C. R. Acad. Sci. Paris, Ser. I 355 (2017) 618–622

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Homological algebra/Functional analysis

The cyclic homology of crossed-product algebras, I

L'homologie cyclique des algèbres produits-croisés, I

Raphaël Ponge [1]

Department of Mathematical Sciences, Seoul National University, 1 Gwanak-ro, Gwanak-gu, Seoul 08826, South Korea

ARTICLE INFO

Article history:
Received 3 March 2017
Accepted 21 April 2017
Available online 9 May 2017

Presented by Alain Connes

ABSTRACT

In this note we produce explicit quasi-isomorphisms computing the cyclic homology of crossed-product algebras.

RÉSUMÉ

Dans cette note, on donne des quasi-isomorphismes explicites calculant l'homologie cyclique des algèbres produits-croisés.

0. Introduction

There is a large amount of work on the cyclic homology of crossed-product algebras (see, e.g., [1–3,6,8,10,11,17]). However, at the exception of the chain map of Connes [6,7] for the homogeneous component of periodic cyclic homology, we do not have explicit quasi-isomorphisms. The aim of this note is to present the construction of explicit quasi-isomorphisms for the cyclic homology and the periodic cyclic homology of crossed-product algebras, including for the localizations at infinite order components. Furthermore, the arguments use only elementary homological algebra. This allows us to bypass the difficult homological arguments involved in some previous approaches to the cyclic homology of crossed-product algebras.

The focus of this note is on algebraic crossed-products $\mathcal{A}_\Gamma = \mathcal{A} \rtimes \Gamma$ associated with the action of an arbitrary group Γ on a unital algebra $\mathcal{A}$ over a commutative ring $k \supset \mathbb{Q}$. In the sequel [18], we will explain how to apply the results in the contexts of group actions on manifolds and varieties.

1. Cyclic homology and triangular S-modules

We refer to [5,7,15] for background on cyclic homology, including cyclic and bi-cyclic modules, mixed complexes, and S-maps. If $C = (C_\bullet, b, B)$ is a mixed complex, we let $C^\natural = (C^\natural_\bullet, b + SB)$ be its cyclic complex, where $C^\natural_m = C_m \oplus C_{m-2} \oplus \cdots$ and $S : C^\natural_\bullet \to C^\natural_{\bullet-2}$ is the periodicity operator. The homology of the chain complex $C^\natural$ is the *cyclic homology* of C and is denoted

E-mail address: ponge.snu@gmail.com.

[1] Research partially supported by grants 2013R1A1A2008802 and 2016R1D1A1B01015971 of National Research Foundation of Korea.

http://dx.doi.org/10.1016/j.crma.2017.04.012
1631-073X/

by $\mathrm{HC}_\bullet(C)$. The *periodic cyclic homology* $\mathrm{HP}_\bullet(C)$ is the homology of the complex $C^\sharp = (C^\sharp_\bullet, b+B)$, where $C^\sharp_i = \prod_{q\geq 0} C_{2q+i}$, $i=0,1$.

We refer to [11] for background on paracyclic, bi-paracyclic and cylindrical modules, parachain and cylindrical complexes, and parachain bicomplexes. When C is a bi-paracyclic module (resp., parachain bicomplex), we denote by $\mathrm{Diag}(C)$ (resp., $\mathrm{Tot}(C)$) its diagonal paracyclic module (resp., total parachain complex). When C is cylindrical, we obtain a cyclic complex (resp., mixed complex) (see [11]).

The S-modules of Jones–Kassel [13,14] encapsulate various approaches to cyclic homology. More generally, by a *para-S-module* we shall mean the datum of $(C_\bullet, b, S)$, where C_m, $m\geq 0$, are k-modules and $d: C_\bullet \to C_{\bullet-1}$ and $S: C_\bullet \to C_{\bullet-2}$ are k-module maps commuting with each other such that $d^2 = (1-T)S$, where $T: C_\bullet \to C_\bullet$ is some k-module map commuting with both d and S. When $d^2=0$, we obtain an S-module. For instance, if $C=(C_\bullet, b, B)$ is a parachain complex, then we can define its cyclic complex of the para-S-module $C^\natural = (C^\natural_\bullet, b+SB, S)$. Notions of para-$S$-module maps and S-homotopies of para-S-module maps make sense in the same way as with S-modules. Therefore, although quasi-isomorphisms do not quite make sense for para-S-modules, the S-homotopy inverse of a para-S-module map and the S-homotopy equivalence of para-S-modules do make sense. This enables us to state a version of the Eilenberg–Zilber theorem for bi-paracyclic modules ([19]). If C is a bi-paracyclic module, then there is an S-homotopy equivalence between $\mathrm{Diag}(C)$ and $\mathrm{Tot}(C)$. It is given by S-maps $\sqcup\!\sqcup^\natural : \mathrm{Tot}_\bullet(C)^\natural \to \mathrm{Diag}_\bullet(C)^\natural$ and $\mathrm{AW}^\natural : \mathrm{Diag}_\bullet(C)^\natural \to \mathrm{Tot}_\bullet(C)^\natural$ whose zeroth degree components are the shuffle and Alexander–Whitney maps.

A *left triangular para-S-module* is given by the datum of $(C_{\bullet,\bullet}, d, b, B, S)$, where $C_{p,q}$, $p,q\geq 0$, are k-modules, $(C_{\bullet,q}, d, S)$ is a para-S-module and $(C_{p,\bullet}, b, B)$ is a parachain complex for all $p,q\geq 0$, the horizontal operators (d,S) both commute with each of the vertical differentials (b,B). We say that we have a *(left) triangular S-module* when $d^2+S(bB+Bb)=0$. There is a similar definition of the right triangular para-S-module and of the S-module $(C_{\bullet,\bullet}, b, B, d, S)$, where the operators (d,S) act vertically and the operators (b,B) act horizontally. The triangular (para-)S-modules provide us with a natural framework for defining the tensor product of (para-)S-modules with mixed and parachain complexes.

Any triangular para-S-module $C=(C_{\bullet,\bullet}, \overline{d}, b, B, S)$ gives rise to a *total para-S-module* $\mathrm{Tot}(C) = (\mathrm{Tot}_\bullet(C), d^\dagger, S)$, where $\mathrm{Tot}_m(C) = \bigoplus_{p+q=m} C_{p,q}$ and $d^\dagger = d + (-1)^p(b+SB)$ on $C_{p,q}$. When C is a triangular S-module, the condition $d^2+S(bB+Bb)=0$ ensures us that $(d^\dagger)^2=0$, and so we actually obtain an S-module. Furthermore, the filtration by columns of $\mathrm{Tot}_\bullet(C)$ is a filtration of chain complexes, and so it gives rise to a spectral sequence converging to $H_\bullet(\mathrm{Tot}(C))$.

We also observe that if $C=(C_{\bullet,\bullet}, \overline{b}, \overline{B}, b, B)$ is a cylindrical complex, then $\mathrm{Tot}(C)^\natural$ is the total S-module of the left triangular S-module $C^{\overline{\sigma}} := (C^{\overline{\sigma}}_{\bullet,\bullet}, \overline{b}+S\overline{B}, b, B, S)$, where $C^{\overline{\sigma}}_{p,q} = C_{p,q}\oplus C_{p-2,q}\oplus\cdots$. It is also the total S-module of the right triangular S-module $C^{\sigma} = (C^{\sigma}_{\bullet,\bullet}, \overline{b}, \overline{B}, b+SB, S)$, where $C^{\sigma}_{p,q} = C_{p,q}\oplus C_{p,q-2}\oplus\cdots$. As a result, we obtain *two* spectral sequences converging to $\mathrm{HC}_\bullet(\mathrm{Tot}(C))$. The spectral sequence of Getzler–Jones [11] is an instance of such a spectral sequence.

2. The cylindrical complexes $C^\phi(\Gamma, \mathscr{C})$

From now on, we assume that Γ is a group acting on a unital algebra $\mathcal{A}$ over a commutative ring $k\supset\mathbb{Q}$. The cyclic $k\Gamma$-module of Γ is $C(\Gamma) = (C_\bullet(\Gamma), d, s, t)$, where $C_m(\Gamma) = k\Gamma^{m+1}$, $m\geq 0$, and (d,s,t) are given by $d(\psi_0,\ldots,\psi_m) = (\psi_0,\ldots,\psi_{m-1})$, $s(\psi_0,\ldots,\psi_m) = (\psi_m,\psi_0,\ldots,\psi_m)$, and $t(\psi_0,\ldots,\psi_m) = (\psi_m,\psi_0,\ldots,\psi_{m-1})$, $\psi_j\in\Gamma$. Its b-differential is given by $\partial(\psi_0,\ldots,\psi_m) = \sum_{0\leq j\leq m}(-1)^j(\psi_0,\ldots,\hat{\psi}_j,\ldots,\psi_m)$. Given any $k\Gamma$-module $\mathscr{M}$, the *group homology* $H_\bullet(\Gamma,\mathscr{M})$ is the homology of the chain complex $(C_\bullet(\Gamma,\mathscr{M}),\partial)$, where $C_m(\Gamma,\mathscr{M}) = C_m(\Gamma)\otimes_\Gamma\mathscr{M}$, $m\geq 0$. The *group cohomology* $H^\bullet(\Gamma,\mathscr{M})$ is the cohomology of the dual cochain complex $(C^\bullet(\Gamma,\mathscr{M}),\partial)$, where $C^m(\Gamma,\mathscr{M})$ consists of all Γ-equivariant maps $u:\Gamma^{m+1}\to\mathscr{M}$ and $\partial u = u\circ\partial$.

Let ϕ be a central element of Γ. This gives rise to the paracyclic $k\Gamma$-module $C^\phi(\Gamma) = (C_\bullet(\Gamma), d, s_\phi, t_\phi)$, where d is as above and (s_ϕ, t_ϕ) are given by $s_\phi(\psi_0,\ldots,\psi_m) = (\phi^{-1}\psi_m,\psi_0,\ldots,\psi_m)$ and $t_\phi(\psi_0,\ldots,\psi_m) = (\phi^{-1}\psi_m,\psi_0,\ldots,\psi_{m-1})$. The simplicial module structure of $C^\phi(\Gamma)$ agrees with that of $C(\Gamma)$, and so its b-differential is the differential ∂ described above. We also note that $t_\phi^{m+1}(\psi_0,\ldots,\psi_m) = (\phi^{-1}\psi_0,\ldots,\phi^{-1}\psi_m)$, i.e. t_ϕ^{m+1} is given by the action of ϕ^{-1} on $C_m(\Gamma)$.

In what follows, by a *ϕ-parachain complex*, we shall mean a parachain complex of $k\Gamma$-modules $\mathscr{C} = (\mathscr{C}_\bullet, b, B)$ such that $T := 1-(bB+Bb)$ is given by the action of ϕ^{-1} on $\mathscr{C}_\bullet$. We also define *ϕ-paracyclic $k\Gamma$-modules* as paracyclic $k\Gamma$-modules $\mathscr{C} = (\mathscr{C}_\bullet, d, s, t)$ such that t^{m+1} is given by the action of ϕ^{-1} on $\mathscr{C}_m$ (so that the associated parachain complex is a ϕ-parachain complex). The paracyclic $k\Gamma$-module $C^\phi(\Gamma)$ is a ϕ-paracyclic module. Another example of ϕ-paracyclic $k\Gamma$-module is the twisted cyclic $k\Gamma$-module $C^\phi(\mathcal{A}) = (C_\bullet(\mathcal{A}), d_\phi, s, t_\phi)$, where $C_m(\mathcal{A}) = \mathcal{A}^{m+1}$ and (d_ϕ, s, t_ϕ) are given by $d_\phi(a^0\otimes\cdots\otimes a^m) = [(\phi^{-1}a^m)a^0]\otimes a^1\otimes\cdots\otimes a^{m-1}$, $s(a^0\otimes\cdots\otimes a^m) = 1\otimes a^0\otimes\cdots\otimes a^m$ and $t_\phi(a^0\otimes\cdots\otimes a^m) = (\phi^{-1}a^m)\otimes a^0\otimes\cdots\otimes a^{m-1}$. For $\phi=1$ we recover the cyclic module $C(\mathcal{A})$ of $\mathcal{A}$.

Given (left) $k\Gamma$-modules $\mathscr{M}_1$ and $\mathscr{M}_2$ we shall denote by $\mathscr{M}_1\otimes_\Gamma\mathscr{M}_2$ their tensor product over $k\Gamma$, i.e. the quotient of $\mathscr{M}_1\otimes_k\mathscr{M}_2$ by the action of Γ. If $\mathscr{C}$ and $\mathscr{C}'$ are parachain complexes of $k\Gamma$-modules (resp., paracyclic $k\Gamma$-modules), then we can form their tensor product $\mathscr{C}\otimes_\Gamma\mathscr{C}'$ so as to get a parachain complex of k-modules (resp., a bi-paracyclic k-modules). When $\mathscr{C}$ and $\mathscr{C}'$ are ϕ-parachain complexes (resp., ϕ-paracyclic modules) the tensor product $\mathscr{C}\otimes_\Gamma\mathscr{C}'$ is cylindrical.

In what follows, when $\mathscr{C}$ is a ϕ-parachain complex (resp., ϕ-paracyclic $k\Gamma$-module) we shall denote by $C^\phi(\Gamma,\mathscr{C})$ the cylindrical complex (resp., cylindrical k-module) $C^\phi(\Gamma)\otimes_\Gamma\mathscr{C}$. We shall use the notation $C^\phi(\Gamma,\mathcal{A})$ for $\mathscr{C} = C^\phi(\mathcal{A})$. It can be shown that if $\alpha:\mathscr{C}_\bullet\to\mathscr{C}'_\bullet$ is a quasi-isomorphism of ϕ-parachain complexes, then $1\otimes\alpha:\mathrm{Tot}_\bullet(C^\phi(\Gamma,\mathscr{C}))\to\mathrm{Tot}_\bullet(C^\phi(\Gamma,\mathscr{C}'))$ is a quasi-isomorphism of mixed complexes.

3. Splitting along conjugacy classes

The crossed-product algebra $\mathcal{A}_\Gamma := \mathcal{A} \rtimes \Gamma$ is the unital k-algebra with generators $a \in \mathcal{A}$ and u_ϕ, $\phi \in \Gamma$ subject to the relations $a^0 u_{\phi_0} a^1 u_{\phi_1} = a^0(\phi_0^{-1} a^1) u_{\phi_0\phi_1}$ for all $a^j \in \mathcal{A}$ and $\phi_j \in \Gamma$. Given any $\phi \in \Gamma$, we shall denote by $[\phi]$ its conjugacy class in Γ. We then denote by $C(\mathcal{A}_\Gamma)_{[\phi]}$ the cyclic submodule of $C(\mathcal{A}_\Gamma)$ generated by chains $a^0 u_{\phi_0} \otimes \cdots \otimes a^m u_{\phi_m}$, with $a^j \in \mathcal{A}$ and $\phi_j \in \Gamma$ such that $\phi_0 \cdots \phi_m \in [\phi]$. We then have a direct-sum decomposition of cyclic k-modules $C_\bullet(\mathcal{A}_\Gamma) = \bigoplus C_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$, where the summation goes over all conjugacy classes. This provides us with the corresponding decomposition and inclusion at the level of the cyclic and periodic complexes. We shall denote by $\mathrm{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ (resp., $\mathrm{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$) the cyclic homology (resp., periodic cyclic homology) of $C(\mathcal{A}_\Gamma)_{[\phi]}$. We then have a splitting $\mathrm{HC}_\bullet(\mathcal{A}_\Gamma) = \bigoplus \mathrm{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$, and an inclusion $\bigoplus \mathrm{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]} \subset \mathrm{HP}_\bullet(\mathcal{A}_\Gamma)$, which is onto when Γ has a finite number of conjugacy classes.

Given $\phi \in \Gamma$, let us denote by Γ_ϕ its centralizer in Γ. As ϕ is a central element of Γ_ϕ, we may form the cylindrical complex $C^\phi(\Gamma_\phi, \mathcal{A})$ as in Section 2. We have a natural embedding of cyclic k-modules $\mu_\phi : \mathrm{Diag}_\bullet(C^\phi(\Gamma_\phi, \mathcal{A})) \to C_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ given by $\mu_\phi\left((\psi_0, \ldots, \psi_m) \otimes_{\Gamma_\phi} (a^0 \otimes \cdots \otimes a^m)\right) = [(\psi_m^{-1}\phi) \cdot a^0] u_{\phi\psi_m^{-1}\psi_0} \otimes (\psi_0^{-1} \cdot a^1) u_{\psi_0^{-1}\psi_1} \otimes \cdots \otimes (\psi_{m-1}^{-1} \cdot a^m) u_{\psi_{m-1}^{-1}\psi_m}$. This embedding can be shown to be a quasi-isomorphism. Combining this with the Eilenberg–Zilber theorem for bi-paracyclic modules, we then obtain explicit quasi-isomorphisms:

$$\mathrm{Tot}_\bullet\left(C^\phi(\Gamma_\phi, \mathcal{A})\right)^\natural \underset{\mathrm{AW}^\natural}{\overset{\sqcup\!\sqcup^\natural}{\rightleftarrows}} \mathrm{Diag}_\bullet\left(C^\phi(\Gamma_\phi, \mathcal{A})\right)^\natural \xrightarrow{\mu_\phi} C_\bullet(\mathcal{A}_\Gamma)^\natural_{[\phi]}. \tag{1}$$

There are analogous quasi-isomorphisms between the corresponding periodic cyclic complexes. All this reduces the study of the cyclic module of $\mathcal{A}_\Gamma$ to that of the mixed complexes $\mathrm{Tot}_\bullet(C^\phi(\Gamma_\phi, \mathcal{A}))$, $\phi \in \Gamma$.

4. The cyclic module $C^\phi(\mathcal{A})_{[\phi]}$. Finite order case

Suppose that ϕ is an element of Γ of finite order r. Let $C^\phi(\Gamma_\phi)$ be the mixed complex $(C(\Gamma_\phi), \partial, 0)$. Given any ϕ-invariant mixed complex of $k\Gamma_\phi$-modules $\mathscr{C} = (\mathscr{C}_\bullet, b, B)$, we shall denote by $C^\flat(\Gamma_\phi, \mathscr{C})$ the mixed bicomplex $C^\flat(\Gamma_\phi) \otimes_{\Gamma_\phi} \mathscr{C}$. More generally, if $\mathscr{C} = (\mathscr{C}_\bullet, b, B)$ is a ϕ-parachain complex, then we have a ϕ-invariant mixed complex $\mathscr{C}^\phi = (\mathscr{C}^\phi_\bullet, b, B)$, where $\mathscr{C}^\phi_m$ is the ϕ-invariant submodule of $\mathscr{C}_m$. We then may form the mixed bicomplex $C^\flat(\Gamma_\phi, \mathscr{C}^\phi)$.

Bearing this in mind, let $\nu_\phi : C_\bullet(\Gamma_\phi) \to C_\bullet(\Gamma_\phi)$ be the $k\Gamma_\phi$-module map defined by $\nu_\phi(\psi_0, \ldots, \psi_m) = \frac{1}{r^{m+1}} \times \sum_{0 \leq \ell_j \leq r-1} (\phi^{\ell_0}\psi_0, \ldots, \phi^{\ell_m}\psi_m)$, $\psi_j \in \Gamma_\phi$. We also let $\varepsilon : C_\bullet(\Gamma_\phi) \to C_\bullet(\Gamma_\phi)$ be the antisymmetrization map $\varepsilon(\psi_0, \ldots, \psi_m) = \frac{1}{(m+1)!} \sum_{\sigma \in \mathfrak{S}_m} (\psi_{\sigma^{-1}(0)}, \ldots, \psi_{\sigma^{-1}(m)})$, where $\mathfrak{S}_m$ is the group of permutations of $\{0, \ldots, m\}$. As ν_ϕ and ε both are projections and commute with each other, the composition $\varepsilon\nu_\phi$ is a projection as well. It can be checked that the composition $\varepsilon\nu_\phi : C^\phi_\bullet(\Gamma_\phi) \to C^\flat(\Gamma_\phi)$ is a map of parachain complexes which is both a projection and ∂-homotopic to the identity (compare [4,16]). By elaborating on the perturbation theory of Kassel [14], it can be shown that the associated S-map $\varepsilon\nu_\phi : C^\phi_\bullet(\Gamma_\phi)^\natural \to C^\flat_\bullet(\Gamma_\phi)^\natural$ has an explicit S-homotopy inverse $(\varepsilon\nu_\phi)^\flat : C^\flat_\bullet(\Gamma_\phi)^\natural \to C^\phi_b t(\Gamma_\phi, \mathscr{C})^\natural$ which is an S-map whose zeroth degree component is $\varepsilon\nu_\phi$. Given any ϕ-parachain complex $\mathscr{C}$, we obtain a parachain bicomplex map $(\varepsilon\nu_\phi) \otimes 1 : C^\phi_{\bullet,\bullet}(\Gamma_\phi, \mathscr{C}) \to C^\flat_{\bullet,\bullet}(\Gamma_\phi, \mathscr{C}^\phi)$. A homotopy inverse of the S-map $(\varepsilon\nu_\phi) \otimes 1 : \mathrm{Tot}_\bullet(C^\phi(\Gamma_\phi, \mathscr{C}))^\natural \to \mathrm{Tot}_\bullet(C^\flat(\Gamma_\phi, \mathscr{C}^\phi))^\natural$ is given by $(\varepsilon\nu_\phi)^\flat \otimes 1 : \mathrm{Tot}_\bullet(C^\flat(\Gamma_\phi, \mathscr{C}^\phi))^\natural \to \mathrm{Tot}_\bullet(C^\phi(\Gamma_\phi, \mathscr{C}))^\natural$. Therefore, we arrive at the following result.

Theorem 4.1. *Let $\phi \in \Gamma$ have finite order. Suppose we are given a quasi-isomorphism of parachain complex $\alpha : C^\phi_\bullet(\mathcal{A}) \to \mathscr{C}_\bullet$, where $\mathscr{C}$ is a ϕ-parachain complex. Then we have quasi-isomorphisms,*

$$\mathrm{Tot}_\bullet\left(C^\flat(\Gamma_\phi, \mathscr{C}^\phi)\right)^\natural \xleftarrow{(\varepsilon\nu_\phi)\otimes\alpha} \mathrm{Tot}_\bullet\left(C^\phi(\Gamma_\phi, \mathcal{A})\right)^\natural \underset{\mathrm{AW}^\natural}{\overset{\sqcup\!\sqcup^\natural}{\rightleftarrows}} \mathrm{Diag}_\bullet\left(C^\phi(\Gamma_\phi, \mathcal{A})\right)^\natural \xrightarrow{\mu_\phi} C_\bullet(\mathcal{A}_\Gamma)^\natural_{[\phi]}.$$

If $\beta : \mathscr{C}_\bullet \to C^\phi_\bullet(\mathcal{A})$ is a quasi-inverse (resp., homotopy inverse) of α, then $(\varepsilon\nu_\phi)^\flat \otimes \beta$ is a quasi-inverse (resp., homotopy inverse) of $(\varepsilon\nu_\phi) \otimes \alpha$. There are analogous statements at the level of the corresponding periodic cyclic complexes. We thus obtain isomorphisms $\mathrm{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]} \simeq \mathrm{HC}_\bullet(\mathrm{Tot}(C^\flat(\Gamma_\phi, \mathscr{C}^\phi)))$ and $\mathrm{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]} \simeq \mathrm{HP}_\bullet(\mathrm{Tot}(C^\flat(\Gamma_\phi, \mathscr{C}^\phi)))$.

Remark 4.2. By interpreting $\mathrm{Tot}_\bullet\left(C^\flat(\Gamma_\phi, \mathscr{C}^\phi)\right)^\natural$ as the total S-module of triangular S-modules (*cf.* Section 1), we obtain spectral sequences converging to $\mathrm{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$. In particular, we obtain a refinement of the spectral sequence of Getzler–Jones [11] and recover the spectral sequence of Feigin–Tsygan [10].

Remark 4.3. When Γ_ϕ is finite, we can construct an explicit pair of S-homotopy inverses between $\mathrm{Tot}(C^\flat(\Gamma_\phi, \mathscr{C}^\phi))$ and the Γ_ϕ-invariant cyclic complex $\mathscr{C}^{\Gamma_\phi,\natural}$. We thus obtain isomorphisms $\mathrm{HC}(\mathcal{A}_\Gamma)_{[\phi]} \simeq \mathrm{HC}_\bullet(\mathscr{C}^{\Gamma_\phi})$ and $\mathrm{HP}(\mathcal{A}_\Gamma)_{[\phi]} \simeq \mathrm{HP}_\bullet(\mathscr{C}^{\Gamma_\phi})$.

5. The cyclic module $C^{\phi}(\mathcal{A})_{[\Gamma]}$. Infinite order case

Suppose that ϕ is an infinite-order element of Γ. Set $\overline{\Gamma}_\phi = \Gamma_\phi/\langle\phi\rangle$, where $\langle\phi\rangle$ is the subgroup generated by ϕ. Composing the natural projection $\overline{\pi}^\natural : C^\phi_\bullet(\Gamma_\phi)^\natural \to C_\bullet(\overline{\Gamma}_\phi)$ with the antisymmetrization map of $C_\bullet(\overline{\Gamma}_\phi)$ defined in the previous section, we get a chain map $\overline{\varepsilon}^\natural := \varepsilon\overline{\pi}^\natural : C^\phi_\bullet(\Gamma_\phi)^\natural \to C_\bullet(\overline{\Gamma}_\phi)$. If $\mathscr{M}$ is a ϕ-invariant $k\Gamma_\phi$-module, then the action of Γ_ϕ on $\mathscr{M}$ descends to an action of $\overline{\Gamma}_\phi$, and so we obtain a chain map $\overline{\varepsilon}^\natural \otimes 1 : C^\phi_\bullet(\Gamma_\phi, \mathscr{M})^\natural \to C_\bullet(\overline{\Gamma}_\phi, \mathscr{M})$. Using results of Marciniak [16] and Kassel [14], this chain map can be shown to give rise to an S-homotopy equivalence. In addition, let $u_\phi \in C^2(\overline{\Gamma}_\phi, \mathbb{Z})$ be a 2-cocycle representing the Euler class $e_\phi \in H^2(\overline{\Gamma}_\phi, \mathbb{Z})$ of the central extension $1 \to \langle\phi\rangle \to \Gamma_\phi \to \overline{\Gamma}_\phi \to 1$. The cap product with u_ϕ then gives rise to a chain map $u_\phi \frown - : C_\bullet(\overline{\Gamma}_\phi) \to C_{\bullet-2}(\overline{\Gamma}_\phi)$. We then can construct an explicit chain homotopy $h_\phi : C^\phi_\bullet(\Gamma_\phi)^\natural \to C_{\bullet-1}(\overline{\Gamma}_\phi)$ such that $\overline{\varepsilon}^\natural S - (u_\phi \frown -)\overline{\varepsilon}^\natural = \partial h_\phi + h_\phi(\partial + B_\phi S)$, where B_ϕ is the B-differential of $C^\phi(\Gamma_\phi)$ (compare [12]).

As $u_\phi \frown -$ is a chain map degree -2, we get an S-module $C^\sigma(\overline{\Gamma}_\phi) = (C_\bullet(\overline{\Gamma}_\phi), \partial, u_\phi \frown -)$. Given any ϕ-invariant mixed complex $\mathscr{C} = (\mathscr{C}_\bullet, b, B)$, we denote by $C^\sigma(\overline{\Gamma}_\phi, \mathscr{C})$ the triangular S-module given by the tensor product $C^\sigma(\overline{\Gamma}_\phi) \otimes_{\overline{\Gamma}_\phi} \mathscr{C}$. Its total S-module is $\operatorname{Tot}(C^\sigma(\overline{\Gamma}_\phi, \mathscr{C})) = (\operatorname{Tot}_\bullet(C^\sigma(\overline{\Gamma}_\phi, \mathscr{C})), d^\dagger, u_\phi \frown -)$, where $\operatorname{Tot}_m(C^\sigma(\overline{\Gamma}_\phi, \mathscr{C})) = \bigoplus_{p+q=m} C_p(\overline{\Gamma}_\phi, \mathscr{C}_q)$ and $d^\dagger = \partial + (-1)^p b + (-1)B(u_\phi \frown -)$. We then obtain a chain map $\theta : \operatorname{Tot}_\bullet(C^\phi(\Gamma_\phi, \mathscr{C}))^\natural \to \operatorname{Tot}_\bullet(C^\sigma(\overline{\Gamma}_\phi, \mathscr{C}))$ by letting $\theta = \overline{\varepsilon}^\natural \otimes 1 + (-1)^{p-1}(1 \otimes B)(h_\phi \otimes 1)$ on $C_{p,q}(\Gamma_\phi, \mathscr{C})$. It can be shown that θ is a quasi-isomorphism and $\theta S - (u_\phi \frown -)S = d^\dagger(h_\phi \otimes 1) + (h_\phi \otimes 1)d^\dagger$, where we have denoted by $d^\dagger$ the differentials of $\operatorname{Tot}(C^\phi(\Gamma_\phi, \mathscr{C}))^\natural$ and $\operatorname{Tot}(C^\sigma(\overline{\Gamma}_\phi, \mathscr{C}))$.

Theorem 5.1. *Let $\phi \in \Gamma$ have infinite order. Suppose we are given a quasi-isomorphism of parachain complexes $\alpha : C^\phi_\bullet(\mathcal{A}) \to \mathscr{C}_\bullet$, where $\mathscr{C}$ is a ϕ-invariant mixed complex. Then we have quasi-isomorphisms,*

$$\operatorname{Tot}_\bullet(C^\sigma(\overline{\Gamma}_\phi, \mathscr{C})) \xleftarrow{\theta(1\otimes\alpha)} \operatorname{Tot}_\bullet\left(C^\phi(\Gamma_\phi, \mathcal{A})\right)^\natural \underset{\mathrm{AW}^\natural}{\overset{\sqcup\!\sqcup^\natural}{\rightleftarrows}} \operatorname{Diag}_\bullet\left(C^\phi(\Gamma_\phi, \mathcal{A})\right)^\natural \xrightarrow{\mu_\phi} C_\bullet(\mathcal{A}_\Gamma)^\natural_{[\phi]}.$$

This gives an isomorphism $\mathrm{HC}_\bullet(\mathcal{A})_{[\phi]} \simeq H_\bullet\left(\operatorname{Tot}(C^\sigma(\overline{\Gamma}_\phi, \mathscr{C}))\right)$, under which the periodicity operator of $\mathrm{HC}_\bullet(\mathcal{A})_{[\phi]}$ is given by the cap product $e_\phi \frown - : H_\bullet(\operatorname{Tot}(C^\sigma(\overline{\Gamma}_\phi, \mathscr{C}))) \longrightarrow H_{\bullet-2}(\operatorname{Tot}(C^\sigma(\overline{\Gamma}_\phi, \mathscr{C})))$.

Actions satisfying the assumptions of Theorem 5.1 naturally appear in the context of group actions on manifolds (*cf.* [18]). In general, we have the following result.

Theorem 5.2. *Let $\phi \in \Gamma$ have infinite order. Suppose we are given a quasi-isomorphism of parachain complexes $\alpha : C^\phi_\bullet(\mathcal{A}) \to \mathscr{C}_\bullet$, where $\mathscr{C} = (\mathscr{C}_\bullet, b, B)$ is a ϕ-parachain complex. Then we have a spectral sequence,*

$$E^2_{p,q} = H_p(\overline{\Gamma}_\phi, H_q(\mathscr{C})) \Longrightarrow \mathrm{HC}_{p+q}(\mathcal{A}_\Gamma)_{[\phi]},$$

where $H_\bullet(\mathscr{C})$ is the homology of $(\mathscr{C}_\bullet, b)$. If $b = 0$, then $E^2_{p,q} = H_p(\overline{\Gamma}_\phi, \mathscr{C}_q)$ and the E^2-differential is given by $(-1)^p B(u_\phi \frown -) : H_p(\overline{\Gamma}_\phi, \mathscr{C}_q) \to H_{p-2}(\overline{\Gamma}_\phi, \mathscr{C}_{q+1})$.

Remark 5.3. When $\mathscr{C} = C^\phi(\mathcal{A})$ and $\alpha = \mathrm{id}$, the above spectral sequence specializes to the spectral sequence of Feigin–Tsygan [10].

Let $\delta : C^\phi_\bullet(\Gamma_\phi) \to \operatorname{Diag}_\bullet(C^\phi(\Gamma_\phi, k) \otimes C^\phi(\Gamma_\phi))$ be the paracyclic k-module map given by $\delta(\psi_0, \ldots, \psi_m) = [(\psi_0, \ldots, \psi_m) \otimes_{\Gamma_\phi} 1] \otimes (\psi_0, \ldots, \psi_m)$. Combining it with the bi-paracyclic Alexander–Whitney map, we obtain a para-S-module map $\mathrm{AW}^\natural \circ \delta : C^\phi_\bullet(\Gamma_\phi)^\natural \to \operatorname{Tot}_\bullet(C^\phi(\Gamma_\phi, k) \otimes C^\phi(\Gamma_\phi))^\natural$. Let $C^\flat(\overline{\Gamma}_\phi, k)$ be the mixed complex $(C_\bullet(\overline{\Gamma}_\phi), \partial, 0)$, and let us form the vertical triangular para-S-module $C^\flat(\overline{\Gamma}_\phi, k) \otimes_{\Gamma_\phi} C^\phi(\Gamma_\phi)^\natural$. We have a para-$S$-module map $(\varepsilon\overline{\pi}) \otimes 1 : \operatorname{Tot}_\bullet(C^\phi_\bullet(\Gamma_\phi, k) \otimes C^\phi(\Gamma_\phi))^\natural \to \operatorname{Tot}_\bullet(C^\flat(\overline{\Gamma}_\phi, k) \otimes C^\phi(\Gamma_\phi)^\natural)$. We thus obtain a para-$S$-module map $\Delta^\natural := ((\varepsilon\overline{\pi}) \otimes 1)\,\mathrm{AW}^\natural \circ \delta : C^\phi(\Gamma_\phi)^\natural \to \operatorname{Tot}_\bullet(C^\flat(\overline{\Gamma}_\phi, k) \otimes C^\phi(\Gamma_\phi)^\natural)$. This gives rise to a bilinear differential graded map,

$$\triangleright : C^\bullet(\overline{\Gamma}_\phi, k) \times \operatorname{Tot}_\bullet\left(C^\phi(\Gamma_\phi, \mathcal{A})\right)^\natural \longrightarrow \operatorname{Tot}_\bullet\left(C^\phi(\Gamma_\phi, \mathcal{A})\right)^\natural. \qquad (2)$$

More precisely, given any cochain $u \in C^p(\overline{\Gamma}_\phi, k)$, $p \geq 0$, and chains $\eta \in C^\phi_\bullet(\Gamma_\phi)^\natural$ and $\xi \in C_\bullet(\mathcal{A})$, we have $u \triangleright (\eta \otimes_\Gamma \xi) = \left[(u \otimes 1)\Delta^\natural\eta\right] \otimes_{\Gamma_\phi} \xi$.

Theorem 5.4. *Let $\phi \in \Gamma$ have infinite order. The quasi-isomorphisms (1) and the bilinear map (2) give rise to an associative action of the cohomology ring $H^\bullet(\overline{\Gamma}_\phi, k)$ on $\mathrm{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$. Moreover, the periodicity operator of $\mathrm{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ is given by the action of the Euler class e_ϕ. In particular, $\mathrm{HP}_\bullet(\mathcal{A}_\Gamma)_{[\phi]} = 0$ whenever e_ϕ is nilpotent in $H^\bullet(\overline{\Gamma}_\phi, k)$.*

Remark 5.5. The result that $\mathrm{HC}_\bullet(\mathcal{A}_\Gamma)_{[\phi]}$ is a module over $H^\bullet(\overline{\Gamma}_\phi, k)$ and the action of e_ϕ gives the periodicity is due to Nistor [17]. The improvement with respect to [17] is twofold. First, we are able to bypass the difficult homological algebra arguments of [17]. Second, we have an explicit description of the action at the level of chains.

Remark 5.6. The nilpotence of e_ϕ is closely related to the Bass and idempotent conjectures ([9,12]). In particular, e_ϕ is rationally nilpotent for every infinite order element $\phi \in \Gamma$ whenever Γ belongs to one of the following classes of groups: free products of Abelian groups, hyperbolic groups, and arithmetic groups.

Acknowledgements

I wish to thank Alain Connes, Sasha Gorokhovsky, Masoud Khalkhali, Henri Moscovici, Victor Nistor, Markus Pflaum, Hessel Posthuma, Bahram Rangipour, Xiang Tang, and Hang Wang for various stimulating discussions related to the subject matter of this note.

References

[1] P. Baum, A. Connes, Chern character for discrete groups, in: A Fête of Topology, Academic Press, Boston, 1988, pp. 163–232.
[2] J. Brodzki, S. Dave, V. Nistor, The periodic cyclic homology of crossed products of finite type algebras, Adv. Math. 306 (2017) 494–523.
[3] J.-L. Brylinski, V. Nistor, Cyclic cohomology of étale groupoids, K-Theory 8 (1994) 341–365.
[4] D. Burghelea, The cyclic homology of the group rings, Comment. Math. Helv. 60 (1985) 354–365.
[5] A. Connes, Noncommutative differential geometry, Inst. Hautes Études Sci. Publ. Math. 62 (1985) 257–360.
[6] A. Connes, Cyclic cohomology and the transverse fundamental class of a foliation, in: Pitman Research Notes in Mathematics, vol. 123, Longman, Harlow, 1986, pp. 52–144.
[7] A. Connes, Noncommutative Geometry, Academic Press, San Diego, 1994.
[8] M. Crainic, Cyclic cohomology of étale groupoids: the general case, K-Theory 17 (1999) 319–362.
[9] I. Emmanouil, On a class of groups satisfying Bass' conjecture, Invent. Math. 132 (1998) 307–330.
[10] B. Feigin, B. Tsygan, Cyclic homology of algebras with quadratic relations, universal enveloping algebras and group algebras, in: Lecture Notes in Math., vol. 1289, Springer, 1987, pp. 210–239.
[11] E. Getzler, J.D.S. Jones, The cyclic cohomology of crossed product algebras, J. Reine Angew. Math. 445 (1993) 161–174.
[12] R. Ji, Nilpotency of Connes' periodicity operator and the idempotent conjectures, K-Theory 9 (1995) 59–76.
[13] J.D.S. Jones, C. Kassel, Bivariant cyclic theory, K-Theory 3 (1989) 339–365.
[14] C. Kassel, Homologie cyclique, caractère de Chern et lemme de perturbation, J. Reine Angew. Math. 408 (1990) 159–180.
[15] J.-L. Loday, Cyclic Homology, Springer, Berlin, 1992.
[16] Z. Marciniak, Cyclic homology of group rings, in: Banach Center Publ., vol. 18, PWN, Warsaw, 1986, pp. 210–239.
[17] V. Nistor, Group cohomology and the cyclic cohomology of crossed products, Invent. Math. 99 (1990) 411–424.
[18] R. Ponge, The cyclic homology of crossed-product algebras, II, C. R. Acad. Sci. Paris, Ser. I 355 (2017) 623–627.
[19] R. Ponge, Perturbation lemma, para-S-modules, and cup product for paracyclic modules, in preparation.