C. R. Acad. Sci. Paris, Ser. I 355 (2017) 596–600

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Dynamical systems

On the Hamilton–Poisson realizations of the integrable deformations of the Maxwell–Bloch equations

Sur les réalisations Hamilton–Poisson des déformations intégrables des équations de Maxwell–Bloch

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ARTICLE INFO

Article history:
Received 11 February 2017
Accepted 6 April 2017
Available online 18 April 2017

Presented by Claire Voisin

ABSTRACT

In this note, we construct integrable deformations of the three-dimensional real valued Maxwell–Bloch equations by modifying their constants of motions. We obtain two Hamilton–Poisson realizations of the new system. Moreover, we prove that the obtained system has infinitely many Hamilton–Poisson realizations. Particularly, we present a Hamilton–Poisson approach of the system obtained considering two concrete deformation functions.

RÉSUMÉ

Dans cette Note, nous construisons des déformations intégrables des équations de Maxwell–Bloch en modifiant leurs constantes de mouvement. Nous obtenons deux réalisations Hamilton–Poisson du nouveau système. De plus, nous prouvons que le système obtenu admet des réalisations Hamilton–Poisson infiniment nombreuses. Nous présentons une approche Hamilton–Poisson du système obtenu en considérant deux fonctions particulières de déformation.

Version française abrégée

La construction des déformations intégrables a été étudiée dans des articles récents [1,5].

Dans cette Note, nous construisons des déformations intégrables des équations de Maxwell–Bloch [4]. Nous montrons qu'une telle déformation intégrable est un système bi-hamiltonien. De plus, nous analysons une déformation intégrable particulière des équations de Maxwell–Bloch. Plus précisément, nous établissons la stabilité de Lyapunov des points d'équi-

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http://dx.doi.org/10.1016/j.crma.2017.04.002

libre et nous prouvons l'existence d'orbites périodiques autour de ces points. Nous présentons les liens entre la fonction énergie-Casimir et les éléments dynamiques susmentionnés.

1. Introduction

In recent papers, the construction of integrable deformations of a given integrable system was studied. In [1], considering Poisson–Lie groups as deformations of Lie–Poisson (co)algebras, integrable deformations of both uncoupled and coupled versions of certain integrable types of Röossler and Lorenz systems were given. In [5], using the fact that the constants of motion uniquely determine the dynamical equations, integrable deformations of the Euler top were constructed.

In this paper, we construct integrable deformations of the three-dimensional real valued Maxwell–Bloch equations. The Maxwell–Bloch equations have significant importance in optics. These equations represent a model used to describe the interaction between laser light and a material sample composed of two-level atoms [4].

The paper is organized as follows.

In the second section, we prove that the constants of motion uniquely determine the Maxwell–Bloch equations, up to a parameterization of time. Using this property, we construct integrable deformations of the Maxwell–Bloch equations. In the third section, we show that such integrable deformation is a bi-Hamiltonian system. Moreover, this system has infinitely many Hamilton–Poisson realizations. In the last section, we analyze a particular integrable deformation of the Maxwell–Bloch equations. More precisely, we establish the Lyapunov stability of the equilibrium points, and we prove the existence of the periodic orbits around these points. We also present the connections between the energy-Casimir mapping and the aforementioned dynamical elements.

2. Integrable deformations of the Maxwell–Bloch equations

In this section, we construct integrable deformations of the three-dimensional real valued Maxwell–Bloch equations. We use the method considered in [5].

We recall that the equations

$$\dot{x}=y\,,\quad \dot{y}=xz\,,\quad \dot{z}=-xy \tag{1}$$

are called the three-dimensional real valued Maxwell–Bloch equations [4]. Moreover, two constants of motion in involution of system (1) are given by

$$I_1(x,y,z)=\frac{1}{2}x^2+z\,,\quad I_2(x,y,z)=\frac{1}{2}y^2+\frac{1}{2}z^2. \tag{2}$$

Let us prove that equations (1) are uniquely determined by these constants of motion, up to a parameterization of time. Indeed, differentiating the above constants of motion (2), we get

$$\dot{x}=-\frac{1}{x}\dot{z}\,,\quad \dot{y}=-\frac{z}{y}\dot{z}.$$

Considering $\dot{z}=-xyf$, where $f=f(t)$ is an arbitrary continuous function, we obtain

$$\dot{x}=yf\,,\quad \dot{y}=xzf\,,\quad \dot{z}=-xyf\,.$$

Using the transformation $t=t(\tau)$, where τ is the new time variable, given by $\tau=\int_0^t f(s)\,\mathrm{d}s$, it follows $\frac{\mathrm{d}x}{\mathrm{d}\tau}=\frac{\mathrm{d}x}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}\tau}=yf\cdot\frac{1}{f}=y(\tau)$. Analogously, we obtain $\frac{\mathrm{d}y}{\mathrm{d}\tau}=x(\tau)z(\tau)$, $\frac{\mathrm{d}z}{\mathrm{d}\tau}=-x(\tau)y(\tau)$, as required.

The above property of equations (1) allows constructing integrable deformations of the Maxwell–Bloch equations altering their constants of motion. More precisely, let us consider the new constants of motion C_1 and C_2 given by

$$C_1(x,y,z)=\frac{1}{2}x^2+z+\alpha(x,y,z)\ \text{ and }\ C_2(x,y,z)=\frac{1}{2}y^2+\frac{1}{2}z^2+\beta(x,y,z), \tag{3}$$

where α and β are arbitrary differentiable functions.

By (3), we have

$$x\dot{x}+\dot{z}+\frac{\partial\alpha}{\partial x}\dot{x}+\frac{\partial\alpha}{\partial y}\dot{y}+\frac{\partial\alpha}{\partial z}\dot{z}=0\,,\quad y\dot{y}+z\dot{z}+\frac{\partial\beta}{\partial x}\dot{x}+\frac{\partial\beta}{\partial y}\dot{y}+\frac{\partial\beta}{\partial z}\dot{z}=0,$$

or equivalent,

$$\left(x+\frac{\partial\alpha}{\partial x}\right)\dot{x}+\frac{\partial\alpha}{\partial y}\dot{y}=-\left(1+\frac{\partial\alpha}{\partial z}\right)\dot{z}\,,\quad \frac{\partial\beta}{\partial x}\dot{x}+\left(y+\frac{\partial\beta}{\partial y}\right)\dot{y}=-\left(z+\frac{\partial\beta}{\partial z}\right)\dot{z}.$$

Solving this algebraic system and denoting $\dot{z}$ as below, we obtain the following integrable deformation of the Maxwell–Bloch equations:

$$\begin{aligned}
\dot{x} &= y - z\frac{\partial\alpha}{\partial y} + y\frac{\partial\alpha}{\partial z} + \frac{\partial\beta}{\partial y} - \frac{\partial\alpha}{\partial y}\cdot\frac{\partial\beta}{\partial z} + \frac{\partial\alpha}{\partial z}\cdot\frac{\partial\beta}{\partial y},\\
\dot{y} &= xz + z\frac{\partial\alpha}{\partial x} - \frac{\partial\beta}{\partial x} + x\frac{\partial\beta}{\partial z} + \frac{\partial\alpha}{\partial x}\cdot\frac{\partial\beta}{\partial z} - \frac{\partial\alpha}{\partial z}\cdot\frac{\partial\beta}{\partial x},\\
\dot{z} &= -xy - y\frac{\partial\alpha}{\partial x} - x\frac{\partial\beta}{\partial y} - \frac{\partial\alpha}{\partial x}\cdot\frac{\partial\beta}{\partial y} + \frac{\partial\alpha}{\partial y}\cdot\frac{\partial\beta}{\partial x}.
\end{aligned} \tag{4}$$

It is obvious if α and β are constant functions, then (4) reduces to (1).

3. Hamilton–Poisson realizations of the integrable deformations of the Maxwell–Bloch equations

In this section, we determine two Poisson brackets on $C^\infty(\mathbb{R}^3, \mathbb{R})$. As a consequence, we obtain two Hamilton–Poisson realizations of system (4). Moreover, since the Poisson brackets are compatible, it follows that the considered system is bi-Hamiltonian. We also prove that system (4) has infinitely many Hamilton–Poisson realizations.

Firstly, we consider that C_1 (3) is a Casimir function of the Poisson bracket $\{.,.\}_1$, denoted in matrix notation by Π_1. Thus, the condition $\Pi_1 \cdot \nabla C_1 = \mathbf{0}$ implies

$$\begin{aligned}
&\frac{\partial\alpha}{\partial y}\{x,y\}_1 + \left(1+\frac{\partial\alpha}{\partial z}\right)\{x,z\}_1 = 0,\\
&\left(x+\frac{\partial\alpha}{\partial x}\right)\{x,y\}_1 = \left(1+\frac{\partial\alpha}{\partial z}\right)\{y,z\}_1,\\
&\left(x+\frac{\partial\alpha}{\partial x}\right)\{x,z\}_1 = -\frac{\partial\alpha}{\partial y}\{y,z\}_1.
\end{aligned} \tag{5}$$

We also impose the condition that C_2 (3) to be the Hamiltonian of system (4), namely $\Pi_1 \cdot \nabla C_2 = (\dot{x}, \dot{y}, \dot{z})^{\mathrm{t}}$:

$$\begin{aligned}
&\left(y+\frac{\partial\beta}{\partial y}\right)\{x,y\}_1 + \left(z+\frac{\partial\beta}{\partial z}\right)\{x,z\}_1 = \dot{x},\\
&-\frac{\partial\beta}{\partial x}\{x,y\}_1 + \left(z+\frac{\partial\beta}{\partial z}\right)\{y,z\}_1 = \dot{y},\\
&-\frac{\partial\beta}{\partial x}\{x,z\}_1 - \left(y+\frac{\partial\beta}{\partial y}\right)\{y,z\}_1 = \dot{z}.
\end{aligned} \tag{6}$$

Considering (5)–(6) as an algebraic system, and using (4), we obtain:

$$\{x,y\}_1 = 1+\frac{\partial\alpha}{\partial z}\,,\quad \{x,z\}_1 = -\frac{\partial\alpha}{\partial y}\,,\quad \{y,z\}_1 = x+\frac{\partial\alpha}{\partial x}. \tag{7}$$

The Jacobi identity is checked in coordinates, thus $\{.,.\}_1$ given by (7) is a Poisson bracket. Consequently, $(\mathbb{R}^3, \{.,.\}_1, C_2)$ is a Hamilton–Poisson realization of system (4).

Analogously, considering C_2 to be a Casimir function of the second Poisson structure Π_2, and C_1 the Hamiltonian, we obtain

$$\{x,y\}_2 = -z-\frac{\partial\beta}{\partial z}\,,\quad \{x,z\}_2 = y+\frac{\partial\beta}{\partial y}\,,\quad \{y,z\}_2 = -\frac{\partial\beta}{\partial x}. \tag{8}$$

Furthermore, $(\mathbb{R}^3, \{.,.\}_2, C_1)$ is a Hamilton–Poisson realization of system (4). In addition, it immediately follows that $\Pi_1+\Pi_2$ is a Poisson structure. Thus, the Poisson brackets $\{.,.\}_1$ and $\{.,.\}_2$ are compatible. Taking into account system (4) has the form $\Pi_1 \cdot \nabla C_2 = \Pi_2 \cdot \nabla C_1 = (\dot{x}, \dot{y}, \dot{z})^{\mathrm{t}}$; it is a bi-Hamiltonian system.

Now, since $\Pi_1+\Pi_2$ is a Poisson structure, it follows that $\Pi_{a,b} = a\Pi_1 + b\Pi_2$ is a Poisson structure for every $a, b \in \mathbb{R}$. Considering $c, d \in \mathbb{R}$ such that $ad-bc=1$, and the functions $H_{c,d} = -cC_1 + dC_2$, $C_{a,b} = aC_1 - bC_2$, we have $\Pi_{a,b} \cdot \nabla C_{a,b} = \mathbf{0}$, and $\Pi_{a,b} \cdot \nabla H_{c,d} = (\dot{x}, \dot{y}, \dot{z})^{\mathrm{t}}$. Consequently, system (4) has infinitely many Hamilton–Poisson realizations $(\mathbb{R}^3, \Pi_{a,b}, H_{c,d})$ indexed by $a, b, c, d \in SL(2, \mathbb{R})$.

4. A particular integrable deformation of the Maxwell–Bloch equations

In this section, we consider particular functions α and β. We study the dynamics of system (4) in this particular case, namely the stability of the equilibrium points and the existence of the periodic orbits around these points. Moreover, we give the image of the energy-Casimir mapping associated with the considered system. We also present the classification of the topology of the fibers of the energy-Casimir mapping.

Let us consider the functions $\alpha(x,y,z)=\dfrac{k}{2}y^2$ and $\beta(x,y,z)=\dfrac{k}{2x^2}$, where $k\in[0,\infty)$ is a deformation parameter. In this case, system (4) becomes

$$\dot{x}=y-kyz\,,\quad \dot{y}=xz+\frac{k}{x^3}\,,\quad \dot{z}=-xy-\frac{k^2y}{x^3}. \tag{9}$$

We observe that system (9) is invariant under the transformation $(x,y,z)\to(-x,-y,z)$. Therefore, we analyze this system in the case $x\in(0,\infty)$.

The constants of motion of system (9) are given by (3):

$$C(x,y,z)=\frac{1}{2}x^2+\frac{k}{2}y^2+z,\ H(x,y,z)=\frac{k}{2x^2}+\frac{1}{2}y^2+\frac{1}{2}z^2.$$

Considering C a Casimir function and H the Hamiltonian, system (9) has the Hamilton–Poisson realization $(e(1,1)^*, \{.,.\}_1, H)$, where the Poisson bracket $\{.,.\}_1$ given by (7) is in fact a modified Lie–Poisson bracket on the dual of the Lie algebra $e(1,1)$ corresponding to the three-dimensional Lie group of rigid motions of the Minkowski plane, $E(1,1)$ (see, for example, [6]). For details about modified Lie–Poisson structures see, for example, [10].

The equilibrium points of system (9) are given by the family $\mathcal{E}=\{(M,0,-\frac{k}{M^4}):M\in(0,\infty)\}$. It is easy to see that all the equilibrium points of $\mathcal{E}$ are Lyapunov stable [11], via the Lyapunov function

$$L(x,y,z)=\frac{k}{2}\left(\frac{1}{x}-\frac{x}{M^2}\right)^2+\frac{M^4+k^2}{2M^4}y^2+\frac{1}{2}\left(z+\frac{k}{M^4}\right)^2. \tag{10}$$

In the following, we prove the existence of the periodic orbits of the considered system. The eigenvalues of the Jacobian matrix corresponding to system (9) at the equilibrium point $e_M:=(M,0,-\frac{k}{M^4})$ are $\lambda_1=0$, $\lambda_{2,3}=\pm i\omega$, where $\omega=\frac{1}{M^4}\sqrt{(M^4+k^2)(M^6+4k)}$. Thus, we can apply a version of the Moser theorem in the case of zero eigenvalue regarding the existence of periodic orbits around a nonlinearly stable equilibrium point (Theorem 2.1, [2]). Indeed, the function L (10) is a constant of motion of system (9) that satisfy $dL(e_M)=0$ and $d^2L(e_M)|_{W\times W}>0$, where $W=\mathrm{span}_{\mathbb{R}}\{(1,0,-M),(0,1,0)\}$. Consequently, for each sufficiently small $\varepsilon\in\mathbb{R}_+^*$, any integral surface

$$\Sigma_\varepsilon^{e_M}\ :\ \frac{k}{2}\left(\frac{1}{x}-\frac{x}{M^2}\right)^2+\frac{M^4+k^2}{2M^4}y^2+\frac{1}{2}\left(z+\frac{k}{M^4}\right)^2=\varepsilon^2$$

contains at least one periodic orbit of system (9) whose period is close to $\frac{2\pi}{\omega}$.

The Poisson geometric frame of the considered system allows considering the energy-Casimir mapping [12] corresponding to (9):

$$\mathcal{EC}:\mathbb{R}^3\to\mathbb{R}^2\,,\ \ \mathcal{EC}(x,y,z)=\left(\frac{k}{2x^2}+\frac{1}{2}y^2+\frac{1}{2}z^2,\frac{1}{2}x^2+\frac{k}{2}y^2+z\right). \tag{11}$$

Taking into account the connections between the images of the energy-Casimir mapping and the dynamics of some particular systems of differential equations [3,6–9,12], it is natural to ask whether the same properties are obtained in our case, namely if the boundary of the set $\mathrm{Im}(\mathcal{EC})\subsetneq\mathbb{R}^2$ is the union of the images of the stable equilibrium points through $\mathcal{EC}$, and the image of the energy-Casimir mapping is convexly generated by these images, as well as if the fibers corresponding to the points that belong to the interior of the set $\mathrm{Im}(\mathcal{EC})$ are periodic orbits.

The image of $\mathcal{EC}$ is the set $\mathrm{Im}(\mathcal{EC})=\{(h,c)\in\mathbb{R}^2|(\exists)(x,y,z)\in\mathbb{R}^3:\mathcal{EC}(x,y,z)=(h,c)\}$. In some cases, this set is obtained from the algebraic system $H(x,y,z)=h$, $C(x,y,z)=c$. From geometric point of view, in our case this system has solutions if and only if the surfaces $\frac{k}{x^2}+y^2+z^2=2h$, $x^2+ky^2+2z=2c$ have nonempty intersection (Fig. 1).

We notice that $\mathcal{EC}(e_M)=(\frac{k}{2M^2}+\frac{k^2}{2M^8},\frac{M^2}{2}-\frac{k}{M^4})\overset{\text{not}}{=}(h_M^e,c_M^e)$. Considering the function $f(x,y,z)=\frac{k}{x^2}+y^2+z^2-2h$ and the constraint $x^2+ky^2+2z=2c_M^e$, we obtain that e_M is a local minimum point of f. Taking into account the behavior of the aforementioned surfaces when h and c vary, we obtain that for every fixed value of c, $c=c_M^e$, $\mathrm{Im}(\mathcal{EC})$ is an empty set for $h<h_M^e$. We conclude that $\mathrm{Im}(\mathcal{EC})$ is convexly generated by the points of the curve $\mathrm{Im}(\mathcal{EC})|_{\mathcal{E}}$ given parametrically by the equations $h=\frac{k}{2M^2}+\frac{k^2}{2M^8}$, $c=\frac{M^2}{2}-\frac{k}{M^4}$, $M\in(0,\infty)$ (Fig. 2).

Now we present the connexions between the image of the energy-Casimir mapping and the dynamics of system (9). For each $(h,c)\in\mathrm{Im}(\mathcal{EC})$, a fiber of the energy-Casimir mapping is the set $\mathcal{F}_{(h,c)}=\{(x,y,z)\in\mathbb{R}^3:\mathcal{EC}(x,y,z)=(h,c)\}$. We deduce that for every $(h,c)\in\partial\mathrm{Im}(\mathcal{EC})$, there is $M\in(0,\infty)$ such that $\mathcal{F}_{(h,c)}=\{e_M\}$. Furthermore, if $(h,c)\in Int(\mathrm{Im}(\mathcal{EC}))$, then $\mathcal{F}_{(h,c)}$ is a periodic orbit (Fig. 1, right). Therefore, there are two types of fibers: nonlinearly stable equilibrium points, and periodic orbits.

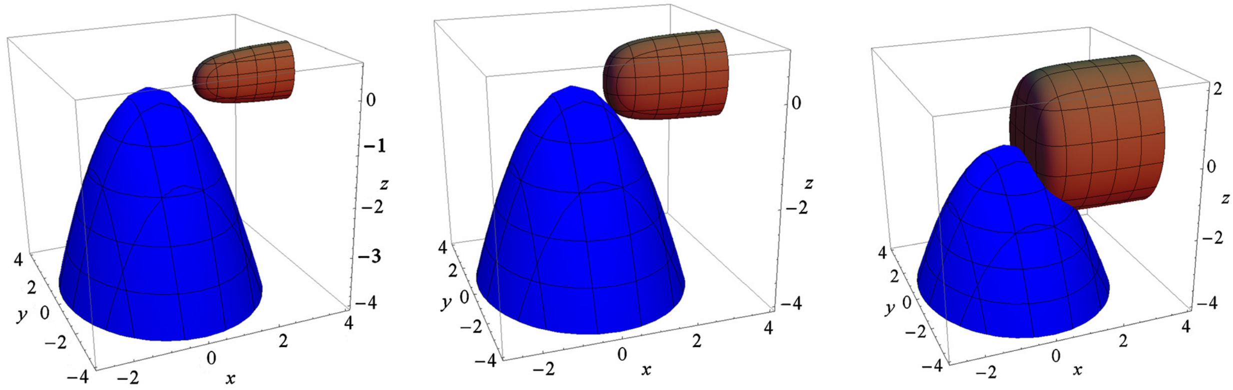

Fig. 1. Relative positions of the surfaces $H(x, y, z) = h$, $C(x, y, z) = c$. (Position relative des surfaces $H(x, y, z) = h$, $C(x, y, z) = c$.)

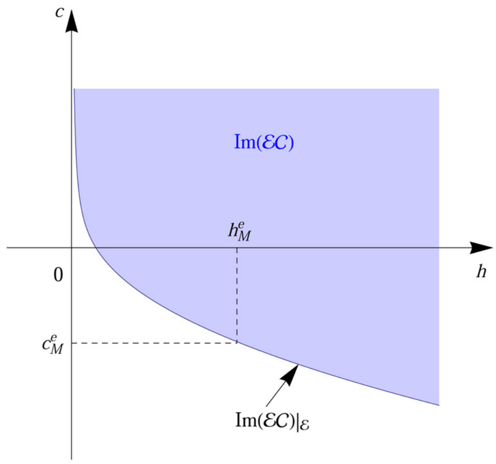

Fig. 2. The image of the energy-Casimir mapping. (L'image de la fonction énergie-Casimir.)

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