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Joint spectrum and large deviation principle for random matrix products

Spectre joint et principe de grandes déviations pour les produits de matrices aléatoires

Cagri Sert

Departement Mathematik, ETHZ, Rämistrasse 101, CH-8092 Zürich, Switzerland

A R T I C L E   I N F O

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A B S T R A C T

The aim of this note is to announce some results about the probabilistic and deterministic asymptotic properties of linear groups. The first one is the analogue, for norms of random matrix products, of the classical theorem of Cramér on large deviation principles (LDP) for sums of iid real random variables. In the second result, we introduce a limit set describing the asymptotic shape of the powers $S^n = \{g_1 \cdot \ldots \cdot g_n \mid g_i \in S\}$ of a subset S of a semisimple linear Lie group G (e.g., $\mathrm{SL}(d, \mathbb{R})$). This limit set has applications, among others, in the study of large deviations.

R É S U M É

Le but de cette note est d'énoncer certains résultats sur les propriétés asymptotiques probabilistes et déterministes des groupes linéaires. Le premier est l'homologue, pour les normes des produits de matrices aléatoires, du théorème classique de Cramér sur le principe de grandes déviations des sommes des variables aléatoires iid. Dans le deuxième résultat, nous introduisons un ensemble limite décrivant la forme asymptotique des puissances $S^n = \{g_1 \cdot \ldots \cdot g_n \mid g_i \in S\}$ d'une partie S d'un groupe de Lie linéaire semisimple (e.g., $\mathrm{SL}(d, \mathbb{R})$). Cet ensemble limite trouve, parmi d'autres, une application dans l'étude des grandes déviations.

E-mail address: Cagri.sert@math.ethz.ch.

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1. Large deviation principle for random matrix products

1.1. Introduction

Let G be a connected semisimple linear real algebraic group (e.g., $SL(d,\mathbb{R})$). A random walk on G is a random process $Y_n = X_n.\ldots.X_1$ where X_i's are independent and identically distributed (iid) G-valued random variables. Starting from Bellman [3], Furstenberg–Kesten [13] and Furstenberg [12], an important aim in the study of these non-commutative random walks was to establish the analogues of the classical limit theorems existing for the iid real random variables. More precisely, one is interested in studying the probabilistic limiting behavior of the (logarithms of the) norms $\log ||Y_n||$ of random matrix products. In fact, we will consider the slightly more general multi-norm given by the Cartan projection that we briefly describe before going on: let $\mathfrak{g}$ be the Lie algebra of the group G, $\mathfrak{a}$ a Cartan subspace of $\mathfrak{g}$ and $\mathfrak{a}^+$ a chosen Weyl chamber. Let K be a maximal compact subgroup of G for which we have the Cartan decomposition $G = K\exp(\mathfrak{a}^+)K$. Then the map $\kappa : G \longrightarrow \mathfrak{a}^+$ is well defined by the following equality and is called the Cartan projection: for all $g \in G$, $g = k\exp(\kappa(g))u$ for some $k, u \in K$. As an example, in the case of $G = SL(d,\mathbb{R})$, the Cartan projection $\kappa(g)$ of a matrix $g \in SL(d,\mathbb{R})$ writes as $\kappa(g) = (\log ||g||, \log \frac{||\wedge^2 g||}{||g||}, \ldots, \log \frac{||\wedge^d g||}{||\wedge^{d-1} g||})$, where $\wedge^k\mathbb{R}^d$'s are considered with their canonical Euclidean structures and $||.||$'s denote the associated operator norms. The components of $\kappa(g)$ are the logarithms of the singular values of g.

The first limit theorem that was proven for random matrix products was the analogue of the law of large numbers. Stating it in our setting, Furstenberg–Kesten's result [13] reads: if μ is a probability measure on G with a finite first moment (i.e. $\int ||\kappa(g)||\mu(dg) < \infty$ for some norm $||.||$ on $\mathfrak{a}$), then the μ-random walk $Y_n = X_n.\ldots.X_1$ (i.e. X_i's are iid of law μ) satisfies

$$\frac{1}{n}\kappa(Y_n) \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} \vec{\lambda}_\mu \in \mathfrak{a}$$

where $\vec{\lambda}_\mu$ can be defined by this and called the Lyapunov vector of μ. Nowadays, this result is a rather straightforward corollary of Kingman's subadditive ergodic theorem. A second important limit theorem that was established in increasing generality by Tutubalin [25], Le Page [18], Goldsheid–Guivarc'h [14], and Benoist–Quint [8,7] is the central limit theorem (CLT). Benoist–Quint's CLT reads: if μ is a probability measure on G of finite second-order moment and such that the support of μ generates a Zariski-dense semigroup in G, then $\frac{1}{\sqrt{n}}(\kappa(Y_n) - n\vec{\lambda}_\mu)$ converges in distribution to a non-degenerate Gaussian law on $\mathfrak{a}$. A feature of this result is the Zariski density assumption, which also appears in our result below. We note that the fact that the support S of the probability measure μ generates a Zariski-dense semigroup can be read as: any polynomial that vanishes on $\cup_{n\geq 1} S^n$ also vanishes on G (we recall that in $\mathbb{R}$, a subset is Zariski dense if and only if it is infinite). Some other limit theorems whose analogues have been obtained are the law of iterated logarithm and local limit theorems for which we refer the reader to the nice books of Bougerol–Lacroix [10] and more recently Benoist–Quint [7].

An essential and, up to our work [20,21], a rather incomplete aspect of these non-commutative limit theorems is concerned with large deviations. The main result in this direction is that of Le Page [18] (see also Bougerol [10]) and its extension by Benoist–Quint [7], stating the exponential decay of large deviation probabilities off the Lyapunov vector. Before stating this result, recall that a probability measure μ on G is said to have a finite exponential moment, if there exists $\alpha > 1$ such that we have $\int \alpha^{||\kappa(g)||}\mu(\mathrm{d}g) < \infty$. We have the following theorem.

Theorem 1.1 *(Le Page [18], Benoist–Quint [7]). Let G be as before, μ be a probability measure of finite exponential moment on G whose support generates a Zariski-dense semigroup in G. Let Y_n denote the nth step of the μ-random walk on G. Then, for all $\epsilon > 0$, we have* $\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(||\frac{\kappa(Y_n)}{n} - \vec{\lambda}_\mu|| > \epsilon) < 0$.

1.2. Statement of the main result

In our first main result, under the usual Zariski density assumption, we prove the matrix analogue of a classical theorem (see below) about large deviations for iid real random variables. Let X be a topological space and $\mathcal{F}$ be a σ-algebra on X.

Definition 1.2. A sequence Z_n of X-valued random variables is said to satisfy a large deviation principle (LDP) with rate function $I : X \longrightarrow [0,\infty]$, if for every measurable subset R of X, we have

$$-\inf_{x\in \mathring{R}} I(x) \leq \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(Z_n \in R) \leq \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(Z_n \in R) \leq -\inf_{x\in\overline{R}} I(x)$$

where, $\mathring{R}$ denotes the interior and $\overline{R}$ the closure of R.

With this definition, the classical Cramér–Chernoff theorem says that the sequence of averages $Y_n = \frac{1}{n}\sum_{i=1}^n X_i$ of real iid random variables of finite exponential moment satisfies an LDP with a proper convex rate function I, given by the convex conjugate (Fenchel–Legendre transform) of the logarithmic moment generating function of X_i's. Our first main result reads as follows.

Theorem 1.3. *Let G be a connected semisimple linear real algebraic group and μ be a probability measure of finite exponential moment on G, whose support generates a Zariski dense sub-semigroup of G. Then, the sequence of random variables $\frac{1}{n}\kappa(Y_n)$ satisfies an LDP with a proper convex rate function $I : \mathfrak{a} \longrightarrow [0,\infty]$ assuming a unique zero on the Lyapunov vector $\vec{\lambda}_\mu$ of μ.*

Remark 1. 1. Without any moment assumptions on μ, we also obtain a weaker result which is an analogue of a result of Bahadur [2] for iid real random variables.

2. Under a stronger exponential moment condition (sometimes called finite super-exponential moment), by exploiting the convexity of I, we are able to identify the rate function I with the convex conjugate of a limiting logarithmic moment generating function of the random variables $\frac{1}{n}\kappa(Y_n)$.

3. It follows by convexity of I that the effective support $D_I := \{x \in \mathfrak{a} \mid I(x) < \infty\}$ of the rate function I is a convex subset of $\mathfrak{a}^+$. We also show that this set D_I depends only on the support S of μ (i.e. not on the particular mass distribution μ on S) and we identify D_I with a Hausdorff limiting set of a deterministic construction, namely the joint spectrum $J(S)$ of S (see Theorem 2.2 below).

4. We conjecture that a similar LDP holds for the Jordan projection $\lambda : G \to \mathfrak{a}^+$ in place of κ (see Def. in paragraph 2.1).

1.3. The idea of the proof

The fact that $I(\vec{\lambda}_\mu) = 0$ follows from the definition of $\vec{\lambda}_\mu$ and I. That this zero is unique is merely the translation of the deep result of Le Page (Theorem 1.1). That I is proper is a rather straightforward consequence of Chernoff's estimates and that one can identify I as a Fenchel–Legendre transform (2. of Remark 1) follows from its convexity using standard techniques, namely the Varadhan's integral lemma (see Theorem 4.3.1. in [11]) and elementary properties of convex conjugation of functions. Moreover, the convexity of the rate function is proven using ideas similar to those used to prove its existence. Therefore, in this note, we will focus on the proof of the existence of an LDP. For simplicity, we shall assume that the measure μ is compactly supported. We make use of the following general fact:

Theorem 1.4 *(see Theorem 4.1.11 in [11]). Let X be a topological space endowed with its Borel σ-algebra β_X, and Z_n be a sequence of X-valued random variables taking values in a compact subset of X. Let $\mathcal{A}$ be a base of open sets for the topology of X. For each $x \in X$, define:*

$$I_{li}(x) := \sup_{\substack{A\in\mathcal{A}\\ x\in A}} -\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}(Z_n \in A) \qquad \textit{and} \qquad I_{ls}(x) := \sup_{\substack{A\in\mathcal{A}\\ x\in A}} -\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}(Z_n \in A)$$

Suppose that for all $x \in X$, we have $I_{li}(x) = I_{ls}(x)$. Then, the sequence Z_n satisfies an LDP with rate function I given by $I(x) := I_{li}(x) = I_{ls}(x)$.

In view of this theorem, to prove the existence of the LDP in Theorem 1.3, we have to show that the equality $I_{li} = I_{ls}$ is satisfied. A first remark is that if κ were an additive mapping (i.e. $\kappa(gh) = \kappa(g) + \kappa(h)$), this would follow rather easily from the independence of random walk increments and uniform continuity of κ. A further important remark is that, in fact, a weaker form of additivity (i.e. $||\kappa(gh) - \kappa(g) - \kappa(h)||$ is uniformly bounded for all $g, h \in \mathrm{supp}(\mu)$) is sufficient to insure this equality. A key result of Benoist [4,5] shows that this weak form of additivity is satisfied in a given (r,ϵ)-Schottky semigroup. This already finishes the proof in the case when μ is supported on such a semigroup. For the general case, we need an argument showing that we can restrict the random walk on Schottky semigroups with no loss in the exponential rate of decay probabilities (as in Lemma 1.5). This is done by using a result of Abels–Margulis–Soifer [1] about proximal elements in Zariski dense semigroups (which in turn uses a result of Benoist–Labourie [6] and Prasad [19]) together with the uniform continuity of the Cartan projection and then a simple partitioning and pigeonhole argument. A key step in this proof is the following Lemma 1.5.

Recall that an element g in $\mathrm{PGL}(d,\mathbb{R})$ is called ϵ-proximal if it has a unique eigenvalue of maximal modulus and the ratio of its first two singular values is at least $\frac{1}{\epsilon}$. It is called (r,ϵ)-proximal if additionally the top eigenvector is at least r away from the projective hyperplane spanned by the other generalized eigenspaces. An element g in G is called (r,ϵ)-loxodromic if it is (r,ϵ)-proximal in each of the $\mathrm{rk}(G)$ fundamental proximal representations of G.

Abels–Margulis–Soifer show that for a Zariski dense semigroup Γ in G, there exists $r > 0$ such that for every $\epsilon > 0$, one can find a *finite* subset $F \subset \Gamma$ with the property that for all $\gamma \in \Gamma$, there exists $f \in F$ such that $\gamma.f$ is (r,ϵ)-loxodromic. We denote by Γ the semigroup generated by the support of μ.

Lemma 1.5. *Let $0 < \epsilon < r = r(\Gamma)$. There exist a compact set $C = C(\Gamma,\epsilon) \subset \mathfrak{a}$, a natural number $i_0 = i_0(\epsilon,\Gamma,\mu)$, and a constant $d_1 = d_1(\epsilon,\Gamma,\mu) > 0$ such that for all $n_0 \in \mathbb{N}$ and subset $R \subset \mathfrak{a}^+$, there exists a natural number $n_1 \geq n_0$ with $n_1 - n_0 \leq i_0$ such that we have*

$$\mathbb{P}(\kappa(Y_{n_1}) \in R + C \textit{ and } Y_{n_1} \textit{ is } (r,\epsilon)\textit{-loxodromic}) \geq d_1.\mathbb{P}(\kappa(Y_{n_0}) \in R)$$

The next step in the proof consists in observing that one can further restrict the random walk to an (r, ϵ)-Schottky semigroup, again losing only a uniformly bounded proportion of the probability (as in Lemma 1.5). By doing so, we reduce the situation to a random walk on a semigroup on which the Cartan projection $\kappa(.)$ is almost additive, and hence we can conclude as we sketched in the beginning of the argument.

Remark 2. In a further work in preparation [24], using essentially the same method as in the proof of Theorem 1.3, we show that the LDP holds for the average word length of random walks on Gromov hyperbolic groups.

2. Joint spectrum

2.1. Introduction

In this second part of this note, we define the notion of *joint spectrum* of a bounded subset of a semisimple Lie group. We then relate this notion to the effective support $D_I = \{x \in \mathfrak{a} \mid I(x) < \infty\}$ of the rate function from Theorem 1.3. Recall the definition of the Jordan projection $\lambda : G \longrightarrow \mathfrak{a}^+$: if $g = g_e g_h g_u$ is the Jordan decomposition of g with g_e elliptic, g_h hyperbolic and g_u is unipotent, then $\lambda(g)$ is defined as $\kappa(g_h)$. Let S be a bounded subset of G and $S^n = \{g_1 \ldots . g_n \mid g_i \in S\}$ denote its nth power. We are interested in the following questions: do the sequences of bounded subsets of $\mathfrak{a}^+$, $\frac{1}{n}\kappa(S^n)$ and $\frac{1}{n}\lambda(S^n)$ have a limit in the Hausdorff topology ? If yes, are the limits the same and can one describe these limit sets?

2.2. Statement of the main results

Regarding the above questions, in [23] we show the following.

Theorem 2.1. *Let G be a connected semisimple linear real algebraic group and S a bounded subset of G generating a Zariski dense sub-semigroup. Then,*

(1) *the following Hausdorff limits exist, and we have the equality:* $\lim_{n\to\infty} \frac{1}{n}\kappa(S^n) = \lim_{n\to\infty} \frac{1}{n}\lambda(S^n)$. *This common limit will be denoted as $J(S)$ and called the* joint spectrum *of S,*
(2) *$J(S)$ is a compact convex subset of $\mathfrak{a}^+$ with non-empty interior.*

Remark 3. 1. It is not hard to see that if two subsets S and S' of G generate the same Zariski dense semigroup Γ in G, then the projective images of $J(S)$ and $J(S')$ are the same. Therefore the corresponding cone in $\mathfrak{a}^+$ only depends on Γ. It turns out that this is precisely the Benoist limit cone of Γ [5].

2. For a Banach algebra $(\mathcal{B}, ||.||)$ and a bounded subset S of $\mathcal{B}$, denote the joint spectral radius of S by $r(S) := \lim_{n\to\infty} \frac{1}{n} \log(\sup_{x\in S^n} ||x||)$. When $\mathcal{B}$ is finite dimensional, this does not depend on the particular norm on $\mathcal{B}$. Let now $\rho : G \to \mathrm{SL}(V)$ be an irreducible rational representation of G and let $\chi_\rho \in \mathfrak{a}^*$ denote the highest weight of ρ. Then we have the equality $\sup_{z\in J(S)} \chi_\rho(z) = r(\rho(S))$. This allows us to derive a multi-dimensional generalization of Berger–Wang identity [9].

We now relate the joint spectrum of the support of a probability measure μ with the effective support D_I of the rate function I for the LDP of $\frac{1}{n}\kappa(Y_n)$, where Y_n denotes as usual the μ-random walk on G.

Theorem 2.2. *Let G and μ be as in Theorem 1.3, and suppose moreover that the support S of μ is bounded. Let I be the rate function given by Theorem 1.3 and $D_I = \{x \in \mathfrak{a} \mid I(x) < \infty\}$ be the effective support of I. We then have*

(1) $\overline{D_I} = J(S)$ *and* $\overset{\circ}{D_I} = \overset{\circ}{J(S)}$. *If S is moreover finite, then $D_I = J(S)$,*
(2) *the Lyapunov vector $\vec{\lambda}_\mu \in \mathfrak{a}^+$ of μ belongs to the interior of $J(S)$.*

For the point (2) of this theorem, we note that, for a probability measure as in Theorem 1.3, the fact that $\vec{\lambda}_\mu$ belongs to the interior of $\mathfrak{a}^+$ was obtained by Guivarc'h–Raugi [17] and Goldsheid–Margulis [15]. Our result gives a more precise location for $\vec{\lambda}_\mu$ in case μ is, moreover, boundedly supported. Our method extends also to the case where μ only has a finite exponential moment to show that $\vec{\lambda}_\mu$ belongs to the Benoist cone of the semigroup generated by the support of μ.

The tools that go into the proof of Theorems 2.1 and 2.2 are mostly similar to those used in the proof of Theorem 1.3. We use an additional tool to prove the fact that the joint spectrum is of non-empty interior: while this result can be directly deduced from the properties of Benoist cone proved in [5], we adopt an indirect approach and use the central limit theorem of Goldsheid–Guivarc'h [14] and Guivarc'h [16] that we combine with Abels–Margulis–Soifer's result [1] and Benoist's estimates [4]. This in turn allows us to derive point (2) of the previous theorem and hence also part of the point (2) of Theorem 2.1.

Remark 4. The notion of joint spectrum plays also an important role in another work in preparation [22], where we study a new exponential counting function for a finite subset S in a group G as before.

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