\left|u(x,t)\right|^2\,\mathrm{d}x$$
$$+C_A\mathrm{e}^{C_{A,b}\delta h}\mathrm{e}^{\frac{\rho^2}{h}\delta C_A}\mathrm{e}^{-\frac{(\rho-\varepsilon)^2}{h}}\mathrm{e}^{\frac{(\rho-2\varepsilon)^2}{h}}\int_{T-\delta h}^T\int_{\Omega\cap B_\rho}\left|u\right|^2\,\mathrm{d}x\,\mathrm{d}s\,.$$

Choose

$$\delta=\frac{1}{C_A}\frac{\varepsilon\left(2\rho-3\varepsilon\right)}{2\rho^2}$$

that is $\delta C_A=\frac{1}{2}\frac{(\rho-\varepsilon)^2-(\rho-2\varepsilon)^2}{\rho^2}\in(0,1/8]$ so that $\rho^2\delta C_A-(\rho-\varepsilon)^2+(\rho-2\varepsilon)^2<0$. Therefore, we get

$$\int_{\Omega\cap B_{\rho-2\varepsilon}}\left|u(x,T)\right|^2\,\mathrm{d}x\leq \mathrm{e}^{C_{A,b}\delta h}\mathrm{e}^{\frac{(\rho-\varepsilon)^2+(\rho-2\varepsilon)^2}{2h}}\int_{\Omega\cap B_\rho}\left|u(x,t)\right|^2\,\mathrm{d}x$$
$$+C_A\mathrm{e}^{C_{A,b}\delta h}\mathrm{e}^{\frac{-(\rho-\varepsilon)^2+(\rho-2\varepsilon)^2}{2h}}\int_{T-\delta h}^T\int_{\Omega\cap B_\rho}\left|u\right|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \mathrm{e}^{C_{A,b}}\mathrm{e}^{\frac{(\rho-\varepsilon)^2+(\rho-2\varepsilon)^2}{2h}}\int_{\Omega\cap B_\rho}\left|u(x,t)\right|^2\,\mathrm{d}x$$
$$+C_A\mathrm{e}^{C_{A,b}}\mathrm{e}^{\frac{-(\rho-\varepsilon)^2+(\rho-2\varepsilon)^2}{2h}}\mathbb{U}$$

where we used $\delta h<\min(1,T/2)$ and have denoted

$$\mathbb{U}=\int_0^T\int_\Omega |u(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s+\int_\Omega |u(x,0)|^2\,\mathrm{d}x\,.$$

Now, choose h such that both $\delta h<\min(1,T/2)$ and

$$(1+C_A)\,\mathrm{e}^{C_{A,b}}\mathrm{e}^{\frac{-(\rho-\varepsilon)^2+(\rho-2\varepsilon)^2}{2h}}\mathbb{U}\le\frac{1}{\mathrm{e}}\int_{\Omega\cap B_{\rho-2\varepsilon}}|u(x,T)|^2\,\mathrm{d}x\,.$$

With such choice, one has

$$\left(1-\frac{1}{\mathrm{e}}\right)\int_{\Omega\cap B_{\rho-2\varepsilon}}|u(x,T)|^2\,\mathrm{d}x\le\mathrm{e}^{C_{A,b}}\mathrm{e}^{\frac{(\rho-\varepsilon)^2+(\rho-2\varepsilon)^2}{2h}}\int_{\Omega\cap B_\rho}|u(x,t)|^2\,\mathrm{d}x$$

and, moreover,

$$\mathbb{U}\le\mathrm{e}^{\frac{(\rho-\varepsilon)^2}{h}}\int_{\Omega\cap B_\rho}|u(x,t)|^2\,\mathrm{d}x$$

for any $T/2<T-\delta h\le t\le T$. Such h exists by choosing

$$h=\frac{\varepsilon\,(2\rho-3\varepsilon)/2}{\ln\left(K\dfrac{(1+C_A)\mathrm{e}^{C_{A,b}}\mathbb{U}}{\frac{1}{\mathrm{e}}\int_{\Omega\cap B_{\rho-2\varepsilon}}|u(x,T)|^2\,\mathrm{d}x}\right)}\text{ with }K=\mathrm{e}^{\varepsilon\frac{(2\rho-3\varepsilon)}{2}\left(\frac{2}{T}+1\right)\delta}\,.$$

Clearly, $\delta h<T/2$ and $\delta h\le 1$. We conclude that, for any $T/2\le T-\theta\le t\le T$,

$$\frac{\mathbb{U}}{\int_{\Omega\cap B_\rho}|u(x,t)|^2\,\mathrm{d}x}\le\mathrm{e}^{\frac{1}{C_A}\frac{\varepsilon(2\rho-3\varepsilon)(\rho-\varepsilon)^2}{2\rho^2}\frac{1}{\theta}}$$

with

$$\frac{1}{\theta}=C_A\frac{4\rho^2}{\varepsilon^2\,(2\rho-3\varepsilon)^2}\ln\left(e\,(1+C_A)\,\mathrm{e}^{C_{A,b}}\mathrm{e}^{\left(\frac{2}{T}+1\right)\frac{1}{C_A}\frac{\varepsilon^2(2\rho-3\varepsilon)^2}{4\rho^2}}\frac{\mathbb{U}}{\int_{\Omega\cap B_{\rho-2\varepsilon}}|u(x,T)|^2\,\mathrm{d}x}\right).$$

This completes the proof.

Acknowledgements

We are very happy to dedicate this article to our friend and colleague Jean-Michel Coron on the occasion of his 60th birthday.

Appendix A

This appendix is devoted to the proof of Proposition 3.2 and of the backward inequality (2.2).

A.1. Trace estimate for the kinetic transport equation (proof of Proposition 3.2)

Denote $\left(\partial\Omega\times\mathbb{S}^{d-1}\right)_+=\left\{(x,v)\in\partial\Omega\times\mathbb{S}^{d-1};\,v\cdot\vec{n}_x\ge 0\right\}$. First, multiplying both sides of the first line of (1.1) by $\eta f\,|f|^{\eta-2}$ and integrating over $\Omega\times\mathbb{S}^{d-1}\times(0,T)$, one has the following a priori estimate, for any $\eta\ge 2$

$$\int_0^T\int_{\left(\partial\Omega\times\mathbb{S}^{d-1}\right)_+}v\cdot\vec{n}_x\,|f|^\eta\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t\le\epsilon\int_{\Omega\times\mathbb{S}^{d-1}}|f_0|^\eta\,\mathrm{d}x\,\mathrm{d}v\,.$$

Secondly, one uses Hölder's inequality to get

$$\int_0^T \int_{\partial\Omega\times\mathbb{S}^{d-1}} |f|^2 \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \leq \left(\int_0^T \int_{\left(\partial\Omega\times\mathbb{S}^{d-1}\right)_+} \frac{\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t}{\left(v\cdot\vec{n}_x\right)^{\frac{1}{p-1}}} \right)^{\frac{p-1}{p}} \left(\int_0^T \int_{\left(\partial\Omega\times\mathbb{S}^{d-1}\right)_+} v\cdot\vec{n}_x \, |f|^{2p} \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \right)^{\frac{1}{p}} .$$

But

$$\int_{\left(\partial\Omega\times\mathbb{S}^{d-1}\right)_+} \frac{\mathrm{d}x\,\mathrm{d}v}{\left(v\cdot\vec{n}_x\right)^{\frac{1}{p-1}}} \leq C\frac{p-1}{p-2} \text{ for any } p>2\,.$$

Hence, as soon as $p>2$, one gets the desired estimate

$$\|f\|_{L^2\left(\partial\Omega\times\mathbb{S}^{d-1}\times(0,T)\right)} \leq CT^{\frac{p-1}{2p}}\epsilon^{\frac{1}{2p}}C_p\|f_0\|_{L^{2p}\left(\Omega\times\mathbb{S}^{d-1}\right)}$$

where $C_p=\left(\frac{p-1}{p-2}\right)^{\frac{p-1}{2p}}$ and $C>0$ only depends on (Ω,d).

A.2. Backward estimate for diffusion equations (proof of (2.2))

Classical energy identities for our parabolic equation are:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u|^2 \,\mathrm{d}x + \int_\Omega \frac{1}{da}|\nabla u|^2 \,\mathrm{d}x = 0\,,$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{da}|\nabla \varphi|^2 \,\mathrm{d}x + \int_\Omega |u|^2 \,\mathrm{d}x = 0\,,$$

where $\varphi(\cdot,t)\in H_0^1(\Omega)$ solves $-\nabla\cdot\left(\frac{1}{da}\nabla\varphi(\cdot,t)\right)=u(\cdot,t)$ in Ω. Now, one can easily check, with $y(t)=\int_\Omega \frac{1}{da(x)}|\nabla\varphi(x,t)|^2\,\mathrm{d}x$ and $N(t)=\dfrac{\int_\Omega |u(x,t)|^2\,\mathrm{d}x}{\int_\Omega \frac{1}{da(x)}|\nabla\varphi(x,t)|^2\,\mathrm{d}x}$, that

$$\begin{cases} \frac{1}{2}y'(t)+N(t)\,y(t)=0 \\ N'(t)\leq 0\,. \end{cases}$$

By solving such differential inequalities, one obtain

$$\int_\Omega \frac{1}{da(x)}|\nabla\varphi(x,0)|^2\,\mathrm{d}x \leq \mathrm{e}^{2TN(0)}\int_\Omega \frac{1}{da(x)}|\nabla\varphi(x,T)|^2\,\mathrm{d}x$$

which implies

$$\|u(\cdot,0)\|^2_{H^{-1}(\Omega)} \leq \frac{c_{\max}}{c_{\min}}\mathrm{e}^{2T\frac{\|u(\cdot,0)\|^2_{L^2(\Omega)}}{dc_{\min}\|u(\cdot,0)\|^2_{H^{-1}(\Omega)}}}\|u(\cdot,T)\|^2_{H^{-1}(\Omega)}\,.$$

One concludes that

$$\|u(\cdot,0)\|_{L^2(\Omega)} = \frac{\|u(\cdot,0)\|_{L^2(\Omega)}}{\|u(\cdot,0)\|^2_{H^{-1}(\Omega)}}\|u(\cdot,0)\|_{H^{-1}(\Omega)} \leq \frac{c}{\sqrt{T}}\mathrm{e}^{cT\frac{\|u(\cdot,0)\|^2_{L^2(\Omega)}}{\|u(\cdot,0)\|^2_{H^{-1}(\Omega)}}}\|u(\cdot,T)\|_{L^2(\Omega)}\,.$$

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